Basic Properties of the Spectral Problem with Spectral Parameter in two-Point Boundary Conditions of the Vibration Problem

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Abstract

In this present work, the properties as completeness, minimafity and basic property are investigated for the eigenfunction of boundary problems for a differential equation of second-order with a spectral parameter in the both of boundary conditions of the vibration problem.

Key word: Spectral problem, Spectral parameters, Vibration problem.

الخلاصة:

في هذا البحث تناولنا مسألة القيم الخاصة المنتظمة لمعادلة تفاضلية اعتيادية من الرتبة الثانية مع وجود البارامیتر الطيفي في الشروط الحدودية لمعادلة الاهتزاز. حيث تم البرهنة على إن الدوال الذاتية للمؤثر التفاضلي الاعتيادي المكافئ لمسالة القيم الخاصة تشكل نظام أصغر ما يمكن وكامل و تكون أساس متعامد في فضاء هلبرت الموسع.

Introduction

A boundary value problem with a spectral parameter in the boundary condition is appeared commonly in mathematical models of mechanic; consider the vibration problem of homogeneous string. Suppose that there is string on the vertical ox-axis. Tension of the string is expressed by Ku_x, where K is the elastic module. If a mass m is attached to the upper and lower ends of a longitudinally vibrating string the Bcs would be

$$mu_{tt}(0,t)=Ku_x(0,t) \qquad t\geq 0$$

$$mu_{tt}(1,t)=Ku_x(1,t) \qquad t\geq 0$$

The mathematical module of this problem is represented by the equation.

$u_{xx} = u_{tt} \quad , 0 < x < 1 \ , 0 < t < \infty$

And boundary conditions

$Ku_x(0,t) - mu_{tt}(0,t) = 0$

$Ku_x(1,t) + mu_{tt}(1,t) = 0$

And the initial condition

$u(x,0) = f(x) \quad and \quad u_t(x,0) = g(x)$

Applying the Fourier method to the boundary-value problem, separating the variable by $u(x,t) = y(x)e^{imt}$ we obtain the boundary-value problem

$$y'' + \lambda y = 0 \quad \ldots\ldots\ldots\ldots.(1)$$

$$\left.\begin{array}{l} y'(0) = -h\lambda y(0) \\ y'(1) = h\lambda y(1) \end{array}\right\} \ldots\ldots\ldots.(2)$$

$h = \frac{m}{K} > 0 \quad and \quad \lambda = m^2$ Where

The application of this boundary problem was given in [1,2].

In general, for the equation (1) when the boundary conditions contain a spectral parameter this problem can't be interpreted an eigenvalue-eigenfunction

$L_2(0,1)$. In this case when the both of boundary problem in the Hilbert space

conditions contained a spectral parameter, it was considered the space $L_2(0,1) \times C \times C \quad (C \ complex \ number)$. [3,4,5,6]

Formulation problem:

Defined Adequate Hilbert space by:

$$H = L_2(0,1) \times C \times C = \{(y,a,b) : y \in L_2(0,1) \ \text{and} \ a,b \in C\}$$

And the inner product by

$$< y(a,b),(y,a,b) > = \int_0^1 y(x)\bar{y}(x)dx + \frac{1}{h}a\bar{a} + \frac{1}{h}b\bar{b}$$

$$= \|y\|^2 + \frac{1}{k}(|a|^2 + |b|^2)$$

the operator is defined in the space H We denote by A,

$$A(y,a,b) = (-y'', -y'(0), y'(1))$$

And its domain $D(A)$ of all elements (y,a,b) in H satisfying the conditions:

1- y *and* y' absolutely continues on $(0,1)$.

2- $a = hy(0)$.

3- $b = hy(1)$.

to the spectral problemWe can easily obtain that the boundary problem $(1)-(2)$ is equivalent

$$A(y,a,b) = \lambda(y,a,b)$$

Lemma (1): The domain $D(A)$ is dense in the Hilbert space H.

Proof: See [7].

Lemma (2): The eigenvalue of the boundary problem $(1)-(2)$ with multiplicity coincide with the eigenvalue of operator A: for every one chain of the eigenfunctions

$$(y_0, a_0, b_0), (y_1, a_1, b_1), \dots\dots\dots (y_n, a_n, b_n)$$

Corresponding to the eigenvalue λ_0 coincide with the eigenfunctions $y_0, y_1, y_2, \dots\dots\dots y_n$ corresponding to the eigenvale λ_0 of the operator A and vice versa.

