The Existence of the Approximate Solution for the Hamilton_Jacobi equation by Using the Dual Dynamic Programming

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ABSTRACT

Stability of an approximation of a minimum for a control problem of Bolza is investigated. The dual dynamic programming method is used. An ε-value function and a dual ε-value function are defined. Several properties of these functions are presented. The verification theorems are proved.

الملخص:- في هذا البحث تم تفحص استقرارية الحل التقريبي لمشكلة بولزا والتي هي احدى مشاكل نظرية السيطرة المثلى عن طريق استخدام طريقة جديدة للتفحص هي طريقة البرمجة الديناميكية المواجهة.

كما تم تعريف الدوال ε-value function وكذلك dule ε-value function وتم تقديم خواص جديدة لتلك الدوال.

واخيراً تم برهان المبرهنة الاساسية في هذا البحث.

Keywords: Calculus of variations and optimal control. Approximate minimum, dual dynamic programming, dual ε - value function.

1. Introduction

We shall study the Bolza functional

$$J(x, y) = \int_a^b L(t, x(t), u(t))dt + \ell(x(b)) \qquad (1)$$

where the absolutely continues trajectory $x:[a,b] \to R^n$, the Lebesgue measurable control function $u:[a,b] \to R^m$, and ℓ is not identically to $+\infty$ are subject to

$$\dot{x} = f(t, x(t)u(t)) \text{ a.e. in } [a,b] \quad (2)$$
$$u(t) \in U(t), \qquad t \in [a,b] \quad (3) \qquad (C)$$
$$x(a) = c \ . \quad (4)$$

Here $f : [a,b] \times R^n \times R^m \to R^n$, $L : [a,b] \times R^n \times R^m \to R$, are given functions; $x : [a,b] \to R^n$, is absolutely continuous functions; $u : [a,b] \to U(t)$, is a measurable function and $U(t)$ is the set of controls with the initial condition $x(a) = c$ which is defined as:

$U(t) = \{u(t)$ measurables; such that $t \in [a,b]$ and $u(t) \in K$, where K is compact subset of $R^m\}$.

Throughout the paper we shall assume the following hypothesis:

(Z) $(t,x,u) \to f(t,x,u)$ and $(t,x,u) \to L(t,x,u)$ are continuous and bounded functions on $[a,b] \times R^n \times K$;they are Lipschitz functions with respect to t, x, u .

A pair $x(t), u(t)$ is admissible if it satisfies (2),(4), and $L(t, x(t), u(t))$ is assumable function; then the corresponding trajectory $x(t)$ will simply be called admissible.

We are looking for an admissible pair $x_\varepsilon(t)$, $u_\varepsilon(t)$, $t \in [a,b]$, $x_\varepsilon(a) = c$ such that

$$J(x_\varepsilon, u_\varepsilon) \le \inf J(x,u) + \varepsilon(b-a) \quad (5)$$

Where the infimum is taken over all admissible pairs satisfying (4) and $\varepsilon > 0$ is any given number.

It is clear that such a pair $x_\varepsilon(t)$, $u_\varepsilon(t)$, satisfying (5) always exists if $\inf J(x,u) > -\infty$.

The main problem considered in the literature is how to existence of an approximate solution for the Hamilton-Jacobi (H-J) equation for the optimality problem by using the non-classical approach to dynamic programming (the dual dynamic programming) (see Nowakowski A. [7]). The first answer for the first problem, at least partially belongs to Ekeland [3,4]. He formulated it in the form of the variational principle and it corresponds to the first variation in the ordinary extremum problem, i.e. for (1)-(4) it is simply the ε-maximum pontryagin principle.

However, from this principle we cannot infer that a pair satisfying it satisfies also (5). The situation is even worse: not every pair satisfying (5) satisfies the $\in$-maximum pontryagin principle.Nowakowski (1998-1990) there are described theories, basing an generalizations of filed of extremals and Hilbert's independence integral, which allow us to state, under additional geometrical assumptions, that a pair $x_\varepsilon(t)$, $u_\varepsilon(t)$ satisfying the ε-maximum principle satisfies (5) with some extra term on the right hand side.

The aim of this article is to descrbe the noclassical dynamic programming method for an approximate minimum of the Bolza functional and to use this method in order to existence of an approximate solution for the Hamilton-Jacobi equation.

The remainder of the paper is organized as follows:

In section 2 we propose the dual dynamic programming method (see, Nowakowski, (1990)). We define a dual ε-value function in the dual space and show that it has properties analogous to the dual value function. Next we describe the necessary method used for the construction of dual ε- value function which is an approximation to the dual value function. Throughout the construction process we check whether the function under construction is a solution of the Hamilton – Jacobi equation.

2. The Dual Dynamic Programming Approach

Let T be a set covered by graphs of admissible trajectories but not necessary with nonempty interior. Next consider a set $\mathrm{P} \subset R^{n+2}$ of points $(t, y^0, y) = (t, P)$, $y^0 \le 0$, with nonempty interior and a function $x(t, p)$ defined in P such that $(t, x(t, p)) \in T$ for $(t, p) \in P$; and assume that satisfies the following:

Define in P the dual value function as

$$S_D(t, p) = \inf\{-y^0 \int_t^b L(s, x(s), u(s)))\ ds - y^0 \ell(x(b))\}. \qquad (6)$$

where the infimum is taken over admissible pairs $x(s), u(s), s \in [t, b]$ whose trajectories start at $(t, x(t, p))$ and their graphs are contained in T.We assume further that $x(t, p)$ is a Borel measurable, locally bounded, Lipschitz function, and such that for each admissible trajectory $x(t)$ lying in T there exists an absolutely continuous function $p(t) = (y^0, y(t))$ lying in P

such that $x(t)=x(t,p)$ and, if all trajectories $x(t)$ start at the same (t_0,x_0), then all the corresponding $p(t)$ have the same first coordinate y^0.

