Z-Transformation for Solving Volterra Integral Equations of Convolution Type

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الخلاصــة

يقدم البحث طريقة مقترحة مع خوارزمية جديدة لحل معادلات فولتيرا التكاملية الخطية من صنف الالتفاف باستخدام محول Z حيث يتم حل معادلات فولتيرا التكاملية الالتفافية عن طريق تحويل المعادلة من الزمن المستمر إلى الزمن المتقطع باستخدام قاعدة اويلر. كما من الممكن ملاحظة كفاءة الطريقة و سهولة الحسابات فيها حيث تمت مقارنة نتائج هذه الطريقة مع نتائج متعددات حدود لاكير من خلال بعض الأمثلة التوضيحية لحل معادلات فولتيرا التكاملية الالتفافية من النوع الأول و الثاني وقد تم الحصول على نتائج جيدة. بالإضافة إلى ذلك لقد تم ذكر الخصائص المهمة لتحويل Z.

Abstract

The proposed method with new algorithm is presented to solve the linear Volterra integral equation (VIE) of convolution type by using the Z-transformation via converting the continuous-time integral equation to a discrete-time equation by using Euler's rule. The paper has useful properties of the Z-transformation. The results of the proposed method is compared with the Laguerre polynomials and good results are obtained. Four illustrative examples are given for conciliated the accuracy of the results of this proposed method.

Key words : Volterra integral equation, Convolution type, Z-Transform, Euler's rule.

1. Introduction

A large number of researchers and scientists published books that we devoted entirely to integral equation methods and their applications.

The advantage of the integral equation is witnessed by the increasing frequency of integral equations in the literature and in many fields, since more problems have their mathematical representation appear directly, and in a very natural way, in terms of integral equations. Other problems, whose direct representation is in terms of differential equations have their auxiliary conditions replaced by integral equations more elegantly than the differential equations [1].

The name integral equation was introduced by Bois-Reymond in 1888. However, the linear integral equation which is Volterra equation, was introduced by Volterra in 1884.

An integral equation is an equation in which the unknown function appears under the integral sign [1,2]. The general form of linear integral equation is [3]:

$$h(x)y(x) = f(x) + \int_a^b k(x,t)y(t)dt \qquad \ldots(1)$$

where $h(x)$ and $f(x)$ are known functions of x , $k(x,t)$ is called the kernel of the integral equation, a and b are the limits of integral either are given constants or functions of x and the function $y(x)$ which appears under the integral sign is to be determined. Integral equations can be classified into different kinds according to the limits of integral and the kernel. If a and b in eq.(1) are constants then equation (1) is called a Fredholm integral equation. If a in eq.(1) is a constant while (b = x), eq.(1) is called a *Volterra integral equation* [3,4]. Volterra integral equation was treated numerically using numerical methods as Laguerre polynomials [5] and Taylor series method [2]. On the other side the integral transformations are very useful for solving many kinds of Volterra integral equations. The most popular transformations are Laplace transformation and Mellin transformation [1,6].

Analytic solutions of the integral equations is not easily obtained by the classical methods [3,4,7], for examples :

- $y(x) = x + \int_0^x y(t)\, J_1(x-t)dt$

  where $J_1(x)$ is the Bessel function of the first kind of order one [4,6].

- $y(x) = -2xe^{-x} + \int_0^x y(t)y(x-t)dt$

- $y(x) = \frac{1}{2}\sin 2x + \int_0^x y(t)y(x-t)dt$

So we need some approximated methods which can solve these Volterra convolution equations.

In this work, an approximated solution of the linear Volterra integral equation of the first and second kinds of convolution type is presented using Z-transformation via discretization of continuous-time integral equation using Euler's rule. Some of definitions and categories of linear integral equations are given in the following section.

2. Classification of Linear Integral Equations :

Integral equations can be classified into different kinds according to the limits of integral and the kernel. Some definitions and a preliminary classification of linear integral equations are introduced [1,3,6].

Definition (1):

If $f(x) \equiv 0$ then eq.(1) is called homogeneous equation, otherwise it is called non-homogeneous.

