Retarded Integral Inequalities with Iterated Integrals

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Abstract

This paper presents an iterated and iterated retarded integral inequalities and explicit bounds to unknown functions on some iterated and iterated retarded integral inequalities are established.

keywords: Integral inequalities, Retarded integral inequalities, Non-decreasing functions, Non-negative continuous functions, partial derivatives, Explicit Bounds.

تكامل المترجحات المتكررة والتباطؤية

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الملخص

ان الهدف الرئيسي من هذا البحث هو تقديم المتراجحات التكاملية المتكررة واعطاء قيود صريحة للدوال المجهولة في بعض المتراجحات التكاملية المتكررة والمتراجحات التكاملية المتكررة التباطؤية .

1. Introduction

Integral inequalities with iterated integrals are indispensable for us in the quantitative study of various differential equations and integral equations. motivated by a desire to apply integral inequalities which provide explicit bounds on unknown functions, in the development of the theory of differential and integral equations with retarded arguments.

Lemma 1 .

Let $u(x)$ and $g(x)$ be nonnegative continuous functions on $I = [0,\infty)$ for the inequality

$$u(x) \le c + \int_a^x g(t)u(t)dt \quad , \quad t \in I,$$

holds, where c is constant. Then

$$u(x) \le c\, e^{\int_a^x g(t)dt} \quad , t \in I .$$

The result was proved by Gronwall [13]. Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types see [2-11, 15].Some Gronwall-Bellman type integral inequalities with fixed delay has been presented in [1].

The aim of the present paper is to establish explicit bounds on more general integral inequalities with iterated and iterated retarded integral inequalities.

The plan of the paper is as follows: Section 2 presents some iterated integral inequalities. Section 3 presents some iterated retarded integral inequalities. Finally, Section 4 presents a short conclusion.

2. Integral Some Iterated Inequalities

This section presents some iterated integral inequalities and then give explicit bounds to unknown functions. Later on $\mathbb{R}_+ = [0,\infty]$, $I = [\alpha,\beta]$ and $I_1 = [t_\circ,\beta]$.

Theorem : (2.1)

Let
$u(t), f(t), a(t), g(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+), k(t,s), k_t(t,s) \in C(D, \mathbb{R}_+)$ and $p > 1$
be real constant, where

$$D = \{(t,s) \in \mathbb{R}_+^2 : 0 \le s \le t \le \infty\}.$$

$(\boldsymbol{a_1})$

If $u^p(t) \le a(t) + \int_0^t f(s)\left[u^p(s) + \int_0^s k(s,\sigma)u(\sigma)d\sigma\right]ds$, …..(2.1)

for $t \in \mathbb{R}_+$, then

$$u(t) \le \left[a(t) + \int_0^t A_1(s)\, exp^{\int_s^t B_1(\tau)d\tau} ds\right]^{\frac{1}{p}} \quad , \quad \text{…… …..} \ (2.2)$$

for $t \in \mathbb{R}_+$, where $A_1(t) = \frac{f(t)}{p}[pa(t) + \int_0^t k(t,\sigma)[(p-1) + a(\sigma)]d\sigma]$ and

$$B_1(t) = f(t)[1 + \int_0^t \frac{k(t,\sigma)}{p} d\sigma].$$

$(\boldsymbol{a_2})$

Let $c(t)$ be real-valued positive continuous and nondecreasing function defined in $\mathbb{R}_+$.

If $u^p(t) \le c^p(t) + \int_0^t f(s)\{u^{\{p\}}(s) + \int_0^s k(s,\sigma)u(\sigma)d\sigma\}ds$, ….. (2.3)

for $t \in \mathbb{R}_+$, then

$$u(t) \le c(t)\left[1 + \int_0^t A_2(s)\ exp^{\int_s^t B\_\{2\}(\tau)d\tau} ds\right]^{\frac{1}{p}} \quad \text{……} \ (2.4)$$

for $t \in \mathbb{R}_+$, where

$$A_2(t) = f(t)\left[1 + \int_0^t k(t,\sigma)\, c^{1-p}(\sigma)d\sigma\right]$$

and

$$B_2(t) = f(t)\left[1 + \int_0^t \frac{\{k(t,\sigma)c^{1-p}(\sigma)\}}{p} d\sigma\right].$$

$(\boldsymbol{a_3})$

If

$$u^p(t) \le a(t) + \int_0^t k(t,s)\ \{u(s) + \int_0^s [g(\sigma)u^p(\sigma) + h(\sigma)u(\sigma)]d\sigma\}ds \ ,$$

….. (2.5)

for $t \in \mathbb{R}_+$, then

$$u(t) \le \left[a(t) + \int_0^t A_3(s)\ exp^{\int_s^t B_3(\tau)d\tau} ds\right]^{\frac{1}{p}} \quad \text{…….} \ (2.6)$$

for $t \in \mathbb{R}_+$, where

$$A_3(t) = \frac{k(t,t)}{p}\left[(p-1) + a(t) + \int_0^t [pg(\sigma)a(\sigma) + h(\sigma)[\,(p-1) + a(\sigma)]]d\sigma\right]$$

$$+ \int_0^t \frac{k_t(t,s)}{p}\left[(p-1) + a(s) + \int_0^s [pg(\sigma)a(\sigma) + h(\sigma)[(p-1) + a(\sigma)]]d\sigma\right] ds,$$

And

$$B_3(t) = \frac{k(t,t)}{p}\left[1 + \int_0^t [pg(\sigma) + h(\sigma)]\, d\sigma\right] + \int_0^t \frac{k_t(t,s)}{p}\left[1 + \int_0^s [pg(\sigma) + h(\sigma)]d\sigma\,\right]ds.$$

Proof:

$(\boldsymbol{a_1})$

Defin a function $z(t)$ by $\quad z(t) = \int_0^t f(s)\left\{u^p(s) + \int_0^s k(s,\sigma)u(\sigma)d\sigma\right\} ds.$ ….. (2.7)

Then $z(t) \geq 0$, $z(t)$ is nondecreasing for $t \in I$ and inequality (2.1) can be written as

$$u^p(t) \leq a(t) + z(t). \qquad ….. (2.8)$$

From inequality (2.8) and using the elementary inequality see [14], [12]

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}$$

Where $x, y \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1,$

we observe that
$$u(t) \leq [\,a(t) + z(t)]^{\frac{1}{p}}\,(1)^{\frac{\frac{1}{p}}{(p-1)}}$$
$$\leq \frac{(p-1)}{p} + \frac{a(t)}{p} + \frac{z(t)}{p}, \qquad ……. (2.9)$$

Differentiating (2.7) and using (2.8), (2.9) we get:

$$z'(t) = f(t)\left\{u^p(t) + \int_0^t k(t,\sigma)u(\sigma)d\sigma\right\}$$

$$\leq f(t)\left\{a(t) + z(t)\right.$$

$$+ \int_0^t k(t,\sigma)\left[\frac{(p-1)}{p} + \frac{a(\sigma)}{p}\right.$$

$$\left.\left.+ \frac{z(\sigma)}{p}\right]d\sigma\right\}$$

$$= \frac{f(t)}{p}\left\{pa(t) + \int_0^t k(t,\sigma)\,[(p-1) + a(\sigma)]d\sigma\right\}$$

$$+ f(t)\left\{1 + \int_0^t \frac{k(t,\sigma)}{p}d\sigma\right\}z(t)$$

$$= A_1(t) + B_1(t)z(t).$$

Integrating both sides of the above inequality from 0 to t we get :

$$z(t) \leq \int_0^t A_1(s)\, exp^{\int_s^t B_1(\tau)d\tau}\, ds \quad \ldots\ldots \quad (2.10)$$

Using (2.10) in (2.8), we get the required inequality in (2.2).