Now, since $S_D(t,p)=-y^0S(t,x(t,p)),(t,p)\in P,t\in[0,b]$ is a lipschitz function for $(t,p)\in P$ and is a solution for the Hamilton-Jacobi equation (see,Nowakowski A., 1992)

$$-y^0S_t(t,x(t,p))+\min\{-y^0S_x(t,x(t,p))f(t,x(t,p)),u)-y^0L(t,x(t,p),u),u\in K\}=0$$

$$a.e.,(t,x(t,p))\in T,t\in(0,b) \qquad (7)$$

with the boundary condition

$$-y^0S(b,x(b,p(b)))=-y^0\ell(x(b)),\ \text{for all}\ (b,x(b,p(b)))\in T \qquad (8)$$

By dual ε - value function we mean any function $S_{\varepsilon D}(t,p)$ satisfying

$$S_D(t,p)\le S_{\varepsilon D}(t,p)\le S_D(t,p)-\varepsilon y^0(b-a) \qquad (9)$$

$$S_{\varepsilon D}(b,p)=-y^0\ell(x(b)),\quad (b,p)\in P \qquad (10)$$

We see that if $S_\varepsilon(t,x)$ is an ε - value function then $-y^0S_\varepsilon(b,x(t,p))$ is a dual ε - value function. Thus we see that the dual ε - value function has properties analogous to ε - value function (see Nowakowski A., 1995). An admissible trajectory $x_\varepsilon(s)$, $s\in[t,b]$, $x_\varepsilon(t)=x$ is called an ε-optimal trajectory if there exists an absolutely continuous function $p_\varepsilon(s)=(y^0_\varepsilon,y_\varepsilon(s))$, $s\in[t,b]$, $p_\varepsilon(t)=p$, lying in P, such that $x_\varepsilon(s)=x(s,p_\varepsilon(s))$, $s\in[t,b]$ and

$$S_{\varepsilon D}(t,p)\ge y^o_\varepsilon\int_t^b L(s,x_\varepsilon(s),u_\varepsilon(s))ds-y^0_\varepsilon\ell(x_\varepsilon(b))$$

for a given fixed $S_{\varepsilon D}(t,p)$.

According to Nowakowski A. and Jacewicz E., in the nonclassical dynamic programming the sufficient condition for optimality of the solution to the considered problem is expressed as the solution to the Hamilton-Jacobi equation so the following Theorem 2.1 holds.

***Theorem* 2.1:** Let $V(t,p)$, $(t,p)\in P$, $t\in[a,b]$, be a Lipschitz function satisfying the dual partial differential inequality of dynamic programming

$$0 \le V_t(t,p) + \sup\{yf(t,-V_y(t,p),u) + y^0 L(t,-V_y(t,p),u) : u \in U(t)\}$$

$$\le -\frac{1}{2} y^\circ \varepsilon \qquad (11)$$

Let E denoted a subset of $[a,b]$ such that if $t_0 \in E$ then for all $(t,p) \in P$, $V_p(t,p)$ exists. We assume that meas $E = b-a$, $b \in E$ and that $V(t,p)$ satisfies the boundary condition $y^0 V_{y^0}(b,p) = y^0 \ell(-V_y(b,p))$,$(b,p) \in P$ and the relation

$$V(t,p) = V_p(t,p)p - \frac{1}{2} y^0 \varepsilon(b-t), t \in E, (t,p) \in P. \qquad (12)$$

Let $x(t), u(t)$ be an admissible pair whose graph of the trajectory $x(t)$ is contained in the closure $\overline{T}$ of $T = \{(t,x) : x = -V_y(t,p), t \in E, \ (t,p) \in P\}$ and such that there is a function of bounded variation $p(t) = (y^0, y(t))$ lying in P, and satisfying $x(t) = -V_y(t,p(t))$ for $t \in E$.

Assume further that then $V_t(t,p(t))$ exists for almost every t. Then

$$-y^0 V_{y^\circ}(t_1, p(t_1)) + y^0 \int_{t_1}^{b} L(t, x(t), u(t))\, dt \le -y^0 V_{y^0}(t_2, p(t_2)) +$$

$$y^0 \int_{t_2}^{b} L(t, x(t), u(t))\, dt - \varepsilon y^0 (b-a) \qquad (13)$$

for all $a \le t_1 \le t_2 \le b$. Let $x_\varepsilon(t), u_\varepsilon(t), t \in [a,b]$, $x_\varepsilon(a) = c$ be an admissible pair with $x_\varepsilon(t)$ lying in $\overline{T}$ and let $p_\varepsilon(t) = (y_\varepsilon^0, y_\varepsilon(t))$,$t \in [a,b]$, be a nonzero absolutely continuous function lying in P such that $x_\varepsilon(t) = -V_y(t, p_\varepsilon(t))$ for all $t \in E$. Suppose that, for almost all t in $[a,b]$,

$$0 \le V_t(t, p_\varepsilon(t)) + y_\varepsilon(t) f(t, -V_y(t, p_\varepsilon(t)), u_\varepsilon(t)) + y_\varepsilon^0 L(t, -V_y(t, p_\varepsilon(t)), u_\varepsilon(t))$$

$$\le -\frac{1}{2} y_\varepsilon^o \qquad (14)$$

Then $x_\varepsilon(t)$ is an ε-optimal trajectory for the dual ε-value function $S_{\varepsilon D}(t, p) = -y_\varepsilon^0 V_{y^0}(t, p)$ relative to all admissible pairs $x\ (t),\ u\ (t),\ t \in [a, b]$, $x(a) = c$, whose graphs of trajectories are contained in $\overline{T}$ and where the corresponding function $p(t) = (y_\varepsilon^0, y(t))$ $(x(t) = -V_y(t, p(t)), t \in E)$ is of bounded variation.

Moreover $-\ y_\varepsilon^0 S_\varepsilon(t, x(t, p)) = -y_\varepsilon^0 V_{y^0}(t, p)$ with $x(t, p) = -V_y(t, p)$.

Proof: see (Nowakosk; A. and Jacewicz .E, 1995).

It can be seen that some regularity of the function $(t, p_\varepsilon) \to S_{\varepsilon D}(t, p_\varepsilon(t))$, being the solution to the dual partial differential inequality of dynamic programming (DPDIDP)(12), is required it must be at least a Lipsohitz function (see Theorem 2.1).

3. Approximating The S$_{\varepsilon D}$ (t,p) Function

This section presents some definitions and Lemmas which will be used in the proof the main theorem in this section.