Definition (2):

The integral equation (1) is called an integral equation of the first kind when $h(x) \equiv 0$, which is :

$$f(x) = \int_a^b k(x,t)y(t)dt \qquad \ldots(2)$$

where $f(x)$ and $k(x,t)$ are known functions.

Definition (3):

The integral equation (1) is called an integral equation of the second kind when $h(x) \equiv 1$, which is :

$$y(x) = f(x) + \int_a^b k(x,t)y(t)dt \qquad \ldots(3)$$

where $f(x)$ and $k(x,t)$ are known functions.

Definition (4):

The integral equations (2) and (3) are called a *Volterra integral equation*s when their upper limits are variable (i.e. b = x). Hence the integral equations :

$$f(x) = \int_a^x k(x,t)y(t)dt \qquad \ldots(4)$$

$$y(x) = f(x) + \int_a^x k(x,t)y(t)dt \qquad \ldots(5)$$

represent Volterra integral equations of the first and the second kinds respectively.

Definition (5):

If the kernel $k(x,t)$ in eq.(1) depends only on the difference $x-t$, such a kernel is called a difference kernel and the eq.(1) with this kind of kernel is called an integral equation of convolution type :

$$h(x)y(x) = f(x) + \int_a^b k(x-t)y(t)dt \qquad \ldots(6)$$

The integral equation of convolution type is an important integral in many applications. Convolution can be found in various places in applied mathematics since it plays an important role in heat conduction, wave motion and time series analysis [3,7].

The convolution of two functions is a way of combining them together. The convolution of the functions $k(x)$ and $g(x)$ is: $\int_a^b k(x-t)g(t)dt$.

Hence, the integral equations :

$$f(x) = \int_0^x k(x-t)y(t)dt$$

…(7)

$$y(x) = f(x) + \int_0^x k(x-t)y(t)dt \qquad \ldots(8)$$

represent Volterra integral equations of convolution type of the first and the second kinds respectively.

<u>3. The Z-Transformation:</u>

The Z-transform is used to transfer sequences of numbers into algebraic equations which, in many cases, help the solution of problems. It is a rule by which a sequence of numbers is converted into a function of the transform variable (z).

The Z-transform of a sequence of numbers $\{f(k)\}$ which is identically zero for negative discrete time (i.e. f (k) = 0 for k = -1,-2,-3,…) is defined by:

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$$

where z is an arbitrary complex variable [8,9].

In the values of the signal $f(t)$ at the sampling instants (i.e. the values of $f(t)$ at $t = kT$ $(k = 0,1,2,...)$) the Z-transform of $f(t)$ is [7,10]:

$$Z\{f(t)\} = Z\{f(kT)\} = F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} \qquad \ldots(9)$$

where $f(t) = 0$ at $t < 0$, T is the sample period and k is a discrete-time, $k = 0,1,2,\ldots.$

Some examples of the Z-transform of discrete signals are given in table (1) [9,10]:

Table (1) Table of commonly used Z-transform		
	$f(kT) \quad ,k \geq 0$	$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}$
1	1	$\frac{z}{(z-1)}$
2	kT	$\frac{Tz}{(z-1)^2}$
3	$(kT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$
4	$(kT)^3$	$\frac{T^3 z(z^2+4z+1)}{(z-1)^4}$
5	e^{-akT}	$\frac{z}{z-e^{-aT}}$
6	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
7	$(kT)^2 e^{-akT}$	$\frac{zT^2 e^{-aT}(z+e^{-aT})}{(z-e^{-aT})^3}$
8	$\sin(kwT)$	$\frac{z\sin(wT)}{(z^2-2z\cos(wT)+1)}$
9	$\cos(kwT)$	$\frac{z(z-\cos(wT))}{(z^2-2z\cos(wT)+1)}$

The Z-transform possesses many important properties. These properties will be proved to be useful in the analysis of discrete systems [8,9].

a) Linearity Property :

If f_1 (k) and f_2 (k) are two discrete signals have Z-transform F_1(z) and F_2(z) respectively, then :

$$Z\{a f_1(k) + b f_2(k)\} = a F_1(z) + b F_2(z) \quad , \qquad k = 0,1,2,\dots$$

where a and b are constants .