$(\boldsymbol{a_2})$

Since $c(t)$ is positive, continuous and nondecreasing function for $t \in \mathbb{R}_+$, from(2.3) then one can get :

$$\left(\frac{u(t)}{c(t)}\right)^p \leq$$

$$1 + \int_0^t f(s)\left\{\left(\frac{u(s)}{c(s)}\right)^p + \int_0^s k(s,\sigma)\, c^{1-p}(\sigma)\frac{u(\sigma)}{c(\sigma)}d\sigma\right\}ds$$

.

Now an application of the inequality given in(a_1) yields the desired result in(2.4).

$(\boldsymbol{a_3})$

Define the function $z(t)$ by

$$z(t) = \int_0^t k(t,s)\left\{u(s) + \int_0^s [g(\sigma)u^p(\sigma) + h(\sigma)u(\sigma)]d\sigma\right\}ds \quad \ldots.(2.11)$$

Then as in the proof of part (a_1), from (2.11) we see that the inequalities (2.8) and (2.9) hold. Differentiating (2.11) and using (2.8), (2.9) and the fact that $z(t)$ is nondecreasing in t we get:

$$z'(t) = k(t,t)\left\{u(t) + \int_0^t [g(\sigma)u^p(\sigma) + h(\sigma)u(\sigma)]\,d\sigma\right\} + \int_0^t k_t(t,s)\left\{u(s) + \int_0^s [g(\sigma)u^p(\sigma) + h(\sigma)u(\sigma)]d\sigma\right\} ds$$

$$\leq k(t,t)\left\{\frac{p-1}{p} + \frac{a(t)}{p} + \frac{z(t)}{p} + \int_0^t \left[g(\sigma)[a(\sigma) + z(\sigma)] + h(\sigma)\left[\frac{p-1}{p} + \frac{a(\sigma)}{p} + \frac{z(\sigma)}{p}\right]\right] d\sigma\right\}$$

$$+ \int_0^t k_t(t,s)\left\{\frac{p-1}{p} + \frac{a(s)}{p} + \frac{z(s)}{p} + \int_0^s [g(\sigma)[a(\sigma) + z(\sigma)] + h(\sigma)[\frac{p-1}{p} + \frac{a(\sigma)}{p} + \frac{z(\sigma)}{p}]]d\sigma\right\} ds$$

$$= \frac{k(t,t)}{p}\left\{(p-1) + a(t) + \int_0^t [pg(\sigma)a(\sigma) + h(\sigma)[(p-1) + a(\sigma)]]\,d\sigma\right\}$$

$$+ \int_0^t \frac{k_t(t,s)}{p}\left\{(p-1) + a(s) + \int_0^s [pg(\sigma)a(\sigma) + h(\sigma)[(p-1) + a(\sigma)]]\,d\sigma\right\} ds + \frac{k(t,t)}{p}$$

$$\left\{z(t)+\int_0^t [\,pg(\sigma)z(\sigma)+h(\sigma)z(\sigma)]d\sigma\right\}$$
$$+\int_0^t \frac{k_t(t,s)}{p}\left\{z(s)\right.$$
$$\left.+\int_0^s [\,pg(\sigma)z(\sigma)+h(\sigma)z(\sigma)]d\sigma\right\}ds$$

$$= \quad A_3(t)+B_3(t)z(t).$$

Integrating both sides of the above inequality from 0 to t yields

$$z(t)\le \int_0^t A_3(s)exp^{\int_s^t B_3(\tau)d\tau}ds\,. \qquad \ldots\ldots \quad (2.12)$$

Using (2.12) in (2.8), we get the required inequality in (2.6).

Theorem : (2.2)

Let $u(t), g(t)\in C(I,\mathbb{R}_+)$, $k(t,s), b(t,s), c(t,s)\in C(D,\mathbb{R}_+)$, $h(t,s,\sigma)\in C(E,\mathbb{R}_+)$ and $a(t), a'(t)\in C(I,\mathbb{R}_+)$, $p>1$ be real constant.

$(\boldsymbol{b_1})$

Let $\phi(t)\in C(I,\mathbb{R}_+)$ and $u^p(t)\le a(t)+\int_\alpha^t \phi(s)u(s)ds+\int_\alpha^t\int_\alpha^s k(s,\tau)u(\tau)d\tau\,ds+$
$$\int_\alpha^t\int_\alpha^s\int_\alpha^\tau h(s,\tau,\sigma)u(\sigma)d\sigma\,d\tau\,ds+\int_\alpha^\beta\int_\alpha^s c(s,\tau)u(\tau)d\tau\,ds$$
$$\ldots.. \quad (2.13)$$

for $t\in I$. If

$$P_1=\frac{1}{p}\int_\alpha^\beta\int_\alpha^s c(s,\tau)\,exp^{\int_\alpha^\tau B_4(\xi)d\xi}d\tau\,ds \;<\; 1 \qquad \ldots\ldots \quad (2.14)$$

then

$$u(t)\le\left[a(t)+M_1 exp^{\int_\alpha^t B_4(\xi)d\xi}+\int_\alpha^t A_4(\eta)\,exp^{\int_\alpha^t B_4(\xi)d\xi}d\eta\right]^{\frac{1}{p}} \qquad \ldots..\ (2.15)$$

for $t\in I$, where

$$A_4(t)=\frac{1}{p}\{\phi(t)[(p-1)+a(t)]+\int_\alpha^t k(t,\tau)\,[\,(p-1)$$
$$+a(\tau)]d\tau$$
$$+\int_\alpha^t\int_\alpha^\tau h(t,\tau,\sigma)\,[(p-1)+a(\sigma)]d\sigma\,d\tau\},$$

$$B_4(t)=\frac{1}{p}\left\{\phi(t)+\int_\alpha^t k(t,\tau)d\tau+\int_\alpha^t\int_\alpha^\tau h(t,\tau,\sigma)d\sigma\,d\tau\right\}$$

and

$M_1 =$

$$\frac{1}{1-P_1}\left\{\frac{1}{p}\int_{\alpha}^{\beta}\int_{\alpha}^{s} c(s,\tau)\,[\,(p-1)+a(\tau) + \int_{\alpha}^{\tau} A_4(\eta)\, exp^{\int_{\alpha}^{\tau} B_4(\xi)d\xi}\, d\eta]d\tau\, ds\right\}$$

$(\boldsymbol{b_2})$

Let $k(t,s),\ b(t,s), h_t(t,s,\sigma)$ are nondecreasing in $t \in I$, for each $s \in I$ and

$$u^p(t) \le a(t) + \int_{\alpha}^{t} k(t,\tau)u(\tau)d\tau + \int_{\alpha}^{t}\int_{\alpha}^{s} h(t,s,\sigma)u(\sigma)d\sigma\, ds + \int_{\alpha}^{\beta} b(t,s)\int_{\alpha}^{s} c(s,\tau)u(\tau)d\tau\, ds$$

for $t \in I$. …… (2.16)

If $P_2 = \frac{1}{p}\int_{\alpha}^{\beta} b(t,s)\int_{\alpha}^{s} c(s,\tau)\, exp^{\int_{\alpha}^{\tau} B_5(\eta,\tau)d\eta} d\tau\, ds < 1$ ….. (2.17)

then

$$u(t) \le \left[a(t) + M_2 exp^{\int_{\alpha}^{t} B_5(\eta,t)d\eta} + \int_{\alpha}^{t} A_5(\xi,t)\, exp^{\int_{\xi}^{t} B_5(\eta,t)d\eta} d\xi\right]^{\frac{1}{p}}$$

….. (2.19)

for $t \in I$, where

$$A_5(t,T) = \frac{1}{p}\left\{k(T,t)[(p-1)+a(t)] + \int_{\alpha}^{t} h(T,t,\sigma)\,[\,(p-1)+a(\sigma)]d\sigma\right\}$$

,

$$B_5(t,T) = \frac{1}{p}\left\{k(T,t) + \int_{\alpha}^{t} h(T,t,\sigma)d\sigma\right\}$$

and

$$M_2 = \frac{1}{1-P_2}\left\{\frac{1}{p}\int_{\alpha}^{\beta} b(t,s)\int_{\alpha}^{s} c(s,\tau)\,[\,(p-1)+a(\tau) + \int_{\alpha}^{\tau} A_5(\xi,\tau)\, exp^{\int_{\xi}^{\tau} B_5(\eta,\tau)d\eta} d\xi]d\tau\, ds\right\}$$

.