Let us define the set *W* as follows:

$W = \{H(t, p) = -y_\varepsilon^0 W_\varepsilon(t, x_\varepsilon(t, p))\ |$ is a Lipschitz for $t, p; (t, p) \in P, t \in [0, b],\quad (t, x_\varepsilon(t, p)) \in T;$ with the boundary condition $H(b, p) = -y_\varepsilon^0 W_\varepsilon(b, x(b, p_\varepsilon)) \le -y_\varepsilon^0 \ell(x_\varepsilon^0(b))$ for all $x_\varepsilon(t, p) \in T$, $(t, p_\varepsilon) \in P$; and

$$-\varepsilon \le -y^0 (\partial/\partial t) W_\varepsilon(t, x(t, p)) + \min\ \{-y^0 (\partial/\partial x) W_\varepsilon(t, x(t, p))$$

$$f(t, x(x, t), u) - y^0 L(t, x(t, p), u) : u \in K\} \le 0 \qquad (15)$$

And we define on the set *W* the lowing partial ordering:

$$H \le \hat{H} \Leftrightarrow H(t, p) \le \hat{H}(t, p), \quad (t, p) \in P, t \in [0, b], \forall H, \hat{H} \in W.$$

Note, from the definition of the function $S_{\varepsilon D}(t,p),(t,p)\in P$, in (9)we observe that the dual ε-value function $S_{\varepsilon D}(t,p)=-y^0 S_\varepsilon(t,x(t,p)),(t,p)\in P$ belongs to the set W of all Lipschitz solutions for the dual partial differential inequality of dynamic programming (DPDIDP)(15), when there exists $x_\varepsilon(t)=x_\varepsilon(t,p_\varepsilon(t)),(t,p)\in P$, lying in T, as a multiplied solution for Bolza problem (C).

Now, let us formulate and prove three lemmas (3.1,3.2,3.3), which will simplify and shorten the proof of the main theorem in this section that the dual ε-value function $S_{\varepsilon D}(t,p),(t,p)\in P$ defined in (9) is

$H(t,p)-\varepsilon(b-t)\le S_{\varepsilon D}(t,p)\le H(t,p)$, for all $H\in W$.

To formulate these lemmas, let us assume that $t_0<b$ and consider $\delta>0$ such that the interval $[t_0+\delta,b-\delta]$ has a nonempty interior.

Now let $x_{0\varepsilon}(t_0)=x_{0\varepsilon}(t_0,p_{0\varepsilon}(t_0))$ be arbitrary and let it belong to T, $u(\cdot)\in U(t)$.

Since $(t,x,u)\to f(t,x,u)$ and $(t,x,u)\to L(t,x,u)$ satisfy assumption (Z), and since $x_\varepsilon(t,p)$, $(t,p_\varepsilon)\in P$ is bounded and Lipschitz with respect to $t,x_\varepsilon(t,p),u_\varepsilon$ in $T\times K$, when $(t,p_\varepsilon)\in P$.

Therefore the response of the system $t\to x_\varepsilon(t)=x_\varepsilon(t,p(t)),t\in[t_0,b]$ with $x_{0\varepsilon}(t_0)=x_{0\varepsilon}(t,p_0(t_0))$, lying in T is bounded, i.e., $x_\varepsilon(t,p_\varepsilon(t))\in Q$, for all $(t,p_\varepsilon(t))\in\hat{Q}$, $t\in[t_0,b]$, where Q and $\hat{Q}$ are compact subsets of T and P respectively.

Now we define a set $\overline{Q}$ as follows: $\overline{Q}=\hat{Q}+B_1(R^{n+2})$, where $B_1(R^{n+2})$ is the sphere centered at the origin having a radius of 1.

For shorter and simpler definition, we propose the following notations:

$$\tilde{f}(t,p,u)=f(t,x_\varepsilon(t,p),u)$$
$$\tilde{L}(t,p,u)=L(t,x_\varepsilon(t,p),u) \qquad (16)$$

Since $f:[a,b]\times R^n \times K \to R^n$ and $L:[a,b]\times R^n \times K \to R$ are Lipschitz satisfies assumptions (Z), and $x_\varepsilon(t,p)$ is a Lipschitz function for $(t,p_\varepsilon)\in P$, then we deduce that $\tilde{f}(t,p,u)$ and $\tilde{L}(t,p,u)$ are also lipschitz functions in $P\times K$.

And since $H(t,p)=-y^0 W_\varepsilon(t,x_\varepsilon(t,p)),(t,p)\in P$ then the dual partial differential inequality of dynamic programming (15) becomes

$$-\varepsilon \le H_t(t,p)+\min\left\{H_x(t,p)\tilde{f}(t,p,u)-y^0\tilde{L}(t,p,u):u\in K\right\}\le 0$$

$$\text{a.e.},(t,p)\in P,t\in[0,b], \qquad (17)$$

with the boundary condition

$$H(b,p(b))\le -y_\varepsilon^0\ell(x_\varepsilon(b)) \ ,(b,p)\in P.$$

Now we propose the function $(t,p)\to F(t,p)$ which is defined as:

$$F(t,p)=H_t(t,p)+\min\left\{H_x(t,p)\tilde{f}(t,p,u)-y^0\tilde{L}(t,p,u):u\in K\right\} \qquad (18)$$

Since the function $(t,p)\to H_t(t,p),(t,p,u)\to \tilde{f}(t,p,u) and (t,p,u)\to \tilde{L}(t,p,u)$ which are used in the definition of the above function $(t,p)\to F(t,p)$ are continuous, and since K is a compact set, then we deduce that $(t,p)\to F(t,p)$ is continuous.

Then according to the basis of the weierstrass's theorem of $(t,p)\to F(t,p)$ bounded on the set $\hat{Q}$. By denoting the infimum and supremum of $(t,p)\to F(t,p)$ over $\hat{Q}$ by h_ℓ and h_u respectively. We can estimate the value of the function $(t,p)\to F(t,p)$ by:

$$h_\ell \le F(t,p)\le h_u,\ \text{for all } (t,p)\in \hat{Q} \qquad (19)$$

The function $(t,p)\to F(t,p)$ may take values of different signs, and we are looking for dual ε – value function which must satisfy the dual partial differential inequality of dynamic programming (15) (see Nowakowski. A and Jacewicz.E, 1995). So we have to define a new function $(t,p)\to H_{1,j}(t,p)$ which depends on the function $H(.,.)$, where $(t,p)\to H_{1,j}(t,p)$ should enable us to estimate non-positive close to zero values of a new function $(t,p) \to F_{1,j}(t,p)$ which is constructed in a similar way to $(t,p) \to F_{1,j}(t,p)$. The function $(t,p) \to F_{1,j}(t,p)$ should also satisfy (15).

