Bilinearization algorithm and KdV- type equations

Inaam A.Malloki University of AL-Mustansiriya
Department of Mathematics
Sheama A. AL-Aubaidee University of AL-Mustansiriya
Department of Mathematics

Abstract :

In this paper , a modification to the basic steps of the bilinearization of evolution equations by Hirota's method is presented by writing the dependent variable u(x,t) and some of its derivatives almost as Hirota polynomials . Then , a definition to modify Peterson definition for the class of KdV-type equations is presented in addition the general work is illustrated by applications to linear equations and to three classes of the nonlinear well-known equations .

(1) Introduction :-

In this work a nonlinear partial differential equation and one of the most analytic methods for solving such types of equations is considered . The approach is called Hirota's method for solving equation which has soliton solution .The soliton was coined to describe a pulse like nonlinear wave which emerges from a collision with a similar pulse having unchanged shape and speed and it was found by Scott-Russell on 1834 [15] .The fundamental idea in Hirota's formalism is to use some dependent variable transformations to put the equation in a form where the unknown function appears bilinearly .In 1976 , the bilinear formalism was introduced by R.Hirota . Hirota's formalism is a very powerful method [14] of simplifying algebraic calculations but one would not do it justice by looking at it only as a purely formal tool for the manipulation of complicated expressions . It also gives a great deal of insight into many problems [14] The basic symbol D of the formalism is defined by :

$$(D_x^a D_y^b D_t^c)\ f(x,y,t)\ .\ g(x,y,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^a \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^b \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^c f(x,y,t)\ g(x',y',t')\ \Big|_{x'=x\ ,\ y'=y,\ t'=\ t}$$

with this symbol , some nonlinear partial differential equations may be represented . The simplest way to represent an equation in this formalism is :

$$P\ (D_x, D_y, D_t)\ f \cdot f = 0$$

where P is some polynomial that can always be taken to be even .

Expanding f in a formal power series in ε , substituting into the bilinear form and equating terms with the same power of ε , we obtain a set of linear equations which is to be solved successively and if the solution series terminates , we have an exact solution to the nonlinear partial differential equation .

The method has been successfully applied to find explicit solutions to many partial differential equations (PDES) see for instance [1] ,[2] ,[3] ,[5] , [7] , [8] , [9] , [12] , [13] , [14] and [15] .

In 1987 Hietarinta et.al . classified such bilinearizable equations into five groups : KdV , MKdV , NLS , SG and BO type equations . This classification is based on the different ways of how the corresponding equations are bi-linearized [7] . This classification , led P.Peterson very recently , to introduce what is called definition of the KdV-type equations [13] (An equation K[u] = 0 is called the KdV-type equation (in the Hirota sense) iff there exist two nontrivial Hirota polynomials P = P(μ,ν,ω) and N = N(μ,ν,ω) such that :

$$P(D_x)(f.f) = 0 \quad \Rightarrow K\left(f^{-2} N(D_x)(f.f)\ \right) = 0$$

holds for all nontrivial positive functions f = f(x). Equation $P(D_x)(f.f) = 0$ is called the Hirota bilinear form of the corresponding equation K[u] = 0) .

In this paper we try to analyse the steps of the bilinear formalism which is the main part of Hirota’s method on KdV equation . This analysis leads us to reconstruct these steps in a different order so that the technique will be easier to apply and the number of steps will be reduced .

Moreover, we redefine P.Peterson definition of KdV-type equations in such a way that it will cover more equations.The new definition is quite practical since it is algorithmically straight-forward to find out if a given nonlinear evolution equation of nth order is included in the class .

This paper consists of six parts .Part one is included the general idea of this work .

In part two is concentrated to the analysis of Hirota's bilinear formalism .

In part three a modification of statement and applicability of Peterson's definition is presented . In the same part , modified definition of KdV- type equation and implementation of the modified definition is presented too .

In part four and five an application of the modified definition if $N = D_x^2$ and $N = D_x D_t$ is presented .This application is applied to some linear partial differential equations and three classes of nonlinear partial differential equations, in each class the method is demonstrated to several known equations .

The last part is included the conclusions of this work .

Note : Through out the paper the symbols will used :

u_{ix} (or u_{it}) , i = 2 , 3 , … to denote the partial ith derivative with respect to x (or t) .