$(\boldsymbol{b_3})$

Let $r(t) \in C(I,\mathbb{R}_+)$ and

$$u^p(t) \le a(t) + \int_\alpha^t g(s) \left\{ u(s) + \int_\sigma^s k(s,\sigma)u(\sigma)d\sigma + \int_\alpha^\beta r(\sigma)u(\sigma)d\sigma \right\}$$

……….(2.20)

for $t \in I$. If

$$P_3 = \int_\alpha^\beta r(\sigma)\, exp^{\int_\alpha^\sigma B_6(\eta)d\eta} d\sigma < 1, \qquad …… (2.21)$$

then

$$u(t) \le \left[a(t) + M_3 exp^{\int_\alpha^t B_6(\eta)d\eta} + \int_\alpha^t A_6(\xi)\, exp^{\int_\alpha^t B_6(\eta)d\eta} d\xi \right]^{\frac{1}{p}}, \qquad … (2.21)$$

for $t \in I$, where

$$A_6(t) = g(t)\frac{1}{p}\left\{ (p-1) + a(t) + \int_\alpha^t k(t,\sigma)\,[(p-1) + a(\sigma)]d\sigma + \int_\alpha^\beta r(\sigma)\,[(p-1) + a(\sigma)]d\sigma \right\}$$

,

$B_6(t) = \frac{1}{p}g(t) + k(t,t) + \int_\alpha^t k_t(t,\sigma)d\sigma$ and

$$M_3 = \frac{1}{1-P_3}\left\{ \int_\alpha^\beta r(\sigma) \int_\alpha^\sigma A_6(\xi)\, exp^{\int_\xi^\tau B_6(\eta)d\eta} d\xi\, d\sigma \right\}.$$

Proof :

$(\boldsymbol{b_1})$

Define a function $z(t)$ by

$$z(t) = \int_\alpha^t \phi(s)u(s)ds + \int_\alpha^t \int_\alpha^s k(s,\tau)u(\tau)d\tau\, ds + \int_\alpha^t \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)u(\sigma)d\sigma\, d\tau\, ds + \int_\alpha^\beta \int_\alpha^s c(s,\tau)u(\tau)d\tau\, ds$$

……… (2.22)

Then $z(t) \ge 0$, $z(t)$ is nondecreasing for $t \in I$

$$z(\alpha) = \int_\alpha^\beta \int_\alpha^s c(s,\tau)u(\tau)d\tau\, ds \qquad …………….(2.23)$$

Then as in the proof of part (a_1), from (2.23)we see that the inequalities(2.8)and(2.9)hold. Differentiating(2.23)and using(2.9)and the fact that $z(t)$ is nondecreasing in t, we get:

$$z'(t) = \phi(t)u(t) + \int_{\alpha}^{t} k(t,\tau)u(\tau)d\tau + \int_{\alpha}^{t}\int_{\alpha}^{\tau} h(t,\tau,\sigma)u(\sigma)d\sigma\, d\tau$$

$$\leq \phi(t)\frac{1}{p}[(p-1) + a(t) + z(t)] + \int_{\alpha}^{t} k(t,\tau)\frac{1}{p}[(p-1) + a(\tau) + z(\tau)]\, d\tau$$

$$+ \int_{\alpha}^{t}\int_{\alpha}^{\tau} h(t,\tau,\sigma)\frac{1}{p}[(p-1) + a(\sigma) + z(\sigma)]d\sigma d\tau$$

$$= \frac{1}{p}\{\phi(t)[(p-1) + a(t)] + \int_{\alpha}^{t} k(t,\tau)[(p-1) + a(\tau)]d\tau + \int_{\alpha}^{t}\int_{\alpha}^{\tau} h(t,\tau,\sigma)[(p-1) + a(\sigma)]d\sigma\, d\tau\}$$

$$+\frac{1}{p}\{\phi(t)z(t) + \int_{\alpha}^{t} k(t,\tau)z(\tau)d\tau + \int_{\alpha}^{t}\int_{\alpha}^{\tau} h(t,\tau,\sigma)z(\sigma)d\sigma\, d\tau\}.$$

But $z(t)$ is nonnegative and nondecreasing for $t \in I$, then
$z'(t) \leq A_4(t) + B_4(t)z(t)$
Therefore, $\alpha \leq \eta \leq t \leq \beta$, one can have :

$$\frac{d}{d\eta}\left[z(\eta)exp^{\int_{\eta}^{t} B_4(\xi)d\xi}\right] \leq A_4(\eta)exp^{\int_{\eta}^{t} B_4(\xi)d\xi}$$

Integrating both sides of the above inequality from α to t, for $t \in I$, we get:

$$z(t) \leq z(\alpha)exp^{\int_{\alpha}^{t} B_4(\xi)d\xi} + \int_{\alpha}^{t} A_4(\eta)\, exp^{\int_{\eta}^{t} B_4(\xi)d\xi} d\eta, \quad \text{………(2.24)}$$

from (2.24) and (2.9) one can get

$$u(t) \leq \frac{1}{p}z(\alpha)exp^{\int_{\alpha}^{t} B_4(\xi)d\xi} + \frac{1}{p}[(p-1) + a(t) + \int_{\alpha}^{t} A_4(\eta)\, exp^{\int_{\eta}^{t} B_4(\xi)d\xi} d\eta$$

.

……(2.25)

From (2.23) and (2.25) Which implies

$$z(\alpha) \le \int_{\alpha}^{\beta}\int_{\alpha}^{s} c(s,\tau)\left\{\frac{1}{p} z(\alpha) exp^{\int_{\alpha}^{\tau} B_4(\xi)d\xi} + \frac{1}{p}\left[(p-1) + a(\tau) + \int_{\alpha}^{\tau} A_4(\eta) exp^{\int_{\alpha}^{\tau} B_4(\xi)d\xi} d\eta\right]\right\} d\tau ds$$

then

$$z(\alpha)\left[1 - \frac{1}{p}\int_{\alpha}^{\beta}\int_{\alpha}^{s} c(s,\tau) exp^{\int_{\alpha}^{\tau} B_4(\xi)d\xi} d\tau ds\right] \le \int_{\alpha}^{\beta}\int_{\alpha}^{s} c(s,\tau)\frac{1}{p}[(p-1) + a(\tau) + \int_{\alpha}^{\tau} A_4(\eta) exp^{\int_{\alpha}^{\tau} B_4(\xi)d\xi} d\eta] d\tau ds$$

from (2.14) we obtain that

$$z(\alpha) \le M_1. \quad …….(2.26)$$

The required inequality (2.15) follows from (2.26), (2.24) and (2.8).