Proof :

From the definition of the Z-transform :

$$Z\{af_1(k)+bf_2(k)\}=\sum_{k=0}^{\infty}\{af_1(k)+bf_2(k)\}z^{-k}$$

$$= a\sum_{k=0}^{\infty} f_1(k)z^{-k} + b\sum_{k=0}^{\infty} f_2(k)z^{-k}$$

$$= a F_1(z) + b F_2(z)$$

b) Right-Shifting Property :

Let m be a positive integer and let f (k) be a sequence which is zero for (k < 0) . Further, let F (z) be the Z-transform of f (k) , then

$$Z\{ f(k-m) \}= z^{-m} F(z) \quad , k = 0,1,2,\dots$$

Proof :

From the definition of the Z-transform :

$$Z\{ f(k-m)\} = \sum_{k=0}^{\infty} f(k-m)z^{-k}$$

$$= f(-m) + f(1-m)z^{-1} +\dots+ f(0)\, z^{-m} + f(1)\, z^{-(m+1)} + \dots$$

$$= z^{-m} [\, f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} +\dots],$$

$(f(k) = 0 \;\; for\; k < 0)$ $\qquad = z^{-m} F(z).$

c) Left-Shifting Property :

Let m be a positive integer and let f (k) be a sequence which is zero for (k < 0). Further, let F (z) be the Z-transform of f (k) , then

$$Z\{f(k+m)\} = z^m F(z) - \sum_{i=0}^{m-1} f(i) z^{m-i} \quad , \text{k} = 0,1,2,\ldots$$

Proof :

From the definition of the Z-transform :

$$Z\{f(k+m)\} = \sum_{k=0}^{\infty} f(k+m) z^{-k} = f(m) + f(m+1)z^{-1} + f(m+2)z^{-2} + \ldots$$

By adding and subtracting terms, we obtain :

$$Z\{f(k+m)\} = z^m[f(0) + f(1)z^{-1} + \ldots + f(m)z^{-m} + f(m+1)z^{-(m+1)} + \ldots$$
$$- f(0) - f(1)z^{-1} - \ldots - f(m-1)z^{-(m-1)}]$$

or

$$Z\{f(k+m)\} = z^m F(z) - \sum_{i=0}^{m-1} f(i) z^{m-i} \quad .$$

Table (2) lists the notable properties enjoyed by the Z-transform

Table (2) Properties of the Z-transform

	Property	Discrete Sequence	Z-Transform
1	Linearity	$af(k)+bg(k)$	$aF(z)+bG(z)$
2	Right-Shifting	$f(k-m)$	$z^{-m}F(z)$
3	Convolution	$\sum_{i=0}^{k} f_1(k-i) f_2(i)$	$F_1(z)F_2(z)$
4	Periodic-Sequence	$f(k) = f(k+N)$	$F(z) = \frac{z^N}{z^N - 1} \sum_{k=0}^{N-1} f(k) z^{-k}$
5	Left-Shifting	$f(k+m)$	$z^m F(z) - \sum_{i=0}^{m-1} f(i) z^{m-i}$
6	Summation	$\sum_{i=0}^{k} f(i)$	$\frac{z}{z-1} F(z)$
7	Multiplication by	$a^k f(k)$	$F(a^{-1}z)$

	a^k		
8	Multiplication by k	$k\, f(k)$	$-z\dfrac{dF(z)}{dz}$

The Z-transform opens up new ways for solving the problems. It is well known that Z-transform is used successfully in many engineering problems. Some applications of Z-transform are [7,11] :

1. Solution of the linear difference equation.
2. Digital filter design.
3. Transfer function.

The Z-transform technique to be a feasible approach in the solution requires methods for determining the inverse Z-transform. Given F(z), there are two methods for obtaining the inverse Z-transform f (k) or f (kT), which will be given here. The two methods are the power series and the inversion integral. In obtaining the inverse Z-transform, we assume as usual that f (k) is zero for (k < 0) . Z^{-1}[F(z)] is denoted as the inverse Z-transform [9,11].