(2) Analysis of Hirota's bilinear formalism :-

For the KdV equation to be bilinearized we need to introduce a new dependent variable . In this case , the most suitable one is Cole-Hopf transformation $u=\frac{\partial^2}{\partial x^2}\ln f(x,t)$. The reason will be soon clear in the following lemma:

Lemma (1) :

If $u=\frac{\partial^2}{\partial x^2}\ln f(x,t)$, then …(1-1)

(I) $$\frac{\partial^n u}{\partial x^n}=\left\{ fp_{nx}-(n+1)\, f_x\, p_n \right\} \Big/ f^{\,n+2}=\frac{\partial}{\partial x}\left(\frac{p_n}{f^{\,n+1}}\right)$$

where $$p_n= fp_{(n-1)x}- nf_x\, p_{(n-1)}$$, n =1,2,3 …(1-2)

$$p_0 = f_x$$

(II) $$\frac{\partial u}{\partial t} = \{ f q_x - 2 f_x q \} / f^3$$

…(1-3)

where $q = f p_{0t} - f_t p_0 \quad , \quad p_0 = f_x$

Proof : (by direct substitution) .

When substituting the Cole-Hopf transformation (1-1) which can be written as

$u(x,t) = \frac{\partial^2}{\partial x^2} \ln f(x,t) = \frac{D_x^2 (f.f)}{f^2}$ in KdV equation

$$K[u] = u_{3x} + a \; u u_x + u_t = 0$$

…(1-4)

where α is “ a constant ” , simplifying and using lemma (1) imply to :

$$K[u] = \frac{\partial}{\partial x}\left(\frac{p_3}{f^4}\right) + a \left\{ \frac{\partial}{\partial x}\left(\frac{p_0}{f}\right) \frac{\partial}{\partial x}\left(\frac{p_1}{f^2}\right) \right\} + \frac{\partial}{\partial x}\left(\frac{q}{f^2}\right) = 0$$

...(1-5)

Integrating with respect to x , simplifying and substituting α = 12 , imply to :

$$\{ f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 \} / f^2 + \{ f f_{xt} - f_x f_t \} / f^2 = 0$$

…(1-6)

One can see that equation (1-6) can be written in the form :

$$P = (D_x^4 + D_x D_t) (f . f) = 0$$

…(1-7)

Therefore we conclude that the KdV equation with α =12 satisfies Peterson’s definition .

(3) Modification of (statement and applicability of Peterson’s

definition):-

The main drawback of Peterson's definition is that it excludes many equations known as KdV-equations. Moreover , when one try to use Peterson's definition to test a given equation two Hirota's polynomials N and P are needed .

In this section , we reconstruct Peterson's definition as follows :

(3-1) Modified definition of KdV-type equation :

The partial differential equation (PDE)

$$K[u] = u_t + h\,(u, u_x, u_{2x}, \ldots) = 0$$

…(1-8a)

or

$$K[u] = u_{2t} + h\,(u, u_x, u_{2x}, \ldots) = 0$$

…(1-8b)

where h is a polynomial ,is called a KdV- type equation if there exist Hirota's polynomials

N ; P_1 , P_2 , … , P_L such that if $u = \frac{N}{f^2}$, then the equation (1-8) will be :

$$f^{k_1} P_1 + f^{k_2} P_2 + \ldots + f^{k_L} P_L = 0$$

for some nonnegative integers k_1 ,k_2 ,… , k_L .

It is clear that the new definition is generalization to Peterson's definition.

This definition guarantees that all the equations which are called KdV equations , will be in the new class .

(3-2) Implementation of the modified definition :

To implement the definition here the same main steps will be used as in the previous section , but in a different order . Explaining most derivatives almost as Hirota's polynomials make the work more easier. To be applicable , the definition may be presented as an algorithm as follows:

Input : Hirota polynomial N .

Output : Hirota polynomials $P_1, P_2, \ldots, P_L$.

(1) Integrate the PDE (1-8a) once and the PDE (1-8b) twice with respect to x .

(2) Substitute the transformation $u = \frac{N}{f^2}$, and its derivatives as Hirota’s polynomials in the integrated partial differential equation .

(3) Simplify the result in a way that it (the PDE) may pass the modified definition of KdV-type equations .