$(\boldsymbol{b_2})$

Fix any T, $\alpha \le T \le \beta$, then for $\alpha \le t \le T$, we have

$$u^p(t) \le a(t) + \int_{\alpha}^{t} k(T,\tau)u(\tau)d\tau + \int_{\alpha}^{t}\int_{\alpha}^{s} h(T,s,\sigma)u(\sigma)d\sigma\, ds + \int_{\alpha}^{\beta} b(T,s)\int_{\alpha}^{s} c(s,\tau)u(\tau)d\tau\, ds,$$

…….. (2.27)

Define a Function $z(t,T)$, $\alpha \le t \le T$ by

$$z(t,T) = \int_{\alpha}^{t} k(T,\tau)u(\tau)d\tau + \int_{\alpha}^{t}\int_{\alpha}^{s} h(T,s,\sigma)u(\sigma)d\sigma\, ds + \int_{\alpha}^{\beta} b(T,s)\int_{\alpha}^{s} c(s,\tau)u(\tau)d\tau\, ds$$

…….(2.28)

Then $z(t,T) \ge 0$, $z(t,T)$ is nondecreasing for $t \in I$,

$z(\alpha,T) = \int_\alpha^\beta b(T,s)\int_\alpha^s c(s,\tau)u(\tau)d\tau\, ds,$ …………(2.29)

and inequality (2.16) can be written as

$u^p(t) \leq\ a(t) + z(t,T), \qquad \alpha \leq\ t\ \leq\ T.$ ……. (2.30)

Then as in the proof of part (a_1), from (2.30) we see that the inequalities (2.31) hold.

$$u(t) \leq \frac{(p-1)}{p} + \frac{a(t)}{p} + \frac{z(t,T)}{p}, \qquad \alpha \leq\ t\ \leq\ T$$

…….. (2.31)

Differentiating (2.28) and using (2.31) and the fact that $z(t,T)$ is nondecreasing in t, we get

$$z'(t,T) = k(T,t)u(t) + \int_\alpha^t h(T,t,\sigma)u(\sigma)d\sigma$$

$$\leq \frac{1}{p}\{\, k(T,t)[(p-1) + a(t)]$$

$$+ \int_\alpha^t h(T,t,\sigma)\,[(p-1) + a(\sigma)]d\sigma\,\} + \frac{1}{p}\{\, k(T,t)$$

$$+ \int_\alpha^t h(T,t,\sigma)d\sigma\,\}\, z(t,T),$$

then

$$z'(t,T) \leq\ A_5(t,T) + B_5(t,T)z(t,T)$$ ……….(2.32)

for $\alpha\ \leq\ T$ by setting $t = \eta$ in (2.32) and integrating it with respect to η from α to T, we get:

$$z(T,T) \leq$$
$$z(\alpha\,,T)\ exp^{\int_\alpha^T B_5(\xi,T)d\xi} + \int_\alpha^T A_5(\eta,T)\ exp^{\int_\eta^T B_5(\xi,T)d\xi}\, d\eta\ ,$$

……..….(2.33)

Since T is arbitrary from (2.33), (2.31), (2.30) and (2.29) with T replaced by t one can get

$$z(t,t) \leq$$
$$z(\alpha,t)\ exp^{\int_\alpha^t B_5(\xi,t)d\xi} + \int_\alpha^t A_5(\eta,t)\ exp^{\int_\alpha^t B_5(\xi,t)d\xi}\, d\eta\ ,$$

………..(2.34)

$$u^p(t) \leq a(t) + z(t,t)\ ,$$ …………(2.35)

$$u(t) \le \frac{1}{p}\{(p-1) + a(t) + z(t,t)\}$$

$$\le \frac{1}{p}\{(p-1) + a(t) + z(\alpha,t)\ exp^{\int_\alpha^t B_5(\xi,t)d\xi} + \int_\alpha^t A_5(\eta,t)\ exp^{\int_\eta^t B_5(\xi,t)d\xi} d\eta\ ,$$

…….(2.36)

Where

$$z(\alpha,t) = \int_\alpha^\beta b(t,s) \int_\alpha^s c(s,\tau)u(\tau)d\tau\ ds \qquad \text{…… ….}\ (2.37)$$

then from (2.37) and (2.36) Which implies

$$z(\alpha,t) \le \int_\alpha^\beta b(t,s) \int_\alpha^s c(s,\tau)\frac{1}{p}\{\ (p-1) + a(\tau) + z(\alpha,\tau)\ exp^{\int_\alpha^\tau B_5(\xi,\tau)d\xi} + \int_\alpha^\tau A_5(\eta,\tau)\ exp^{\int_\eta^\tau B_5(\xi,\tau)d\xi} d\eta\}\ d\tau\ ds.$$

But $z(t,T)$ is nonnegative and nondecreasing for $t\ \in I$, then

$$z(\alpha,t)\left[1 - \frac{1}{p}\int_\alpha^\beta b(t,s)\int_\alpha^s c(s,\tau)\ exp^{\int_\alpha^\tau B_5(\xi,\tau)d\xi}\ d\tau\ ds\right] \le$$

$$\int_\alpha^\beta b(t,s) \int_\alpha^s c(s,\tau)\frac{1}{p}\Big\{(p-1) + a(\tau)\ + + \int_\alpha^\tau A_5(\eta,\tau)\ exp^{\int_\eta^\tau B_5(\xi,\tau)d\xi} d\eta\Big\} d\tau\ ds$$

from (2.17) we obtain that

$$z(\alpha,t) \le\ M_2\ . \qquad \text{………..(2.38)}$$

The required inequality (2.18) follows from (2.38), (2.34) and (2.35).

$(\boldsymbol{b_3})$

Define the function $z(t)$ by

$$z(t) = \int_\alpha^t g(s)\left\{u(s) + \int_\alpha^s k(s,\sigma)u(\sigma)d\sigma + \int_\alpha^\beta r(\sigma)u(\sigma)d\sigma\right\}ds \quad \text{.......} (2.39)$$

Then $z(t) \geq 0$, $z(t)$ is non-decreasing for $t \in I, z(\alpha) = 0$.
Then as in the proof of part (a_1). From (2.39) we see that the inequalities (2.8) and (2.9) hold. Differentiating (2.39) and using (2.9) and the fact that $z(t)$ is nondecreasing in t, we get

$$z'(t) = g(t)[u(t) + \int_\alpha^t k(t,\sigma)u(\sigma)d\sigma + \int_\alpha^\beta r(\sigma)u(\sigma)d\sigma]$$

$$\leq \frac{1}{p}g(t)\{(p-1) + a(t) + \int_\alpha^t k(t,\sigma)[(p-1) + a(\sigma)]d\sigma + \int_\alpha^\beta r(\sigma)[(p-1) + a(\sigma)]d\sigma\}$$

$$+\frac{1}{p}g(t)\{z(t) + \int_\alpha^t k(t,\sigma)z(\sigma)d\sigma + \int_\alpha^\beta r(\sigma)z(\sigma)d\sigma\}$$

$$= A_6(t) + \frac{1}{p}g(t)\left\{z(t) + \int_\alpha^t k(t,\sigma)z(\sigma)d\sigma + \int_\alpha^\beta r(\sigma)z(\sigma)d\sigma\right\}.$$

Let

$$v(t) = z(t) + \int_\alpha^t k(t,\sigma)z(\sigma)d\sigma + \int_\alpha^\beta r(\sigma)z(\sigma)d\sigma \quad \text{..........} (2.40)$$

then $v(t) \geq 0$ and nondecreasing for $t \in I$, and since $z(\alpha) = 0$ then

$$v(\alpha) = \int_\alpha^\beta r(\sigma)z(\sigma)d\sigma \quad \text{..........} (2.41)$$

$$z(t) \leq v(t) \quad \text{.........} (2.42)$$

$$z'(t) \leq A_6(t) + \frac{1}{p}g(t)v(t). \quad \text{..........} (2.43)$$

Differentiating both sides of (2.40) and using (2.42) and (2.43), We get:

$$v'(t) = z'(t) + k(t,t)z(t)d\sigma + \int_\alpha^t k_t(t,\sigma)z(\sigma)d\sigma$$

$$\leq A_6(t) + \frac{1}{p} g(t)v(t) + k(t,t)v(t)d\sigma + \int_{\alpha}^{t} k_t(t,\sigma)\, v(\sigma)d\sigma$$

then

$$v'(t) \leq A_6(t) + B_6(t)v(t) \qquad \text{……….. (2.44)}$$

Integrating both sides of (2.44) from α to t , for $t \in I$, and using (2.42), we get:

$$z(t) \leq v(\alpha) exp^{\int_{\alpha}^{t} B_6(\xi)d\xi} + \int_{\alpha}^{t} A_6(\eta)\, exp^{\int_{\eta}^{t} B_6(\xi)d\xi} d\eta \qquad \text{……… (2.45)}$$

from (2.45) and (2.41), one can get:

$$v(\alpha)\left[1 - \int_{\alpha}^{\beta} r(\sigma)\, exp^{\int_{\alpha}^{\sigma} B_6(\xi)d\xi} d\sigma\right]$$

$$\leq \int_{\alpha}^{\beta} r(\sigma) \int_{\alpha}^{\sigma} A_6(\eta)\, exp^{\int_{\eta}^{\sigma} B_6(\xi)d\xi} d\eta d\sigma$$

from (2.20) which implies

$$v(\alpha) \leq M_3 \qquad \text{……….. (2.46)}$$

The required inequality (2.21) follows from (2.46), (2.45) and (2.9). From the hypotheses, we observe that $\alpha'(t) \geq 0$ for $t \in I_1$.