Now a new function $(t,p) \to H_{1,j}(t,p),\ j\{-q,\ldots,-1\} \cup \{1,\ldots,r\}, q, r \in N$, must be defined on disjoint subsets $\hat{Q}_j$ which covers completely the compact set $\hat{Q}$.

First, the domain of this function must be constructed. We shall divide the interval $[h_\ell, h_u] \subset R$ which is the set of values of function $(t,p) \to F(t,p)$ in to $q+r$ subintervals.

The points of the partition are determined as follows:

(a) if infimum h_ℓ and supremum h_u are of different signs, then:

$$h_\ell = z_{-q} < z_{-q+1} < \ldots < z_{-1} < z_0 < z_1 < \ldots < z_{r-1} < z_r = h_u ,$$

where $z_0 = 0$ and $q, r \in N$;

(b) if infimum h_ℓ is non – negative, then:

$$0 = z_0 \le h_\ell < z_1 < \ldots < z_{r-1} < z_r = h_u ,$$

where $r \in N$;

(c) if supremum h_u is not positive then:

$$h_\ell = z_{-q} < z_{-q+1} < \ldots < z_{-1} < z_0 = h_u ,$$

where $q \in N$.

We will take into consideration the first case (a) where h_ℓ and h_u are of different signs, while the other two cases (b) and (c) are not considered since they are simpler.

As in the definition of a Lebesgue integral, we will define subsets of $\hat{Q}$:

$$\hat{Q}_j = \{(t,p) \in \hat{Q} : z_j \le F(t,p) < z_{j+1}\},\ j \in \{-q,\ldots,-1\}$$

$$\hat{Q}_j = \{(t,p) \in \hat{Q} : z_{j-1} \le F(t,p) \le z_j\},\ j = 1$$

$$\hat{Q}_j = \{(t,p) \in \hat{Q} : z_{j-1} < F(t,p) \le z_j\},\ j \in \{2,\ldots,r\}.$$

It is easily seen that the sets $\hat{Q}_j,\ j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$ are disjointed for all $i, j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}, i \neq j,$ and they cover whole set $\hat{Q}$.

Morever let $\overline{Q}_j$ denotes the closures of the sets $\hat{Q}_j, j \in \{-q,\ldots,-1\} \cup \{i,\ldots,r\}$. On such subsets $\hat{Q}_j, j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$ of $\hat{Q}$,we construct a new function $(t,p) \to H_{1,j}(t,p)$.

Since the function $(t,p) \to F(t,p)$ may have values of different signs in $\hat{Q}$, we have to consider the following two cases:

I. let $F(t,p) \geq 0, (t,p) \in \hat{Q}_j \subset \hat{Q}, j \in \{1,\ldots,r\}$, which means that

$$h_\ell \leq z_{j-1} < F(t,p) \leq z_j \leq h_u, (t,p) \in \hat{Q}_j \subset \hat{Q},\ j \in \{1,\ldots,r\} \tag{20}$$

We define on $\hat{Q}_j \subset \hat{Q}, j \in \{1,\ldots,r\}$, a new function by shifiting the function $(t,p) \to H(t,p)$ as follows:

$$H_{1,j}(t,p) = H(t,p) + \alpha_j z_j (b-t) \tag{21}$$

where $\alpha_j \in (0,1), j \in \{1,\ldots,r\}$ are chosen to enable us estimate non-negative close to zero values of the new function $(t,p) \to F_{1,j}(t,p)$ defined on $\hat{Q}_j \subset \hat{Q}, j \in \{1,\ldots,r\}$ by the following formula:

$$F_{1,j}(t,p) = H_{1,jt} + \min\{H_{1,jx}(t,p)\tilde{f}(t,p,u) - y^0 \tilde{L}(t,p,u) : u \in K\} \tag{22}$$

where $(t,p) \to H_{1,j}(t,p), (t,p,u) \to \tilde{f}(t,p,u)$ and $(t,p,u) \to \tilde{L}(t,p,u)$ be as defined in (21), and (16) respectively.

By using (22) of $(t,p) \to F_{1,j}(t,p)$ and (21) of $(t,p) \to H_{1,j}(t,p)$,we obtain the following formula which shows the relation between the functions $(t,p) \to F(t,p)$ and $(t,p) \to F_{1,j}(t,p)$ where both functions are defined on $\hat{Q}_j \subset \hat{Q},\quad j \in \{1,\ldots,r\}$The following formula shows this relation:

$$\begin{aligned} F_{1,j}(t,p) &= H_t + \alpha_j z_j + \min\{H_x \tilde{f}(t,p,u) - y^\circ \tilde{L}(t,p,u) : u \in K\} \\ &= F(t,p) - \alpha_j z_j \end{aligned}$$

From inequality (20) and the above formula it follows that the following estimation holds:

$$-\mu_j \le F_{1,j}(t,p) \le -\eta_j, \quad (t,p)\in \hat{Q}_j, \quad j\in\{1,\dots,r\} \tag{23}$$

where $-\mu_j = z_{j-1} - \alpha_j z_j, \quad -\eta_j = z_j - \alpha_j z_j, 1<\alpha_j<2, \quad j\in\{1,\dots,r\}$.

It is obvious that this estimation will be better if we choose the numbers α_j sufficiently close to 1, i.e. $\alpha_j = 1.01$.

It is easily seen that the function $(t,p)\to H_{1,j}(t,p)$ is defined and continuous on $\hat{Q}_j$, $j\in\{1,\dots,r\}$, so it may be extended in to $\overline{Q}_j \subset \hat{Q}$, $j\in\{1,\dots r\}$ by putting

$$H_{1,j}(t,p) = H(t,p) + \alpha_j(b-t)$$

for each $(t,p)\in \overline{Q}_j \setminus \hat{Q}_j$, $j\in\{1,\dots,r\}$ and $\alpha_j\in(1,2)$.