Proof: Let $(y_i, a_i, b_i) \in D(a)$, $i = 0,1,2,\dots,n$

The eigenfunctions corresponding to the eigenvalue λ_0 of the operator A is

$$A(y_i, a_i, b_i) = \lambda_0(y_i, a_i, b_i)$$

$$(-y_i'', -y_i'(0), y_i'(1)) = \lambda_0(y_i, hy_i(0), hy_i(1))$$

$$= (\lambda_0 y_i, \lambda_0 hy_i(0), \lambda_0 hy_i(1))$$

We can obtain

$$-y_i'' = \lambda_0 y_i \quad \text{and}$$

$$-y_i'(0) = \lambda_0 hy_i(0)$$

$$y_i'(1) = \lambda_0 hy_i(1)$$

Then

$$y_i'' + \lambda_0 y_i = 0$$

$$y_i'(0) = -h\lambda_0 y_i(0)$$

$$y_i'(1) = h\lambda_0 y_i(1)$$

The lemma is proved.

Lemma (3): The operator A is semi-bounded from below in Hilbert space H.

Proof: See [7].

Lemma (4): Let $\{e_i\}_0^\infty$ and $\{e_i^*\}_0^\infty$ be complete orthogonal system in the Hilbert space H.

If p is an orthogonal projection and codim $p = N$ then it can be the omitted N elements from the system$\{p_{ei}\}_0^\infty$ and the rest of the elements of the system $\{p_{ei}\}_0^\infty$ from a minimal and completed system.

Proof: Seen [8].

Lemma (5): The operator A is symmetric in the Hilbert space H

Proof: Let (y,a,b) and $(z,e,f) \in D(A)$

$$< A(y,a,b),(z,e,f) > =< A\big(y,hy(0),hy(1)\big),\big(z,hz(0),hz(1)\big) >$$

$$=< (-y'',y'(0),y'(1)),(z,hz(0),hz(1)) >$$

$$=-\int_0^1 y''(x)\,\bar{z}(x)dx-\frac{1}{h}hy'(0)\,\bar{z}(0)+\frac{1}{h}y(1)\,h\bar{z}(1)$$

$$=-\int_0^1 y''(x)\,\bar{z}(x)dx-y'(0)\,\bar{z}(0)+y(1)\,\bar{z}(1)$$

Using two times the integrations by parts we obtain

$$< A(y,a,b),(z,e,f) >= -\int_0^1 y(x)\bar{z}''(x)dx - \bar{z}'(0)y(0) + \bar{z}'(1)y(1)$$

$$=-\int_0^1 y(x)\bar{z}''(x)dx+\frac{1}{h}(-\bar{z}'(0))hy(0)+\frac{1}{h}\bar{z}'(1)hy(1)$$

$$=< \big(y,hy(0),hy(1)\big),(-\bar{z}'',-\bar{z}'(0),\bar{z}'(1)) >$$

$$=< (y,a,b), A(z,e,f) >$$

The lemma is proved.

Theorem (1): There is an unboundedly increasing sequence $\{\lambda_n\}_{n=0}^{\infty}$ of eigenvalues of the boundary value problem (1) —(2):

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$$

Moreover, the eigenfunction $y_n(x)$ corresponding to λ_n has exactly n simple zeros in the interval [0,1].

Proof: See [10].

Lemma (6): The operator A is invertible if and only if $\lambda = 0$ is a not eigenvalue of A.

Proof: See [9].

Lemma (7): The operator $(A-\lambda I)^{-1}$ is compact if λ is not eigenvalue of A, (where I is the unit operator in the space H).

Proof: See [11].

Lemma (8): The operator A is selfadjiont in the Hilbert space H.

Proof: from lemma (5) the operator A is symmetric and to prove that

$$(A-\lambda I)^{-1}H = D(A)$$

Let $Y = (y,a,b) \in D(A)$ and satisfying

$$(A-\lambda I)Y = F \quad \ldots\ldots\ldots \quad (3)$$

Where $F = (f,\alpha,\beta)$

The equation (3) is a nonhomogeneous differential equation has a solution:[12].

$$y(x) = c_1\varphi_\lambda(x) + c_2\Psi_\lambda(x) + \int_0^1 G(x,t,\lambda)f(t)dt$$

$$a = hy(0)$$

$$b = hy(1)$$

Defined by:

$$G = (x,t,\lambda) = \begin{cases} \frac{\varphi_\lambda(x)\Psi_\lambda(x)}{c} & 0 \le t \le x \le 1 \\ \frac{\Psi_\lambda(x)\varphi_\lambda(x)}{c} & 0 \le x \le t \le 1 \end{cases}$$

And $G(x,t,\lambda)$ satisfying the boundary condition (2) and $\varphi_\lambda(x), \Psi_\lambda(x)$ two solution for the equation (1) and satisfying the initial conditions

$$\varphi_\lambda(0) = 1$$

$$\varphi_\lambda'(0) = -\lambda h$$

$$\Psi_\lambda(1) = 1$$

$$\Psi_\lambda'(1) = \lambda h$$

We obtain $c_1 = \frac{\alpha}{c}$ and $c_2 = \frac{\beta}{c}$ where c constant.