3. Iterated Retarded Integral Inequalities

In this section we prove obtain explicit bounds to unknown functions in the some iterated retarded integral inequalities ,in the following theorem we take the single integral inequalities and in another theorem we take the double and triple integral inequalities.

Theorem : (3.1)

Let $u(t), f(t), a(t), g(t)$ and $h(t) \in C(I, \mathbb{R}_+)$, $k(t,s) \in C(I^2, \mathbb{R}_+)$ for $\$t_\circ \leq s \leq t \leq T$, $\alpha(t) \in C^1(I,I)$ be non-decreasing with $\alpha(t) \leq t$ on I and $p > 1$ be real constant.

(c_1)

If

$$u^p(t) \leq a(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} f(s) \left\{u^p(s) + \int_{\alpha(t_\circ)}^{s} k(s,\sigma)u(\sigma)d\sigma\right\} ds \qquad \text{…… (3.1)}$$

for $t \in I$, then

$$u(t) \leq \left[a(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} D_1(s)\, exp^{\int_{s}^{\alpha(t)} E_1(\xi)d\xi} ds\right]^{\frac{1}{p}}, \qquad \text{…… (3.2)}$$

for $t \in I$, where

$D_1(t) = f(t)\left[a(t) + \int_{\alpha(t_0)}^{t} k(t,\sigma)\left[\frac{p-1}{p} + \frac{a(\sigma)}{p}\right]d\sigma\right]$ and

$E_1(t) = f(t)[1 + \int_{\alpha(t_0)}^{t} \frac{k(t,\sigma)}{p} d\sigma]$.

(c_2)

Let $c(t)$ be real-valued positive continuous and nondecreasing function defined in I. If

$$u^p(t) \le c^p(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)\left\{u^p(s) + \int_{\alpha(t_0)}^{s} k(s,\sigma)u(\sigma)d\sigma\right\}ds, \quad \ldots\ldots.(3.3)$$

for $t \in I$, then

$$u(t) \le c(t)\left[1 + \int_{\alpha(t_0)}^{\alpha(t)} D_2(s)\, exp^{\int_s^{\alpha(t)} E_2(\xi)d\xi} ds\right]^{\frac{1}{p}} \quad \ldots\ldots (3.4)$$

for $t \in I$, where

$D_2(t) = f(t)\left[1 + \int_{\alpha(t_0)}^{t} k(t,\sigma)\, c^{1-p}(\sigma)\, d\sigma\right]$ and

$E_2(t) = f(t)[1 + \int_{\alpha(t_0)}^{t} \frac{k(t,\sigma)c^{1-p}(\sigma)}{p} d\sigma]$.

Proof :

(c_1)

Define a function $z(t)$ by $z(t) = \int_{\alpha(t_0)}^{\alpha(t)} f(s)\left\{u^p(s) + \int_{\alpha(t_0)}^{s} k(s,\sigma)u(\sigma)d\sigma\right\} ds.$ …… (3.5)

Then $z(t_0) = 0$ and as in the proof of part (a_1), we get

$$z'(t) \le \left[f(\alpha(t))\left\{a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} k(\alpha(t),\sigma)\left[\frac{p-1}{p} + \frac{a(\sigma)}{p}\right]d\sigma\right\}\right]\alpha'(t) + f(\alpha(t))\left\{1 + \int_{\alpha(t_0)}^{\alpha(t)} \frac{k(\alpha(t),\sigma)}{p} d\sigma\right\}\alpha'(t)z(t).$$

Therefore, $t_0 \le \eta \le t \le T$, one can have:

$$\frac{d}{d\eta}\left[z(\eta)exp^{\int_\eta^t f(\alpha(\tau))\left\{1+\int_{\alpha(t_0)}^{\alpha(\tau)}\frac{k(\alpha(\tau),\sigma)}{p}d\sigma\right\}\alpha'(\tau)d\tau}\right]$$

$$\leq f(\alpha(\eta))\left\{a(\alpha(\eta))\int_{\alpha(t_\circ)}^{\alpha(\eta)} k(\alpha(\eta),\sigma)\left[\frac{p-1}{p}\right.\right.$$

$$\left.\left.+\frac{a(\sigma)}{p}\right]d\sigma\right\}\alpha'(\eta)exp^{\int_\eta^t f(\alpha(\tau))\left\{1+\int_{\alpha(t_\circ)}^{\alpha(\tau)}\frac{k(\alpha(\tau),\sigma)}{p}d\sigma\right\}\alpha'(\tau)d\tau}$$

Integrating both side of the above inequality from $t_\circ$ to t , $t \in I$ and by making the change of variables we get:

$$z(t) \leq \int_{\alpha(t_\circ)}^{\alpha(t)} D_1(s)\, exp^{\int_s^{\alpha(t)} E_1(\xi)d\xi} ds \qquad \text{for } t \in I \text{ . } \quad ..... \quad (3.6)$$

Using (3.6) in (2.8), yields the required inequality in (3.2).u

(c_2)

Since $c(t)$ is positive continuous and nondecreasing function for $t \in I$, then inequality (3.3) can be written as

$$\left[\frac{u(t)}{c(t)}\right]^p \leq$$

$$1+\int_{\alpha(t_\circ)}^{\alpha(t)} f(s)\left\{\left[\frac{u(s)}{c(s)}\right]^p+\int_{\alpha(t_\circ)}^{s} k(s,\sigma)\, c^{1-p}(\sigma)\left[\frac{u(\sigma)}{c(\sigma)}\right]d\sigma\right\} ds \ .$$

.

Now an application of the inequality given in (c_1) yields desired result in (3.4).

Theorem : (3.2)

Let
$u(t),\ g(t) \in C(I_1,\mathbb{R}_+),\ k(t,s), b(t,s), c(t,s) \in C(D_1,\mathbb{R}_+),$
$h(t,s,\sigma) \in C(E_1,\mathbb{R}_+)\ and\ \ a(t),\ a'(t) \in C(I_1,\mathbb{R}_+),\ p > 1$
be a real constant, $\alpha(t) \in C^1(I_1,I_1)$ be nondecreasing with $\alpha(t) \leq t\ on\ I_1$.