(4) Application of the modified definition if $N = D_x^2$

In this case , first the following lemma is stated which represented the dependent variable u(x,t) and some of its derivatives almost as Hirota polynomials :

Lemma (2) :- If $$u(x,t) = \frac{\partial^2}{\partial x^2} \ln f(x,t) = \frac{D_x^2(f.f)}{f^2}$$ …(1-9)

then :-

(I) $$u_{2x} = \frac{D_x^4(f.f)}{f^2} - 6\frac{[D_x^2(f.f)]^2}{f^4}$$ …(1-10a)

(II) $$u_{4x} = \frac{D_x^6(f.f)}{f^2} - 60u^3 - 30uu_{2x}$$ …(1-10b)

(III) $$u_t = \frac{\partial}{\partial x}\left(\frac{D_x D_t(f.f)}{f^2}\right)$$ …(1-10c)

(IV) $$u_{2t} = \frac{\partial^2}{\partial x^2}\left(\frac{D_t^2(f.f)}{f^2}\right)$$ …(1-10d)

Proof :- By differentiating , adding and subtractings terms .

Proposition : The linear PDES :

$u_t = \sum_{n=1}^{3} a_n u_{(2n-1)x}$ and $u_{2t} = \sum_{n=1}^{3} b_n u_{(2n)x}$

where α_n , β_n (n = 1,2,3) are constants ,are of KdV-type equations .

Proof : Straight forward

Examples : The linearized KdV equation $u_t + u_{3x} = 0$ and the wave equation $u_{2t} - c^2 u_{2x} = 0$ are of KdV-type equation .

the definition is applied to three classes of nonlinear partial differential equations :-

(A) Equations satisfying Peterson's definition ,and of course they do satisfy the modified one,such as KdV ,
Sawada , Boussinesq and Kp equations

(B) Equations satisfying the modified definition only , namely MKdV, GKdV and Benjamin equations .

(C) Equations which do not satisfy any one of the definitions for example Burger , Fisher and Transonic
equations .

the definition will be demonstrated to KdV equation only :

(4-1) Class (A) :

(A1) : KdV-equations :-

In the following the algorithm of the modified definition is applied on KdV equation as an example :

$$u_{3x} + a\, uu_x + u_t = 0 \qquad (\alpha \text{ constant}) \qquad \ldots(1\text{-}11)$$

Integrate the given equation (1-11) one time with respect to x , imply to

$$u_{2x} + (a/2)\, u^2 + \int u_t\, dx = c(t) \qquad \ldots(1\text{-}12)$$

where c is an arbitrary function of t and it was taken to be zero. Substituting the transformation

$$u = \frac{D_x^2}{f^2} \qquad \ldots(1\text{-}13)$$

using lemma (2) , imply to :

$$\left\{ D_x^4(f.f) \Big/ f^2 - 6\,[\,D_x^2(f.f)\,]^2 \Big/ f^4 \right\} + (a\ /\ 2)\,[\,D_x^2(f.f)]^2 \Big/ f^4$$
$$+ D_x D_t(f.f) \Big/ f^2 = 0$$

…(1-14)
If α =12 then (1-14) may be written in the form :
$$\left\{(\,D_x^4 + D_x D_t)(f.f)\right\} \Big/ f^2 = 0$$
…(1-15)
and so
$$(\,D_x^4 + D_x D_t)(f.f) = 0$$
…(1-16)
which means that there exists a Hirota polynomial $P = D_x^4 + D_x D_t$ and so Peterson's definition is satisfied .
If $\alpha \neq 12$, imply to :
$$\left\{(\,D_x^4 + D_x D_t)(f.f)\right\} \Big/ f^2 + (\frac{a}{2} - 6)\,[\,D_x^2(f.f)\,]^2 \Big/ f^4 = 0$$
…(1-17)
i.e. there exist two Hirota's polynomials P_1 and P_2 such that
$$f^2 P_1 + P_2 = 0$$
…(1-18)
This is , in fact a fourth order partial differential equation in f which is more complicated than the original one , yet it may be solved if the two relations $P_1 = 0$, $P_2 = 0$ are compatible . In this case , a contradiction will appear since $P_2 = 0$ implies u = 0 , so perturbation technique must be used for equation (1-17) as follows :
It is obvious that f = 1 is a solution of the bilinear equation (1-17)
$$f(x,t) = 1 + e^1 f^{(1)}(x,t) + e^2 f^{(2)}(x,t) + ...$$
…(1-19)
where ε is our perturbation parameter .
Substitute (1-19) in (1-17) , then equating the coefficients of powers of ε , imply to :
$$e^1 : (\,D_x^4 + D_x D_t)\,(1.\,f^{(1)} + f^{(1)}.1\,) = 0$$
…(1-20)

and this equation is similar to one which is get by solving equation (1-16) . So the same solution will be get

$$u = \frac{1}{4} k^2 \operatorname{sech}^2(q / 2)$$

…(1-21)