(1) Power Series Method :

The power series method for finding the inverse Z-transform of a function F(z) which is expressed as the ratio of two polynomials in z domain involves dividing the numerator of F(z) by the denominator such that a power series of the form

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots$$

…(10)

is obtained. From the definition of the Z-transform , it can be seen that the values of f (k) are simply the coefficients in the power series.

(2) Inversion-Integral Method :

The most general technique for obtaining the inverse of the Z-transform is the inversion integral. This integral is :

$$f(k) = \frac{1}{2pj}\oint_m F(z) z^{k-1} dz \quad , \quad j = \sqrt{-1}$$

…(11)

where μ is any closed curve which encloses all the poles of F(z) z^{k-1} and the poles of F(z) z^{k-1} are the values of z in the denominator which sets F(z) z^{k-1} to infinity [9,11].

The integral in Eq.(11) can be evaluated via the expression :

$$f(k)=\sum[\text{residues of } F(z)z^{k-1} \text{ at the poles of } F(z)z^{k-1}]$$

…(12)

If the function F(z) z^{k-1} has a simple pole at (z = a), the residue is evaluated as:

$$(residue)_{z=a} = \lim_{z\to a}\left((z-a)F(z)z^{k-1}\right)$$

…(13)

For a pole of order *m* at z = a [12], the residue is calculated using the expression :

$$(residue)_{z=a} = \lim_{z\to a}\left(\frac{1}{(m-1)!}\ \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m F(z)z^{k-1}]\right)$$

…(14)

4. Solution of Linear Volterra Integral Equations of Convolution Type Using Z-Transformation :

Consider the linear Volterra integral equation (VIE) of the convolution type of the form :

$$y(x)= f(x)+\int_0^x \mathrm{K}(x-t)y(t)dt \qquad \text{…(15)}$$

In order to obtain an approximated solution to eq.(15) by using Z-transformation, we first convert the continuous-time integral equation to a discrete-time equation. Integration in continuous-time is considered to be equivalent to summation in discrete-time, where the discrete-time signal is assumed to be generated by sampling a continuous-time signal. Hence, by Euler’s rule [7,12] we see that :

$$\int_{-\infty}^{x} g(u)du \Leftrightarrow \mathrm{T}\sum_{n=-\infty}^{k} g(n\mathrm{T}) \qquad \ldots(16)$$

where x at the sampling instants $x = x_k = k\,\mathrm{T}$, T is the sample period and k is a discrete-time, $k \geq 0$.

Hence, VIE in eq.(15) can be solved using Z-transform as follows :

By discretization the continuous-time of eq.(15) using Euler's rule, one can get the following :

$$\int_{0}^{x} \mathrm{K}(x-t)y(t)dt \to \mathrm{T}\sum_{m=0}^{k} \mathrm{K}(T(k-m))y(mT) \qquad \ldots(17)$$

The convolution integral is converted to a convolution summation in a discrete-time. For $x = x_k = k\,\mathrm{T}$, eq.(17) can be written as :

$$\int_{0}^{x} \mathrm{K}(x-t)y(t)dt \to \mathrm{T}\sum_{m=0}^{k} \mathrm{K}(k\mathrm{T}-m\mathrm{T})y(m\mathrm{T}) \qquad \ldots(18)$$

and the functions y(x) and f(x) in eq.(15) is converted to a discrete-time as follows:

$$y(x) \to y(k\mathrm{T}) \quad \textit{and} \quad f(x) \to f(k\mathrm{T})$$

where T is the sample period and k is a discrete-time, $k = 0,1,2,\ldots.$

Hence, VIE of the convolution type in eq.(15) becomes :

$$y(k\mathrm{T}) = f(k\mathrm{T}) + \mathrm{T}\sum_{m=0}^{k} \mathrm{K}(k\mathrm{T}-m\mathrm{T})y(m\mathrm{T}) \qquad \ldots(19)$$

The following algorithm summarizes the steps for finding the solution of the linear *VIE of the convolution type* by using *Z-transform*.