(d_1)

Let $\phi(t) \in C(I_1,\mathbb{R}_+)$ and

$$u^p(t) \leq a(t)+\int_{\alpha(t_\circ)}^{\alpha(t)}\phi(s)u(s)ds$$

$$+\int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{s} k(s,\tau)u(\tau)d\tau\, ds$$

$$+\int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{s}\int_{\alpha(t_\circ)}^{\tau} h(s,\tau,\sigma)u(\sigma)d\sigma\, d\tau\, ds$$

$$+\int_{\alpha(t_\circ)}^{\beta}\int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds, \qquad ..... \quad (3.7)$$

for $t \in I_1$. If

$$q_1 = \frac{1}{p}\int_{\alpha(t_\circ)}^{\beta}\int_{\alpha(t_\circ)}^{s} c(s,\tau)\, exp^{\int_{\alpha(t_\circ)}^{\alpha(\tau)} E_3(\theta)d\theta}\, d\tau\, ds \ < 1 \qquad \ldots.. \quad (3.8)$$

then

$$u(t) \le \left[a(t) + N_1 exp^{\int_{\alpha(t_\circ)}^{\alpha(t)} E_3(\theta)d\theta} + \int_{\alpha(t_\circ)}^{\alpha(t)} D_3(\psi) exp^{\int_{\psi}^{\alpha(\tau)} E_3(\theta)d\theta}\, d\psi\right]^{\frac{1}{p}}$$

,
for $t \in I_1$, where

$$D_3(t) = \frac{1}{p}\left\{\phi(t)[(p-1)+a(t)] + \int_{\alpha(t_\circ)}^{t} k(t,\tau)\,[(p-1) + a(\tau)]d\tau + \int_{\alpha(t_\circ)}^{t}\int_{\alpha(t_\circ)}^{\tau} h(t,\tau,\sigma)\,[(p-1) + a(\sigma)]d\sigma\, d\tau\right\},$$

$$E_3(t) = \frac{1}{p}\left\{\phi(t) + \int_{\alpha(t_\circ)}^{t} k(t,\tau)d\tau + \int_{\alpha(t_\circ)}^{t}\int_{\alpha(t_\circ)}^{\tau} h(t,\tau,\sigma)\, d\sigma\, d\tau\right\}$$

,
and

$$N_1 = \frac{1}{1-q_1}\left\{\frac{1}{p}\int_{\alpha(t_\circ)}^{\beta}\int_{\alpha(t_\circ)}^{s} c(s,\tau)\left[(p-1)+a(\tau) + \int_{\alpha(t_\circ)}^{\alpha(\tau)} D_3(\psi)\, exp^{\int_{\psi}^{\alpha(\tau)} E_3(\theta)d\theta}\, d\psi\right] d\tau\, ds\right\}$$

$(\boldsymbol{d_2})$

Let $k(t,s),\ b(t,s), h(t,s,\sigma)$ are nondecreasing in $t \in I_1$, for each $s \in I_1$ and

$$u^p(t) \le a(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} k(t,\tau)u(\tau)d\tau + \int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{s} h(t,s,\sigma)u(\sigma)d\sigma\, ds + \int_{\alpha(t_\circ)}^{\beta} b(t,s)\int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds\ ,$$

……. (3.10)

For $t \in I_1$. If

$$q_2 = \frac{1}{p}\int_{\alpha(t_\circ)}^{\beta} b(t,s) \int_{\alpha(t_\circ)}^{s} c(s,\tau)\, exp^{\int_{\alpha(t_\circ)}^{\alpha(\tau)} E_4(\theta)d\theta}\, d\tau\, ds \; < 1$$

,

……. (3.11)

then

$$u(t) \le \left[a(t) + N_2 exp^{\int_{\alpha(t_\circ)}^{\alpha(t)} E_4(\theta)d\theta} + \int_{\alpha(t_\circ)}^{\alpha(t)} D_4(\psi) exp^{\int_{\psi}^{\alpha(t)} E_4(\theta)d\theta}\, d\psi\right]^{\frac{1}{p}}$$

…… (3.12)

for $t \in I_1$, where

$$D_4(\psi) = \frac{1}{p}\left\{k(t,\psi)[(p-1)+a(\psi)] + \int_{\alpha(t_\circ)}^{\psi} h(t,\psi,\sigma)\,[(p-1)+a(\sigma)]d\sigma\right\},$$

$$E_4(\theta) = \frac{1}{p}\left\{k(t,\theta) + \int_{\alpha(t_\circ)}^{\theta} h(t,\theta,\sigma)d\sigma\right\}$$

and

$$N_2 = \frac{1}{1-q_2}\left[\frac{1}{p}\int_{\alpha(t_\circ)}^{\beta} b(t,s) \int_{\alpha(t_\circ)}^{s} c(s,\tau)\left[(p-1)+a(\tau) + \int_{\alpha(t_\circ)}^{\alpha(\tau)} D_4(\psi)\, exp^{\int_{\psi}^{\alpha(\tau)} E_4(\theta)d\theta}\, d\psi\right] d\tau\, ds\right].$$

$(\boldsymbol{d_3})$

Let $e(t,s) \in C(D_1, \mathbb{R}_+)$ and if

$$u^p(t) \le a(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} g(s)\left\{u(s) + \int_{\alpha(t_\circ)}^{s} k(s,\sigma)u(\sigma)d\sigma + \int_{\alpha(t_\circ)}^{\beta} e(s,\sigma)u(\sigma)d\sigma\right\}ds$$

,

….. (3.13)

for $t \in I_1$. Then

$$u(t) \leq \left[a(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} D_5(\psi) exp^{\int_{\psi}^{\alpha(\tau)} E_5(\theta)d\theta} d\psi\right]^{\frac{1}{p}} \quad \ldots\ldots\ (3.14)$$

For $t \in I_1$, where

$$D_5(t) = \frac{1}{p} g(t) \left\{(p-1) + a(t) + \int_{\alpha(t_\circ)}^{t} k(t,\sigma)\,[(p-1) + a(\sigma)]d\sigma + \int_{\alpha(t_\circ)}^{\beta} e(t,\sigma)\,[(p-1) + a(\sigma)]d\sigma\right\},$$

and

$$E_5(t) = \frac{1}{p} g(t) \left[1 + k(t,\sigma)d\sigma + \int_{\alpha(t_\circ)}^{\beta} e(t,\sigma)\, d\sigma\right].$$

Proof:

$(\boldsymbol{d_1})$

Define a function $z(t)$ by

$$z(t) = \int_{\alpha(t_\circ)}^{\alpha(t)} \phi(s)u(s)ds + \int_{\alpha(t_\circ)}^{\alpha(t)} \int_{\alpha(t_\circ)}^{s} k(s,\tau)u(\tau)d\tau\, ds + \int_{\alpha(t_\circ)}^{\alpha(t)} \int_{\alpha(t_\circ)}^{s} \int_{\alpha(t_\circ)}^{\tau} h(s,\tau,\sigma)u(\sigma)d\sigma\, d\tau\, ds + \int_{\alpha(t_\circ)}^{\beta} \int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds, \quad \ldots\ldots\ldots\ldots\ (3.15)$$

Then $z(t) \geq 0$, $z(t)$ is nondecreasing for $t \in I_1$

$$z(t_\circ) = \int_{\alpha(t_\circ)}^{\beta} \int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds \quad \ldots\ldots\ldots\ (3.16)$$

Then as in the proof of part (a_1) , from (3.15) we see that the inequalities (2.8) and (2.9) hold. Differentiating (3.15) and using (2.9) and the fact that $z(t)$ is nondecreasing in t , we get:

$$z'(t) = \phi(\alpha(t))u(\alpha(t))\alpha'(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} k(\alpha(t),\tau)u(\tau)d\tau\, \alpha'(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} \int_{\alpha(t_\circ)}^{\tau} h(\alpha(t),\tau,\sigma)u(\sigma)d\sigma\, d\tau\, \alpha'(t)$$

$$\leq \frac{1}{p}\Bigg\{\phi(\alpha(t))[(p-1)+a(\alpha(t))] + \int_{\alpha(t_\circ)}^{\alpha(t)} k(\alpha(t),\tau)\,[(p-1)+a(\tau)]d\tau + \int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{\tau} h(\alpha(t),\tau,\sigma)\,[(p-1) + a(\sigma)]d\sigma\, d\tau\Bigg\}\alpha'(t)$$
$$+\frac{1}{p}\Bigg\{\phi(\alpha(t)) + \int_{\alpha(t_\circ)}^{\alpha(t)} k(\alpha(t),\tau)d\tau + \int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{\tau} h(\alpha(t),\tau,\sigma)\, d\sigma\, d\tau\Bigg\}\alpha'(t)\, z(t).$$