Certainly such an extended function $(t,p)\to H_{1,j}(t,p)$ is also absolutely continuous in $\overline{Q}_j \subset \hat{Q}, \quad j\in\{1,\dots,r\}$, since the function $(t,p)\to H(t,p)$ is absolutely continuous. Furthermore we have noticed that $(t,p,u)\to \tilde{f}(t,p,u)$ and $(t,p,u)\to \tilde{L}(t,p,u)$ are continuous functions in $\hat{Q}_j \times K$, so we deduce that $(t,p)\to F_{1,j}(t,p)$ is also continuous on $\overline{Q}_j$.

II. Let $F(t,p)<0, (t,p)\in \hat{Q}_j \subset \hat{Q}, j\in\{-q,\dots,-1\}$, which means that

$$(t,p)\in \hat{Q}_j \subset \hat{Q}, j\in\{-q,\dots,-1\}.\ h_\ell \le z_j \le F(t,p) < z_{j+1} \le h_u, \tag{24}$$

Likewise as in the previous case we define on $\hat{Q}_j \subset \hat{Q}, j\in\{-q,\dots,-1\}$ a new function by shifting the function $(t,p)\to H(t,p)$ as follows:

$$H_{1,j}(t,p) = H(t,p) + \gamma_j z_{j+1}(b-t) \tag{25}$$

where the function $(t,p)\to H(t,p)$ was chosen earlier and can be seen in the definition of the function $(t,p)\to F(t,p)$ satisfying (24).

Negative numbers $z_j, j \in \{-q, \ldots, -1\}$ are the points from the division of the interval $[h_\ell, h_u] \subset R$. The numbers $0 < \gamma_j < 1$ were chosen estimate the non egative values, close to zero, of a new function $(t, p) \to F_{1,j}(t, p)$. This new function is defined in subset $\hat{Q}_j, j \in \{-q, \ldots, -1\}$ as follows:

$$F_{1,j}(t, p) = H_{1,jt}(t, p) + \min\{H_{1,jx}(t, p)\tilde{f}(t, p, u) - y^0 \tilde{L}(t, p, u) \; ; u \in K\} \quad (26)$$

where $(t, p) \to H_{1,j}(t, p), (t, p, u) \to \tilde{f}(t, p, u)$ and $(t, p, u) \to \tilde{L}(t, p, u)$ be as defined in (25) and (16) respectively.

By using (26) of $(t, p) \to F_{1,j}(t, p)$ and (25) of $(t, p) \to H_{1,j}(t, p)$, we obtain the following formula which shows the relation between the functions $(t, p) \to F(t, p)$ and $(t, p) \to F_{1,j}(t, p)$ where both functions are defined on $\hat{Q}_j \subset \hat{Q}, j \in \{-q, \ldots, -1\}$.

The following formula shows this relation:

$$\begin{aligned} F_{1,j}(t, p) &= H_t(t, p) - \gamma_j z_{j+1} + \min\{H_x(t, p)\tilde{f}(t, p, u) - y^0 \tilde{L}(t, p, u); u \in K\} \\ &= F(t, p) - \gamma_j z_{j+1} \end{aligned}$$

From inequality (24) and the above formula it follows that the following estimation holds:

$$-\mu_j \le F_{1,j}(t, p) \le -\eta_j, (t, p) \in \hat{Q}_j, j \in \{-q, \ldots, -1\} \quad (27)$$

where $-\mu_j = z_j - \gamma_j z_{j+1}, -\eta_j = z_{j+1} - \gamma_j z_{j+1}, \; 0 < \gamma_j < 1$.

Notice that this estimation will be better if we choose the numbers γ_j sufficiently close to 1, i.e. γ_j=0.99.

It is easily seen that the function $(t, p) \to H_{1,j}(t, p)$ is defined and continuous on $\hat{Q}_j, j \in \{-q, \ldots, -1\}$ so it may extended into $\overline{Q}_j \subset \hat{Q}_j, j \in \{-q, \ldots, -1\}$ by putting

$$H_{1,j}(t, p) = H(t, p) - \gamma_j z_{j+1}(b - t),$$

for each $(t, p) \in \overline{Q}_j / \hat{Q}_j, j \in \{-q, \ldots, -1\}$ and $\gamma_j \in (0, 1)$.

Certainly such an extended function $(t,p)\to H_{1,j}(t,p)$ is also lipschitz function, $j\in\{-q,\dots,-1\}$, since the function $(t,p)\to H(t,p)$ is a lipschitz function.

Furthermore we have noticed that $(t,p,u)\to \tilde{f}(t,p,u)$ and $(t,p,u)\to \tilde{L}(t,p,u)$ are continuous in $\hat{Q}\times K$, so we deduce that $(t,p)\to F_{1,j}(t,p)$ is also continuous function on $\hat{Q}_j\subset\hat{Q}$, $j\in\{-q,\dots,-1\}$.

In the first step of our work, we have constructed the function $(t,p)\to H_{1,j}(t,p)$ on all subset $\overline{Q}_j\subset\hat{Q}$, $j\in\{-q,\dots,-1\}\cup\{1,\dots,r\}$ and $(t,p)\to F_{1,j}(t,p)$ defined on the same domain and take only non-positive values which are close to zero, so $(t,p)\to H_{1,j}(t,p)$ is approximating the dual $\varepsilon-$value function.

Thus we can estimate the values of the function $(t,p)\to F_{1,j}(t,p)$ as follows:

$$-\mu_j\le F_{1,j}(t,p)\le-\eta_j,(t,p)\in\hat{Q}_j\subset\hat{Q} \qquad (28)$$

where

$$\mu_j=\begin{cases}-z_{j-1}+\alpha_j z_j & if \quad j\in\{1,\dots,r\}\\ \\ -z_j+\gamma_j z_{j+1} & if \quad j\in\{-q,\dots,-1\},\end{cases} \qquad \eta_j=\begin{cases}-z_j+\alpha_j z_j & if \quad j\in\{1,\dots,r\}\\ \\ -z_{j+1}+\gamma_j z_{j+1} & if \quad j\in\{-q,\dots,-1\}\end{cases}$$

where $1<\alpha_j<2$ for $j\in\{1,\dots,r\}, 0<\gamma_j<1$ for $j\in\{-q,\dots,-1\}$. If all the numbers α_j and γ_j are sufficiently close to 1, then the numbers μ_j and η_j are non- positive and close to zero.