VIEC-ZT Algorithm

Step 1:

Convert the continuous-time of eq.(15) to a discrete-time as follows :

(a) Put $x = x_k = k\,\mathrm{T},\ k \ge 0$

(b) Convert $\int_0^x \mathrm{K}(x-t)\,y(t)\,dt \to \mathrm{T}\sum_{m=0}^{k} \mathrm{K}(k\mathrm{T}-m\mathrm{T})\,y(m\mathrm{T})$

(c) Convert $y(x) \to y(k\mathrm{T})$ *and* $f(x) \to f(k\mathrm{T})$.

Step 2:

Substitute (a), (b) and (c) of (step 1) into eq.(15) to obtain :

$$y(k\mathrm{T}) = f(k\mathrm{T}) + \mathrm{T}\sum_{m=0}^{k} \mathrm{K}(k\mathrm{T}-m\mathrm{T})\,y(m\mathrm{T})$$

Step 3:

Use eq.(9) and the convolution property of the Z-transform in table(2) for taking the Z-transform to both sides to eq.(19) to get :

$$Y(z) = F(z) + T\,K(z)Y(z)$$

Step 4:

Take the inverse Z-transform to both sides to the equation in step (3) to find y(kT).

Step 5:

For all $k = 0,1,2,\ldots,$n compute y(kT), where

$y(k\mathrm{T}) = y(x_k) = y(x_0), y(x_1), \ldots, y(x_n)$, n is the number of knots and T is the sample period.

5. Illustrative Examples :

Example (1) :

Consider the following linear VIE of convolution type of the first kind :

$$\int_0^x (x-t)\,y(t)\,dt = x^3 \qquad\qquad 0 \le x \le 1$$

which has the exact solution [7]: $y(x) = 6x$.

In this example the Z-transform is used to solve the above VIE. Hence, by applying the algorithm (VIEC-ZT) we get :

$$\int_0^x (x-t)y(t)dt = x^3 \rightarrow \mathrm{T}\sum_{m=0}^{k}(k\mathrm{T}-m\mathrm{T})y(m\mathrm{T}) = (k\mathrm{T})^3$$

Taking the Z-transform to both sides using eq.(9) and the tables (1) and (2) yields :

$$T\left[\frac{zT}{(z-1)^2}Y(z)\right] = \frac{T^3 z(z^2+4z+1)}{(z-1)^4}$$

$$\therefore Y(z) = \frac{T(z^2+4z+1)}{(z-1)^2}$$

Then, taking the inverse Z-transform to both sides using eq.(14) gives :

$$y(kT) = 6kT$$

Table (3) shows the comparison between the exact solution and the approximated solution by using VIEC-ZT algorithm when T=0.1 , $x = x_k = k\mathrm{T}$ and k = 0,1,2,…,10 depending on the least square error (L.S.E.). For the comparison of computing accuracy depending on L.S.E., the solution obtained by using the Laguerre polynomials [5] is also tabulated.

Table (3) The solution of example(1)

x	Exact solution	The Z-Transform	Laguerre polynomials [5]
0	0	0	0.0000
٠.1	0.6000	0.6000	0.6000
٠.2	1.2000	1.2000	1.2000
٠.3	1.8000	1.8000	1.8000
٠.4	2.4000	2.4000	2.4000
٠.5	3.0000	3.0000	3.0000
٠.6	3.6000	3.6000	3.6001
٠.7	4.2000	4.2000	4.2001
٠.8	4.8000	4.8000	4.8002
٠.9	5.4000	5.4000	5.4002

1	6.0000	6.0000	6.0001
L.S.E.		0.0000000	0.0000012

<u>Example (2) :</u>

Consider the following linear VIE of convolution type of the second kind :

$$y(x) = x^2 + \int_0^x \sin(x-t)\, y(t)\,dt \qquad 0 \le x \le 1$$

which has the exact solution : $y(x) = x^2 + \dfrac{x^4}{12}$.