Then

$$y(x) = \frac{\alpha\varphi_\lambda(x)}{c} + \frac{\beta\Psi_\lambda(x)}{c} + \int_0^1 G(x,t,\lambda)f(t)dt$$

$$a = hy(0)$$

$$b = hy(1)$$

From [9]

$$Y = (A - \lambda I)^{-1}F$$

So $$Y \epsilon (A - \lambda I)^{-1}H \quad \dots\dots\dots \quad (4)$$

Since λ is not eigenvalue of A, for all $F = (f, \propto, \beta)$ in H there exist $y = (y, a, b)$ such that

$$y(y, a, b) \epsilon D(A)$$

From [9]

$$Y = (A - \lambda I)^{-1}F$$

Then

$$(A - \lambda I)^{-1}F \epsilon D(A)$$

So

$$(A - \lambda I)^{-1}H \in D(A) \quad \dots \quad \dots \quad \dots. (5)$$

From (4) and (5) we get

$$(A-\lambda I)^{-1}H = D(A)$$

The lemma is proved.

Theorem (2): The eigenfunction of the operator A form an orthonormal basis in the Hilbert space $H = L_2 \times c \times c$.

Proof: The eigenvalue of the boundary problem (1) — (2) we must find the intersection of the curve $\tan(m)$ and $\dfrac{2mh}{m^2h^2-1}$, $(m^2h^2 \neq 1)$

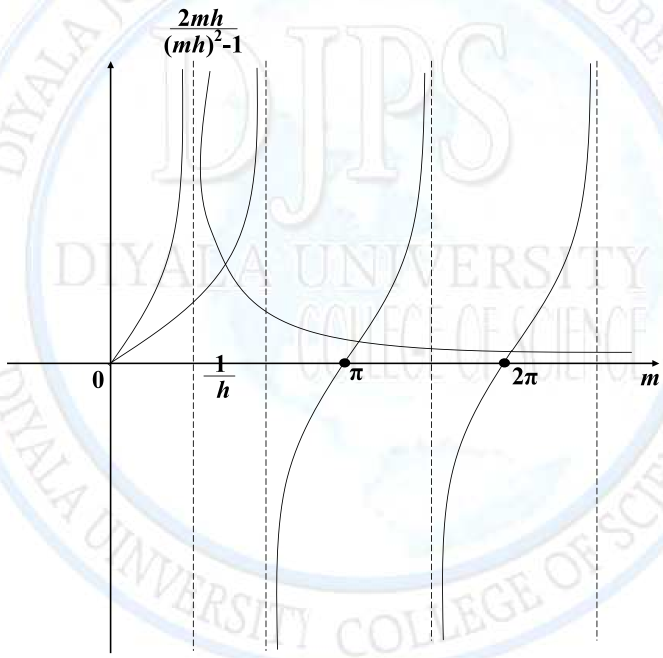

Figure (1): Graph showing intersection of $\tan(m)$ **and** $\dfrac{2mh}{m^2h^2-1}$ **.**

It can be easily obtained that the operator A has at most countable eigenvalue $\lambda_n = M_n^2$ which have the asymptotic form

$$\lambda_k = (k\pi)^2 + o(1)f \qquad \text{as} \quad k \to \infty \text{ . [10]}$$

Then, for any number λ which is a not eigenvalue and arbitrary $F \in H$ it can be found $y \in D(A)$ satisfying the condition

$(A - \lambda I)Y = F$ (from lemma 8).

Thus, the operator $A - \lambda I$ is invertible except for the isolated eigenvalues. Since $\lambda = 0$ is not eigenvalue then the operator A is selfadjiont (from lemma 8). Thus, the selfadjiont operator A^{-1} has countable many eigenvalues which are convergent to zero as infinity. So, the selfadjoint operator A^{-1} is compact (from lemma 7). Applying the Hilbert-schmit theorem to this operator we obtain that the eigenfunction of the operator A form an orthonormal basis in the Hilbert space H. The theorem is proved.

Theorem (3): Let k_0 be an arbitrary fixed nonnegative integer. The system of eigenfunction $\{y_n\}_0^\infty$,$(n \neq k_0)$ of the boundary problem (1) — (2) form a basis in $L_2(0,1)$, complete and minimal system.

Proof: See [13,14].

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