As the estimation of the values of the function $F_{1,j}(.,.)$ given by (28) is valid, the function $H_{1,j}(.,.)$ defined by (21)and (25) and being used in formula (22)and (26) for the function $F_{1,j}(.,.)$ would satisfy the dual dynamic programming inequality (15), i.e., it would approximate the dual $\varepsilon-$ value function for the Bolza problem (C), if only it had been at least of a Lipschitz function. So the construction would have been finished.

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Althought the function $H_{1,j}(.,.)$ is defined and continuous on $Q_j \subset Q,\ j \in \{-q,...,-1\} \cup \{1,..., r\}$ and even continuous on their closures, they are only piecewise continuous in set Q and thus it could not be sufficiently regular.

In order to make the function $H_{1,j}(.,.)$ sufficiently regular, we have to define new function using convolution of the function $H_{1,j}(.,.)$ with a function of class $C_o^\infty(R^{n+2})$ having a compact support.

So we will define a new function $(t, p) \to H_{2,j}^{\beta,i}(t, p),\ j \in \{-q,...,-1\} \cup \{1,...,r\},\ (t, p) \in \hat{Q}_j \subset \hat{Q}, t \in [t_0 + \delta, b - \delta]$ for arbitrary fixed $\beta < \min(1, \delta), i \in N/\{0,1,2,3\}$. To define new function $(t, p) \to H_{2,j}^{\beta,i}(t, p)$ we convolute a function $(t, p) \to H_{1,j}(t, p),\ j \in \{-q,...,-1\} \cup \{1,..., r\}$ with a function $(t, p) \to \rho_\beta(t, p)$ of class $C_0^\infty(R^{n+2})$ having compact support and then translating the convolution to the left as shown below:

$$H_{2,j}^{\beta,i}(t,p) = (H_{1,j} * \rho_\beta)(t,p) - (i-2/i)\eta_j(b-t), (t,p) \in Q_j \subset Q \qquad (29)$$

where the function $(t, p) \to H_{1,j}(t, p)$ as defined in (21) and (25), $\eta_j,\ j \in \{-q,...,-1\} \cup \{1,..., r\}$, is the upper bound of the function $(t, p) \to F_{1,j}(t, p)$ as in (27), $i \in N/\{0,1,2,3\}$, when $i \to +\infty$, then $(i-2/i) \to 1$; $\rho_1 : R \times R^{n+1} \to R$ is a function of class $C_0^\infty(R^{n+1})$ having compact support; $\int_{R^{n+2}} \rho_1(t, p)\ dt\ dp = 1$;

$\rho_\beta(t, p) = (1/\beta^{n+2}\rho_1)(t/\beta, p/\beta) \in C_0^\infty(R^{n+2})$; $\sup\ \rho_1 \subset B_1(R^{n+2})$; $B_1(R^{n+2})$ is a sphere centered at the origin having a radius of 1.

As in the previous parts of construction we define a new function

$$(t, p) \to F_{1,j}^{\beta,i}(t, p), j \in \{-q,...,-1\} \cup \{1,...,r\}, \beta < \min(1, \delta), i \in N/\{0,1,2,3\},$$

on Q_j as follows:

$$F_{1,j}^{\beta,i}(t,p)=(\partial/\partial t)H_{2,j}^{\beta,i}(t,p)+\min\{(\partial/\partial x)H_{2,j}^{\beta,i}(t,p)\tilde{f}(t,p,u)-y^{0}\tilde{L}(t,p,u)$$
$$:u\in K\} \quad (30)$$

where the function $(t,p)\to H_{2,j}^{\beta,i}(t,p)$ is defined by (29). Clearly, this function $(t,p)\to H_{2,j}^{\beta,i}(t,p)$ will also be a Lipschitz function, because the function $(H_{1,j}*\rho_{\beta})(.,.)$ is Lipschitz for *t,p*.

We deduce that $(t,p)\to F_{2,j}^{\beta,i}(t,p)$, $j\in\{-q,\ldots,-1\}\cup\{1,\ldots,r\}$ $\beta<\min(1,\delta)$, $i\in N/\{0,1,2,3\}$ is continuous on $\hat{Q}_j\subset\hat{Q}$,since $(t,p)\to H_{2,j}^{\beta,i}(t,p)$ and $(\partial/\partial t)H_{2,j}^{\beta,i}(t,p)$ are continuous on $\hat{Q}_j$, and since $(t,p,u)\to\tilde{f}(t,p,u)$ and $(t,p,u)\to\tilde{L}(t,p,u)$ are continuous on $\overline{Q}\times K$.

In the next part of this paper we will estimate the values of the function $F_{2,j}^{\beta,i}(.,.)$. It will apper that these values are close to zero, but of different signs. So the function $H_{2,j}^{\beta,i}(.,.)$ satisfy the DPDIDP(15). Thus, we will obtain a new function $H_{2,j}^{\beta,i}(.,.)$ that will approximate the dual ε-value function.

Let us formulate and prove three lemmas, which will simplify and shorten the proof of Theorem 3.1.

Because in theorem 3.1 we will make use of the fact that the functions $-y^{0}\tilde{L}(.,.,.)$ and $-y^{0}(\tilde{L}*\rho_{\beta})(.,.,.)$ have values arbitrary close to each other is needed.

Therefore lemma 3.1 should be proved first. This gives an estimate of the difference of the values between these two functions by arbitrary close to zero on $\overline{Q}_j\times K$, $j\in\{-q,\ldots,-1\}\cup\{1,\ldots,r\}$.