Therefore, $t_\circ \leq \eta \leq t \leq \beta$, one can have:

$$\frac{d}{d\eta}\left\{z(\eta)exp^{\int_\eta^t \frac{1}{p}\left[\phi(\alpha(\xi))+\int_{\alpha(t_\circ)}^{\alpha(\xi)} k(\alpha(\xi),\tau)d\tau+\int_{\alpha(t_\circ)}^{\alpha(\xi)}\int_{\alpha(t_\circ)}^{\tau} h(\alpha(\xi),\tau,\sigma)d\sigma\, d\tau\right]\alpha'(\xi)d\xi}\right\}$$
$$\leq \frac{1}{p}\{\phi(\alpha(\eta))[(p-1)+a(\alpha(\eta))] + \int_{\alpha(t_\circ)}^{\alpha(\eta)} k(\alpha(\eta),\tau)[(p-1)+a(\tau)]d\tau + \int_{\alpha(t_\circ)}^{\alpha(\eta)}\int_{\alpha(t_\circ)}^{\tau} h(\alpha(\eta),\tau,\sigma)\,[(p-1) + a(\sigma)]d\sigma\, d\tau\}$$
$$exp^{\int_\eta^t \frac{1}{p}\left[\phi(\alpha(\xi))+\int_{\alpha(t_\circ)}^{\alpha(\xi)} k(\alpha(\xi),\tau)d\tau+\int_{\alpha(t_\circ)}^{\alpha(\xi)}\int_{\alpha(t_\circ)}^{\tau} h(\alpha(\xi),\tau,\sigma)d\sigma\, d\tau\right]\alpha'(\xi)d\xi}\ \alpha'(\eta)$$

Integrating both sides of the above inequality from $t_\circ$ to t, $t \in I_1$ and by making the change of variables, we get:

$$z(t) \leq z(t_\circ)exp^{\int_{\alpha(t_\circ)}^{\alpha(t)} E_3(\theta)d\theta} + \int_{\alpha(t_\circ)}^{\alpha(t)} D_3(\psi)exp^{\int_\psi^{\alpha(\tau)} E_3(\theta)d\theta}\, d\psi \quad .. (3.17)$$

from (3.17), (2.9) and (3.16), we get:

$$z(t_\circ) \leq \int_{\alpha(t_\circ)}^{\beta} \int_{\alpha(t_\circ)}^{s} c(s,\tau)\frac{1}{p}[(p-1)+a(\tau)$$
$$+ z(t_\circ)exp^{\int_{\alpha(t_\circ)}^{\alpha(\tau)} E_3(\theta)d\theta}$$
$$+ \int_{\alpha(t_\circ)}^{\alpha(\tau)} D_3(\psi)exp^{\int_{\psi}^{\alpha(\tau)} E_3(\theta)d\theta} d\psi]d\tau\, ds,$$

then

$$z(t_\circ)\left[1-\frac{1}{p}\int_{\alpha(t_\circ)}^{\beta}\int_{\alpha(t_\circ)}^{s} c(s,\tau)\, exp^{\int_{\alpha(t_\circ)}^{\alpha(\tau)} E_3(\theta)d\theta} d\tau\, ds\right]$$
$$\leq \int_{\alpha(t_\circ)}^{\beta}\int_{\alpha(t_\circ)}^{s} c(s,\tau)\frac{1}{p}\left[(p-1)+a(\tau)\right.$$
$$\left.+\int_{\alpha(t_\circ)}^{\alpha(\tau)} D_3(\psi)exp^{\int_{\psi}^{\alpha(\tau)} E_3(\theta)d\theta} d\psi\right] d\tau ds\,.$$

from (3.8) we obtain that

$$z(t_\circ) \leq N_1\,. \qquad \ldots\ldots\ldots \quad (3.18)$$

The required inequality (3.9) follows from (3.18), (3.17) and (2.8).

$(\boldsymbol{d_2})$

Fix any $T,\ t_\circ \leq T \leq \beta$, then for $t_\circ \leq t \leq T$, we have

$$u^p(t) \leq a(t) + \int_{\alpha(t_\circ)}^{\alpha(t)} k(T,\tau)u(\tau)d\tau$$
$$+ \int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{s} h(T,s,\sigma)u(\sigma)d\sigma\, ds$$
$$+ \int_{\alpha(t_\circ)}^{\beta} b(T,s)\int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds$$

……….(3.19)

Define a function $z(t,T),\ t_\circ \leq t \leq T$ by

$$z(t,T) = \int_{\alpha(t_\circ)}^{\alpha(t)} k(T,\tau)u(\tau)d\tau + \int_{\alpha(t_\circ)}^{\alpha(t)}\int_{\alpha(t_\circ)}^{s} h(T,s,\sigma)u(\sigma)d\sigma\, ds + \int_{\alpha(t_\circ)}^{\beta} b(T,s)\int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds$$

…..…..(3.20)

Then $z(t,T) \leq 0$, $z(t,T)$ is nondecreasing for $t \in I_1$,

$$z(t_\circ,T) = \int_{\alpha(t_\circ)}^{\beta} b(T,s) \int_{\alpha(t_\circ)}^{s} c(s,\tau)u(\tau)d\tau\, ds \quad , \qquad \text{….. (3.21)}$$

and inequality (3.10) can be written as

$$u^p(t) \leq a(t) + z(t,T), \qquad t_\circ \leq t \leq T \qquad \text{…….(3.22)}$$

Then as in the proof of part (a_1) , from (3.22) we see that the inequalities (3.23) hold.

$$u(t) \leq \frac{(p-1)}{p} + \frac{a(t)}{p} + \frac{z(t,T)}{p} \ , \qquad t_\circ \leq t \leq T \qquad \text{……..(3.23)}$$

Differentiating (3.20) and using (3.23) and the fact that $z(t,T)$ is nondecreasing in t , we get:

$$z'(t,T) = k(T,\alpha(t))u(\alpha(t))\alpha'(t) + \int_{\alpha}^{t} h(T,\alpha(t),\sigma)u(\sigma)d\sigma\, \alpha'(t)$$

$$\leq \frac{1}{p}\left\{ k(T,\alpha(t))[(p-1)+a(\alpha(t))] + \int_{\alpha(t_\circ)}^{\alpha(t)} h(T,\alpha(t),\sigma)\,[(p-1)+a(\sigma)]d\sigma \right\}\alpha'(t)$$

$$+\frac{1}{p}\left\{ k(T,\alpha(t)) + \int_{\alpha(t_\circ)}^{\alpha(t)} h(T,\alpha(t),\sigma)\,d\sigma\right\}\alpha'(t)z(t,T) \, , \qquad \text{…….. (3.24)}$$

for $t_\circ \leq T$ by setting $t = \xi$ in (3.24) and integrating it with respect to ξ from $t_\circ$ to T and by making change of variables we get:

$$z(T,T) \leq z(t_\circ,T)exp^{\int_{\alpha(t_\circ)}^{\alpha(T)}\frac{1}{p}\left\{ k(T,\theta)+\int_{\alpha(t_\circ)}^{\theta} h(T,\theta,\sigma)d\sigma\right\}d\theta}$$

$$+ \int_{\alpha(t_\circ)}^{\alpha(T)} \frac{1}{p}\{ k(T,\psi)[(p-1)+a(\psi)] + \int_{\alpha(t_\circ)}^{\psi} h(T,\psi,\sigma)[(p-1)+a(\sigma)]d\sigma\}$$

$$exp^{\int_{\alpha(t_\circ)}^{\alpha(T)}\frac{1}{p}\left\{ k(T,\theta)+\int_{\alpha(t_\circ)}^{\theta} h(T,\theta,\sigma)d\sigma\right\}d\theta}\, d\psi.$$