This problem is solved by using the Z-transform. Hence, by applying (VIEC-ZT) algorithm we get the following results:

$$y(x) = x^2 + \int_0^x \sin(x-t)\, y(t)\,dt \rightarrow y(kT) = (kT)^2 + T\sum_{m=0}^{k} \sin(kT - mT)\, y(mT)$$

where $x = x_k = k\,\mathrm{T}$.

Taking the Z-transform to both sides using eq.(9) and the tables (1) and (2) yields :

$$Y(z) = \frac{z(z+1)T^2}{(z-1)^3} + T\frac{z\sin T}{(z^2 - 2z\cos T + 1)} Y(z)$$

$$\therefore Y(z) = \frac{T^2 z(z+1)(z^2 - 2z\cos T + 1)}{(z-1)^3 (z^2 - z(2\cos T + T\sin T) + 1)}$$

Then, taking the inverse Z-transform to both sides using eq.(13) and eq.(14) gives :

$$y(kT) = \frac{2T^2(k^2 a(\cos T - 1) + 2k^2(1-\cos T) - a + 2\cos T)}{(2-a)^2} + T^2 C_1 + T^2 C_2$$

where,

$$C_1 = \frac{\left(\frac{a+\sqrt{a^2-4}}{2}\right)^k \left(\frac{a+\sqrt{a^2-4}}{2}+1\right)\left[\frac{(a+\sqrt{a^2-4})^2}{4}-(a+\sqrt{a^2-4})\cos T+1\right]}{\sqrt{a^2-4}\left(\frac{a+\sqrt{a^2-4}}{2}-1\right)^3},$$

$$C_2 = \frac{\left(\frac{a-\sqrt{a^2-4}}{2}\right)^k \left(\frac{a-\sqrt{a^2-4}}{2}+1\right)\left[\frac{(a-\sqrt{a^2-4})^2}{4}-(a-\sqrt{a^2-4})\cos T+1\right]}{-\sqrt{a^2-4}\left(\frac{a-\sqrt{a^2-4}}{2}-1\right)^3}$$

and

$$a = 2\cos T + T\sin T \,.$$

Table (4) shows the comparison between the exact and the approximated solution by using VIEC-ZT algorithm depends on the least square error (L.S.E.) when T=0.1, $x = x_k = k\,\mathrm{T}$ and k = 0,1,2,…,10. Again, the results obtained by the Laguerre polynomials [5] is also listed in table (4) for comparison with the exact solution by depending on L.S.E.

Table (4) The solution of example (2)

x	Exact solution	The Z-Transform	Laguerre polynomials [5]
0	0	0.0000	0.0000
٠.1	٠.٠١٠٠	٠.٠١٠٠	٠.٠١٠٠
٠.2	٠.٠٤٠١	٠.٠٤٠١	٠.٠٤٠١
٠.3	٠.٠٩٠٧	٠.٠٩٠٦	٠.٠٩٠٦

٠.4	٠.١٦٢١	٠.١٦٢٠	٠.١٦٢١
٠.5	٠.٢٥٥٢	٠.٢٥٥١	٠.٢٥٥١
٠.6	٠.٣٧٠٨	٠.٣٧٠٦	٠.٣٧٠٧
٠.7	٠.٥١٠٠	٠.٥٠٩٦	٠.٥٠٩١
٠.8	٠.٦٧٤١	٠.٦٧٤٠	٠.٦٧٤٠
٠.9	٠.٨٦٤٧	0.8٦٤٦	٠.٨٦٤٦
1	١.٠٨٣٣	١.٠٨٣٣	١.٠٨٣٢
L.S.E.		٠.٠٠٠٠٠١٢	0.000182

Figure (1) shows the solution of the VIE of the convolution type which was given in example (2) by using Z-transform (VIEC-ZT algorithm) with the exact solution and the Laguerre polynomials.