Lemma 3.1: Let $\tilde{L}(.,.,.)$ be a function as defined in (16) and satisfying the assumptions (*Z*), and $\rho_{\beta}(.,.)$ be the function of class $C_0^{\infty}(R^{n+2})$ defined above. Then for arbitrary $i\in N/\{0,1,2,3\}$ and η_j, $j\in\{-q,\ldots,-1\}\cup\{1,\ldots,r\}$ described during construction of the function

$H_{1,j}(.,.)$, there exist $\beta_i^j > 0$ such that for any $\beta \le \beta_i^j$, and for all $(t,p,u) \in \hat{Q}_j \times K, t \in [t_0+\delta, b-\delta]$ $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$ the following estimation holds:

$$\left| -y^0 \tilde{L}(t,p,u) - \left(-y^0 \tilde{L} * \rho_\beta\right)(t,p,u) \right| < \frac{1}{i}\eta_j .$$

Proof: For $(t,p,u) \in \hat{Q}_j \times K$, the following estimation holds:

$$\left| -y^0 \tilde{L}(t,p,u) - \left(-y^0 \tilde{L} * \rho_\beta\right)(t,p,u) \right|$$

$$= \left| -y^0 \right| \left| \tilde{L}(t,p,u) - \left(\tilde{L} * \rho_\beta\right)(t,p,u) \right|$$

$$= \left| -y^0 \right| \left| \int_{B_\beta(R^{n+2})} \left[\tilde{L}(t,p,u) - \tilde{L}(t-s,p-p',u)\right] \rho_\beta(s,p')\, ds\, dp' \right|$$

$$\le \left| -y^0 \right| \int_{B_\beta(R^{n+2})} \left| \left[\tilde{L}(t,p,u) - \tilde{L}(t-s,p-p',u)\right] \rho_\beta(s,p') \right| ds\, dp'$$

$$\le \left| -y^0 \right| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}_j,\ t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\beta(R^{n+2})}} \left| \tilde{L}(t,p,u) - \tilde{L}(t-s,p-p',u) \right|,$$

because the function $\tilde{L}(.,.,.)$ is uniformly continuous in the compact sets $\overline{Q}_j \times K, t \in [0,b]$

$$\left| -y^0 \right| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}_j,\ t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\beta(R^{n+2})}} \left| \tilde{L}(t,p,u) - \tilde{L}(t-s,p-p',u) \right| \to 0 \ as\ \beta \to 0,$$

and consequently,

$$\left| -y^0 \tilde{L}(t,p,u) - \left(-y^0 \tilde{L} * \rho_\beta\right)(t,p,u)) \right| \to 0 \ as\ \beta \to 0.$$

Hence, for an arbitrary $i \in N/\{0,1,2,3\}$ and η_j, $j \in \{-q,...,-1\} \cup \{1,..., r\}$ where exists $\beta_i^j > 0$ such that for every $\beta \le \beta_i^j$ and for all $(t, p, u) \in \hat{Q}_j \times K, t \in [t_0 + \delta, b - \delta]$ the following holds:

$$\left| - y^0 \tilde{L}(t, p, u) - \left(- y^0 \tilde{L} * \rho_\beta\right) (t, p, u) \right| < \frac{1}{i} \eta_j . \qquad \blacksquare$$

The fact that the functions $(\partial/\partial x) H_{2,j}^{\beta,i}(.,.) \tilde{f}(.,.,.)$ and $\left[\left((\partial/\partial x) H_{1,j} \tilde{f}(.,.,.)\right) * \rho_\beta\right](.,.)$ have values arbitrarily close will be needed in the proof of theorem 3.1, so lemma 3.2 must be proved. This gives an estimate of difference between the values of these two functions by a real number arbitrarily close to zero.

Lemma 3.2: Let $H_{1,j}(.,.), H_{2,j}^{\beta,i}(.,.)$ and $\rho_\beta(.,.)$ be functions defined in $\hat{Q}_j$, $j \in \{-q,...,-1\} \cup \{1,..., r\}$ (see (29)) and let $\tilde{f}(.,.,.)$ be a function as defined in (16) satisfying the assumptions (Z). Then for an arbitrary number $i \in N/\{0,1,2,3\}$ and $\eta_j, \in \{-q,...,-r\} \cup \{1,..., r\}$ described during the construction of the function $H_{1,j}(.,.)$, there exists $\tilde{\beta}_i^j > 0$ such that for all $\beta \le \tilde{\beta}_i^j$ and for all $(t, p, u) \in \hat{Q}_j \times K$, $j \in \{-q,...,-1\} \cup \{1,..., r\}$, $t \in [t_0 + \delta, b - \delta]$ the following inequality hold:

$$\left|(\partial/\partial x) H_{2,j}^{\beta,i}(t, p) \tilde{f}(t, p, u) - \left[\left((\partial/\partial x) H_{1,j} \tilde{f}(.,.,u)\right) * \rho_\beta\right](t, p)\right| < \frac{1}{i} \eta_j$$

Proof: Since the function $H_{1,j}(.,.)$ is a Lipschitz function, then it is satisfies the Lipschitz condition, i.e.,

$\left|(\partial/\partial x) H_{1,j}(.,.)\right| \le M_{1,j}$ for some constant $M_{1,j} > 0$. Thus for all $(t, p, u) \in \hat{Q}_j \times K$, $t \in [t_0 + \delta, b - \delta]$, and by using the definitions of $H_{2,j}^{\beta,i}(.,.)$ and the convolution, the following holds:

$$\left|(\partial/\partial x) H_{2,j}^{\beta,i}(t, p) \tilde{f}(t, p, u) - \left[\left((\partial/\partial x) H_{1,j} \tilde{f}(.,.,u)\right) * \rho_\beta\right] (t, p) \right|$$

$$=\left|(\partial/\partial x)\left(H_{1,j} * p_\beta\right)\ (t,p)\tilde{f}(t,p,u)-\left[\left((\partial/\partial x)H_{1,j}\tilde{f}(.,.,u)\right)*\rho_\beta\right](t,p)\ \right|$$

$$=\left|\int_{B_\beta\left(R^{n++2}\right)}(\partial/\partial x)H_{1,j}(t-s,p-p')\tilde{f}(t,p,u)\rho_\beta(s,p')\ ds\ dp' -\right.$$

$$\left.\int_{B_\beta\left(R^{n+2}\right)}(\partial/\partial x)H_{1,j}(t-s,p-p')\tilde{f}(t-s,p-p',u)\rho_\beta(s,p')\ ds\ dp'\ \right|$$

$$\leq\int_{\beta_\beta\left(R^{n+2}\right)}\left|(\partial/\partial x)H_{1,j}(t-s,p-p')\right|\ \left|\tilde{f}(t,p,u)-\tilde{f}(t-s,p-p',u)\right|\rho_\beta(s,p')\ ds\ dp'$$

$$\leq M_{1,j}\sup_{\substack{u\in K\\ (t,p)\in\hat{Q}_j,\ t\in[t_0+\delta,b-\delta]\\ (s,p')\in B_\beta\left(R^{n+2}\right)}}\left|\tilde{f}(t,p,u)-\tilde{f}(t-s,p-s,u)\right|.$$