(6) Conclusions:

The effects of the modification definition implies that the class of KdV-type equations becomes wide and includes more KdV equations for instance the closely related equation MKdV and every KdV equation in the literature.In this paper we concentrated on the transformation $u = \frac{D_x^2}{f^2}$, one can see that other transformation which is of aquadratic type may be used . In general if

$$N = a_1 D_x^2 + a_2 D_x D_t + a_3 D_t^2 + b_1 D_x + b_2 D_t + b_3$$

and N is assumed to be Hirota's polynomial , hence

$$N(0,0) = b_3 = 0 \quad and \quad N(-D_x, -D_t) = N(D_x, D_t)$$

which implies $b_1 = b_2 = 0$.

if $u = a_1 D_x^2 + a_2 D_x D_t + a_3 D_t^2 / f^2$

then

$$u_t = \frac{\partial}{\partial t}\left\{\left[a_1 \frac{\partial}{\partial x}\left(\frac{f_x}{f}\right) + a_2 \frac{\partial}{\partial x}\left(\frac{f_t}{f}\right) + a_3 \frac{\partial}{\partial t}\left(\frac{f_t}{f}\right)\right] / f^2 \right\}$$

Clearly , the third term in u_t expression can not be integrated with respect to x . Hence a_3 must be equal to zero , and u should be of the form :

$$u = a_1 D_x^2 + a_2 D_x D_t / f^2$$

References:-

[1] Caudrey P.J. , Hirota's Method of Solving Soliton-type Equations Mancheste Institute of Science and Technology , England 1977 .

[2] Cheng H.X. , A two-Parameter Backlund Transformation for Boussinesq Equation , Shanghai Institute of Computing Technique , Shanghai , China 1982.

[3] Dodd , R.K. , Eilbeck , J.C. , Gibbon , J.D. and Morris , H.C. , Solitons and Nonlinear Wave Equations , Academic Press , London , 1982 .

[4] Fokas A.S. , Asymmetry Approach to Exactly Solvable Evolution Equations , California Institute of Technology , Pasadena , California 1980 .

[5] Gibbon J.D. and Tabor M., Hirota's Method and The Painleve Property, Imperial College , London and Columbia University , New York.

[6] Grauel A. , Nonlinear Wave Equation of KdV-type , Integrability and Solutions , Paderborn , Theoretische Physik 1984.

[7] Hietarinta J. , A search for Bilinear Equations Passing Hirota's Three Soliton Condition .I . KdV-Type Bilinear Equations ,Turku University, Finland 1987.

[8] Hietarinta J. , Asearch for Bilinear Equations Passing Hirota's Three Soliton Condition . II . MKdV-type

Bilinear Equations , Turku University , Finland 1987 .

[9] Hirota R . , Fundamental Properties of the Binary Operators in Soliton Theory and their Generalization , Hiroshima University , Japan .

[10] Hongsheng C., Dispersive Smoothing Effects for KdV-type Equations, Brown University , Providence , Rhode Island 1994 .

[11] Kawamoto S. , Solitary Wave Solutions of the Korteweg de-Vries Equation with Higher Order Nonlinearity , Kanmaki , Kitakatsuragi Gun 1984 .

[12] Peterson P. , Construction and Decomposition of Multi-Soliton Solutions of KdV-type Equations , Internet on Line , 2001 .

[13] Peterson P. , a thesis Multi-Soliton Interactions and the Inverse problem of Wave Cresets , Tallin Technical University , Estona 2001 .

[14] Ramani A. , Inverse Scattering , Ordinary Differential Equations Painleve-type and Hirota’s Bilinear Formalism , Paris – Sud , France 1981 .

[15] Scott A.C. , CHU F.Y.F. And Mclaughlin D.W. , The Soliton : A new Concept in applied Science , Froceedings of the Ieee , Vol.61 , No.10 , October 1973 .