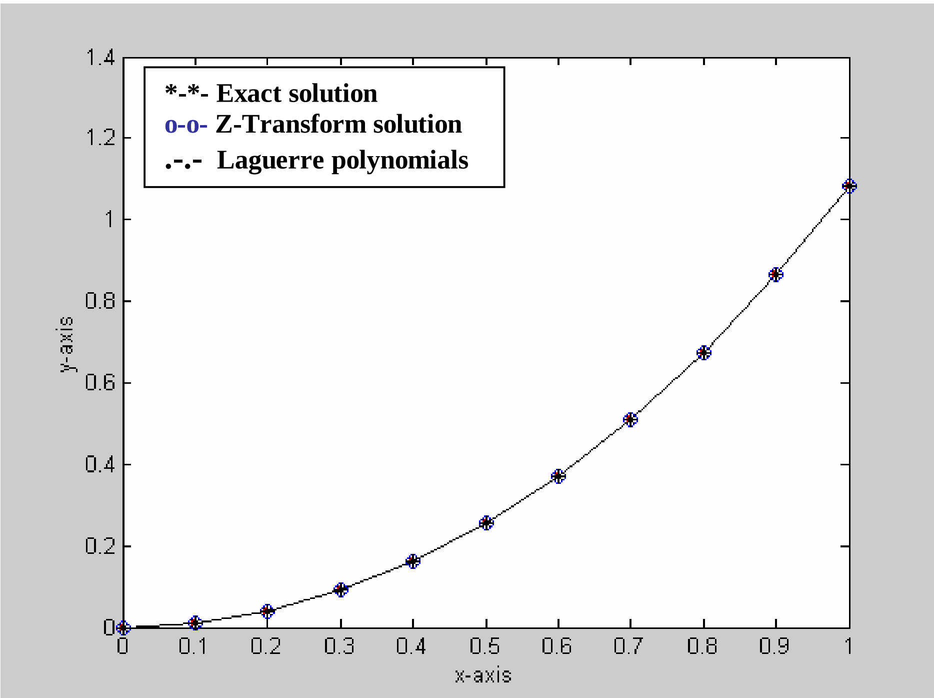

Fig.(1) The comparison between the exact solution, Z-transform and Laguerre polynomials for Volterra integral of convolution type.

The approximated solution by using Z-transform (VIEC-ZT algorithm) is good when taking T small. Different values of T and the corresponding L.S.E. coming from our choices are listed in table (5) where $0 \le x \le 1$.

Table(5) The L.S.E. of Ex.(2) when T=0.05 and 0.01

The Z-Transform solution (VIEC-ZT algorithm)		
T	k	L.S.E.
0.05	0,1,…,20	0.0000011
0.01	0,1,…,100	0.000000002

Example (3) :

Consider the following linear VIE of convolution type of the second kind :

$$y(x) = x + \int_0^x y(t)\ J_1(x-t)\,dt \qquad\qquad 0 \le x \le 0.1$$

which has the exact solution [4,7]:

$$y(x) = \tfrac{1}{2}(x^2+1)\int_0^x J_0(t)dt + \tfrac{1}{2}xJ_0(x) - \tfrac{1}{2}x^2 J_1(x)$$

where $J_0(x)$ is the Bessel function of the first kind of order zero [4] and $J_1(x)$ is the Bessel function of the first kind of order one [6]. For the Bessel function see Ref.'s [4,6,7].

In this example the Z-transform is used to solve this VIE. Hence, by applying the algorithm (VIEC-ZT) we get :

$$y(x) = x + \int_0^x y(t)\ J_1(x-t)\,dt \rightarrow y(kT) = kT + T\sum_{m=0}^{k} y(mT)J_1(kT-mT)$$

Taking the Z-transform to both sides using eq.(9) and the tables (1) and (2) yields :

$$Y(z)=\frac{Tz}{(z-1)^2}+T[Y(z)J_1(z)]$$

$$\therefore Y(z)=\frac{Tz}{(z-1)^2(1-TJ_1(z))}$$

Then, taking the inverse Z-transform to both sides using eq.(14) gives :

$$y(kT)=\frac{kT(1-TJ_1(kT))-T^2kJ_1(kT)}{(1-TJ_1(kT))^2}$$

Table (6) shows the comparison between the exact and the approximated solution by using VIEC-ZT algorithm depends on least square error (L.S.E.) when T=0.01 , $x=x_k=k\mathrm{T}$ and k = 0,1,2,…,10.