Since T is arbitrary from (3.25), (3.23), (3.22) and (3.21) with T replaced by t , one can get:

$$z(t,t) \leq z(t_\circ,t)exp^{\int_{\alpha(t_\circ)}^{\alpha(t)}\frac{1}{p}\left\{k(t,\theta)+\int_{\alpha(t_\circ)}^{\theta}h(t,\theta,\sigma)\,d\sigma\right\}d\theta} + \int_{\alpha(t_\circ)}^{\alpha(t)}\frac{1}{p}\{ k(t,\psi)[(p-1)+a(\psi)] + \int_{\alpha(t_\circ)}^{\psi}h(t,\psi,\sigma)\,[(p-1)+ a(\sigma)]d\sigma\}\ exp^{\int_{\psi}^{\alpha(t)}\frac{1}{p}\left\{k(t,\theta)+\int_{\alpha(t_\circ)}^{\theta}h(t,\theta,\sigma)\,d\sigma\right\}d\theta}\,d\psi,$$

……….(3.26)

$$u^p(t) \leq a(t) + z(t,t) \quad , \qquad \text{……… (3.27)}$$

and

$$u(t) \leq \frac{1}{p}\{(p-1)+a(t)+z(t,t)\} \leq$$

$$\frac{1}{p}\left\{(p-1)+a(t)+z(t_\circ,t)exp^{\int_{\alpha(t_\circ)}^{\alpha(t)}E_4(\theta)d\theta} + \int_{\alpha(t_\circ)}^{\alpha(t)}D_4(\psi)exp^{\int_{\psi}^{\alpha(t)}E_4(\theta)d\theta}\,d\psi\right\},$$

…….(3.28)

and

$$z(t_\circ,t) = \int_{\alpha(t_\circ)}^{\beta}b(t,s)\int_{\alpha(t_\circ)}^{s}c(s,\tau)u(\tau)d\tau\,ds \qquad \text{……….(3.29)}$$

from (3.29) and (3.28) one can get:

$$z(t_\circ,t) = \int_{\alpha(t_\circ)}^{\beta} b(t,s) \int_{\alpha(t_\circ)}^{s} c(s,\tau)\frac{1}{p}\left\{(p-1)+a(\tau) + z(t_\circ,\tau)exp^{\int_{\alpha(t_\circ)}^{\alpha(\tau)}E_4(\theta)d\theta} + \int_{\alpha(t_\circ)}^{\alpha(\tau)} D_4(\psi)exp^{\int_{\psi}^{\alpha(\tau)}E_4(\theta)d\theta}\,d\psi\right\}d\tau\,ds.$$

Since $z(t,T)$ is nondecreasing and nonnegative for $t \in I_1$ and $\tau \leq s \leq t \leq \beta$ then

$$z(t_\circ,t)\,[1-\frac{1}{p}\int_{\alpha(t_\circ)}^{\beta} b(t,s)\int_{\alpha(t_\circ)}^{s} c(s,\tau)\,exp^{\int_{\alpha(t_\circ)}^{\alpha(\tau)} E_4(\theta)d\theta}\,d\tau\,ds\,]$$
$$\leq \int_{\alpha(t_\circ)}^{\beta} b(t,s)\int_{\alpha(t_\circ)}^{s} c(s,\tau)\frac{1}{p}\{(p-1)+a(\tau)$$
$$+\int_{\alpha(t_\circ)}^{\alpha(\tau)} D_4(\psi)exp^{\int_{\psi}^{\alpha(\tau)} E_4(\theta)d\theta}\,d\psi\}\,d\tau\,ds.$$

From (3.11) which implies

$$z(t_\circ,t)\leq N_2 \qquad \text{……….(3.30)}$$

The required inequality (3.12) follows from (3.30), (3.26), and (3.27).

$(\boldsymbol{d_3})$

Define a function $z(t)$ by

$$z(t)=\int_{\alpha(t_\circ)}^{\alpha(t)} g(s)\,\Big\{u(s)+\int_{\alpha(t_\circ)}^{s} k(s,\sigma)u(\sigma)d\sigma+\int_{\alpha(t_\circ)}^{\beta} e(s,\sigma)u(\sigma)d\sigma\,\Big\}ds \qquad \text{……..(3.31)}$$

Then $z(t)\geq 0$, $z(t)$ is nondecreasing for $t\in I_1$, $z(t_\circ)=0$.
Then as in the proof of part (a_1), from (3.31) we see that the inequalities (2.8) and (2.9) hold.
Differentiating (3.31) and using (2.9) and the fact that $z(t)$ is nondecreasing in t, we get:

$$z'(t)=g\big(\alpha(t)\big)\left[u\big(\alpha(t)\big)+\int_{\alpha(t_\circ)}^{\alpha(t)} k(\alpha(t),\sigma)u(\sigma)d\sigma+\int_{\alpha(t_\circ)}^{\beta} e(\alpha(t),\sigma)u(\sigma)d\sigma\right]\alpha'(t)$$

$$\leq \frac{1}{p}g(\alpha(t))\{\,(p-1)+a(\alpha(t))+\int_{\alpha(t_\circ)}^{\alpha(t)} k(\alpha(t),\sigma)\,[(p-1)+a(\sigma)]d\sigma+\int_{\alpha(t_\circ)}^{\beta} e(\alpha(t),\sigma)\,[(p-1)+a(\sigma)]d\sigma\}\,\alpha'(t)+\frac{1}{p}g\big(\alpha(t)\big)\Big\{1+\int_{\alpha(t_\circ)}^{\alpha(t)} k(\alpha(t),\sigma)d\sigma+\int_{\alpha(t_\circ)}^{\beta} e(\alpha(t),\sigma)d\sigma\Big\}\,\alpha'(t)z(t)$$

.

Therefore, $t_\circ \leq \eta \leq t \leq \beta$, one can have

$$\frac{d}{d\eta}\left\{z(\eta)exp^{\int_{\eta p}^{t1} g(\alpha(\xi))[1+\int_{\alpha(t_\circ)}^{\alpha(\xi)} k(\alpha(\xi),\sigma)d\sigma+\int_{\alpha(t_\circ)}^{\beta} e(\alpha(\xi),\sigma)d\sigma]\alpha'(\xi)d\xi}\right\}$$

$$\left\{(p-1)+a(\alpha(\eta))+\int_{\alpha(t_\circ)}^{\alpha(\eta)} k(\alpha(\eta),\sigma)\,[(p-1)+a(\sigma)]d\sigma\right.$$

$$\left.+\int_{\alpha(t_\circ)}^{\beta} e(\alpha(\eta),\sigma)[(p-1)+a(\sigma)]d\sigma\right\}$$

$$exp^{\int_{\eta p}^{t1} g(\alpha(\xi))[1+\int_{\alpha(t_\circ)}^{\alpha(\xi)} k(\alpha(\xi),\sigma)d\sigma+\int_{\alpha(t_\circ)}^{\beta} e(\alpha(\xi),\sigma)d\sigma]\alpha'(\xi)d\xi} \quad \alpha'(\eta).$$

Integrating both sides of the above inequality from $t_\circ$ to t, $t \in I_1$, since $z(t_\circ) = 0$, and by making the change of variables we get:

$$z(t) \leq \int_{\alpha(t_\circ)}^{\alpha(t)} D_5(\psi)\, exp^{\int_{\psi}^{\alpha(t)} E_5(\theta)d\theta}\, d\psi. \qquad \text{….. (3.32)}$$

The required inequality (3.14) follows from (3.32) and (2.9).

4. Conclusions

We have constructed some iterated integral inequalities then extended to the iterated retarded integral inequalities. And also explicit bounds to unknown functions in each integral inequalities are given.

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