Since the function $\tilde{f}(.,.,.)$ is uniformly continuous on the compact $\overline{Q}_j\times K, t\in[0,b]$, then we obtain the last inequality tends to zero as $\beta\to 0$, that is

$$\sup_{\substack{u\in K\\ (t,p)\in\hat{Q}_j,\ t\in[t_0+\delta,b-\delta]\\ (s,p')\in B_\beta\left(R^{n+2}\right)}}\left|\tilde{f}(t,p,u)-\tilde{f}(t-s,p-s,u)\right|\to 0 \quad \text{as} \quad \beta\to 0,$$

and consequently,

$$\left|(\partial/\partial x)H_{2,j}^{\beta,i}(t,p)\tilde{f}(t,p,u)-\left[\left((\partial/\partial x)H_{1,j}\tilde{f}(.,.,u)\right)*\rho_\beta\right](t,p)\right|\to 0\ \ as\ \ \beta\to 0$$

Thus, for arbitrary $i \in N/\{0,1,2,3\}$ and η_j, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$ there exists $\tilde{\beta}_i^j > 0$ such that for all $\beta \le \tilde{\beta}_i^j$, and for all $(t,p,u) \in \hat{Q}_j \times K$, $t \in [t_0+\delta, b-\delta]$, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$, the following holds:

$$\left|(\partial/\partial x) H_{2,j}^{\beta,i}(t,p)\tilde{f}(t,p,u) - \left[\left((\partial/\partial x) H_{1,j}\tilde{f}(.,.,u)\right) * \rho_\beta\right](t,p)\right| \frac{1}{i}\eta_j . \blacksquare$$

In the proof of theorem 3.1, the uniform convergence of the sequence $\{H_{2,j}^{\beta,i}(t,p)\}$ *to* $H_{1,j}(t,p)$ as β converges to zero, for all $(t,p) \in \hat{Q}_j$, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$, $t \in [t_0+\delta, b-\delta]$ is also required as is shown in the following result.

***Lemma* 3.3**: Let $H_{1,j}(.,.)$, $H_{2,j}^{\beta,i}(.,.)$ and $\rho_\beta(.,.)$ be functions defined in $\hat{Q}_j$ $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$ (see (29)). Then for all $(t,p) \in \hat{Q}_j$, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$, $t \in [t_0+\delta, b-\delta]$, we have

$$\lim_{\beta \to o} H_{2,j}^{\beta,i}(t,p) = H_{1,j}(t,p),$$

and this convergence is uniform.

Proof: By definition of uniformly convergent sequence of functions to prove that this lemma holds, it is sufficient to show that for arbitrary $\varepsilon_j > 0$, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$ a $\beta_i^j > 0$ exists such that for every $\beta < \beta_i^j$ and for all $(t,p) \in \hat{Q}_j$, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$, $t \in [t_0+\delta, b-\delta]$ the following holds:

$$\left|H_{2,j}^{\beta,i}(t,p) - H_{1,j}(t,p)\right| \le \varepsilon_j$$

Now by using the definitions of the function $H_{2,j}^{\beta,i}(.,.)$ and the convolution, for all $(t,p) \in \hat{Q}_j$, $j \in \{-q,\ldots,-1\} \cup \{1,\ldots,r\}$, $t \in [t_0+\delta, b-\delta]$, the following holds:

$$\left|H_{2,j}^{\beta,i}(t,p) - H_{1,j}(t,p)\right|$$

$$=\left|\left(H_{1,j} * \rho_\beta\right)(t,p) - H_{1,j}(t,p)\right|$$

$$=\left|\int_{B_\beta(R^{n+2})}\left[H_{1,i}(t-s,p-p') - H_{1,j}(t,p)\right]\rho_\beta(s,p')\ ds\ dp'\right|$$

$$\le \int_{B_\beta(R^{n+2})}\left|H_{1,j}(t-s,p-p') - H_{1,j}(t,p)\rho_\beta(s,p')\right| ds\ dp'$$

$$\le \sup_{\substack{u\in K \\ (t,p)\in \hat{Q}_j,\ t\in[t_0+\delta,b-\delta] \\ (s,p')\in B_\beta(R^{n+2})}} \left|H_{1,j}(t-s,p-p') - H_{1,j}(t,p)\right|.$$

Since the function $H_{1,j}(.,.)$ is uniformly continuous in the compact $\overline{Q}_j,\ j\in\{-q,\dots,-1\}\cup\{1,\dots,r\}$, then we have

$$\le \sup_{\substack{u\in K \\ (t,p)\in \hat{Q}_j,\ t\in[t_0+\delta,b-\delta] \\ (s,p')\in B_\beta(R^{n+2})}} \left|H_{1,j}(t-s,p-p') - H_{1,j}(t,p)\right| \to 0 \ as\ \beta\to 0.$$

Consequently,

$$\left|H_{2,j}^{\beta,i}(t,p) - H_{1,j}(t,p)\right| \to 0\ \ as\ \ \beta\to 0.$$

Therefore, for an arbitrary $\varepsilon_j > 0$ a $\beta_i^j > 0$ exists such that for all $\beta \le \beta_i^j$ and for all $(t,p)\in \hat{Q}_j,\ j\in\{-q,\dots,-1\}\cup\{1,\dots,r\}$, $t\in[t_0+\delta, b-\delta]$ the following holds:

$$\left|H_{2,j}^{\beta,i}(t,p) - H_{1,j}(t,p)\right| \le \varepsilon_j$$

■

The main result of this work is formulated in theorem 3.1, which ensures that the dual ε -value function $S_{\varepsilon D}(t,p),(t,p)$ belongs to set W and satisfies

$$H(t,p)+\upsilon_i(b-t)\le S_{\varepsilon D}(t,p)\le H(t,p) \qquad \text{for all } H\in W.$$

Theorem 3.1: The dual ε -value function $S_{\varepsilon D}(t,p)$, $(t,p)\in P$, $t\in[0,b]$, for problem (C) (see (9) and (10)) satisfies the following

$$H(t,p)+\upsilon_i(b-t)\le S_{\varepsilon D}(t,p)\le H(t,p), \upsilon_i=-\mu+\frac{i-4}{i}\eta_j,\ j\in\{-q,\dots,-1\}\cup\{1,\dots,r\}$$

for all $H(t,p)\in W$, $(t,p)\in P$.

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