Table (6) The solution of example(3) using VIEC-ZT algorithm

x	The Z-Transform	Exact
0	0	0
٠.01	٠.0100	٠.0100
٠.02	٠.0200	٠.0200
٠.03	٠.0300	٠.0300
٠.04	٠.0400	٠.0400
٠.05	٠.0500	٠.0500
٠.06	٠.0600	٠.0600
٠.07	٠.0700	٠.0700
٠.08	٠.0800	٠.0800
٠.09	٠.0900	٠.0901
0.1	0.1000	0.1001

L.S.E.	0.0000000068	

The approximated solution by using Z-transform (VIEC-ZT algorithm) is good when taking T small. Different values of T and the corresponding L.S.E. coming from our choices are listed in table (7).

Table(7) The L.S.E. of Ex.(3) when T=0.001 and 0.0001

The Z-Transform solution(VIEC-ZT algorithm)	
T	L.S.E.
0.001	0.679e-15
0.0001	0.68e-20

<u>Example (4)</u> :

Consider the following VIE of convolution type of second kind :

$$y(x) = -2xe^{-x} + \int_0^x y(t)y(x-t)dt \qquad 0 \le x \le 1$$

which has the exact solution [3]: $y(x) = \mathbf{m}\sqrt{2}\,e^{-x}$.

In this example the Z-transform is used to solve the above VIE. Hence, by applying the algorithm (VIEC-ZT) we get :

$$y(kT) = -2(kT)e^{-kT} + T\sum_{m=0}^{k} y(kT)y(kT - mT)$$

Taking the Z-transform to both sides using eq.(9) and the tables (1) and (2) yields :

$$Y(z) = -2\frac{zTe^{-T}}{(z-e^{-T})^2} + T\left(Y(z)\right)^2$$

$$\therefore Y(z) = \frac{\frac{1}{T} \mp \sqrt{\frac{1}{T^2} + \frac{8ze^{-T}}{(z-e^{-T})^2}}}{2}$$

Then, taking the inverse Z-transform to both sides using eq.(13) gives :

$$y(kT) = \mp\sqrt{2}\, e^{-kT}$$

Table (8) shows the comparison between the exact and the approximated solution by using VIEC-ZT algorithm depending on the least square error (L.S.E.), when T=0.1 , $x = x_k = k\,\mathrm{T}$ and k = 0,1,2,…,10 .

Table (8) The solution of example(4)

X	$Exact_1$	The Z-Transform	$Exact_2$	The Z-Transform
0	1.4142	1.4142	-1.4142	-1.4142
٠.1	1.2796	1.2796	-1.2796	-1.2796
٠.2	1.1579	1.1579	-1.1579	-1.1579
٠.3	1.0477	1.0477	-1.0477	-1.0477
٠.4	0.9480	0.9480	-0.9480	-0.9480
٠.5	0.8578	0.8578	-0.8578	-0.8578
٠.6	0.776	0.7761	-0.7761	-0.7761

	1			
٠.7	0.7023	0.7023	-0.7023	-0.7023
٠.8	0.6354	0.6354	-0.6354	-0.6354
٠.9	0.5750	0.5750	-0.5750	-0.5750
1	0.5203	0.5203	-0.5203	-0.5203
L.S.E.		0.0000	L.S.E.	0.0000

6. Conclusion :

Z-Transform method has been presented for solving linear Volterra integral equation of convolution type. It has been shown that the proposed method is comparable in accuracy with Laguerre polynomials [5]. The results show a marked improvement in the least square errors (L.S.E.). From solving some test examples the following points are included :

1- Z-Transform solves the linear VIE of the convolution type by converting the continuous-time integral equation to a discrete-time equation by using Euler's rule.
2- The Z-transform method gives a better accuracy and consistent than Laguerre polynomials for solving VIE of convolution type.
3- The good approximation depends on the size of T, if T is decreased then the number of points (knots) increases and the L.S.E. approaches to zero.

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