The asymptotic normality for the simulation method of the repeated measures model

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Abstract

Using the simulation method is important for evaluating estimator properties, and through asymptotic normality in simulation, we can approximate estimator distributions to make important statistical inferences and informed statistical decisions. When estimating parameters of any statistical model, we search for estimators that are both unbiased and efficient, which is the main focus of our research. Although many methods like the maximum likelihood technique are effective in estimating the parameters of the repeated measures model, the power of these methods is limited by estimating bias variance in the model's random components .This study aims to address the limitations of current methods by minimizing the bias in variance estimations of a repeated measures model. We utilize the maximum likelihood approach (mean bias reducing method) along with the simulation technique to analyze the performance of the new estimator by verifying its asymptotic normality. This study aims to address the limitations of current methods by minimizing the bias in variance estimations of a repeated measures model. We utilize the maximum likelihood approach (mean bias reducing method) along with the simulation technique to analyze the performance of the new estimator by verifying its asymptotic normality. As a consequence, the new estimator was normality convergent.

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1. INTRODUCTION

Over the last few decades, the high-dimensional data revolution has impacted various aspects of everyday life and business operations including artificial intelligence algorithms. A significant amount of research has focused on analyzing big data and extracting value from it (hidden patterns and insights) [1].
Data modeling is a crucial tool for data analysis, as it represents data using mathematical models to meet specific conditions based on the data's state [2]. Most big data can be formulated as repeated measurement models, so we focus on studying it from all aspects [3].
A repeated measures model is a statistical technique for data collected at different times or under different conditions on the same experimental unit [4]. Several factors make these models more significant than other models for statistical analyses [5]. For instance, they work with data linked to the experimental unit's measurements and examine variance within units to obtain more precise and effective estimates of the effects on the experimental units. Furthermore, it enables the examination of how experimental conditions change over time, such as the progression of a disease, or the development of a skill [6].

One of the main challenges in repeated measurement models is obtaining biased estimates for the variance components, which can influence the outcome of analysis and decision-making using the studied data [7], [8]. Hence, our research aimed to reduce bias in variance component estimates in the repeated-measures model by modifying the maximum likelihood method and conducting simulations of the modified method. To depend on the modified method's simulations, we need to prove its asymptotic normality.
This study consisted of three primary elements. The initial step involves creating a mathematical model (known as a repeated measures model) with two normally distributed random effects and specifying the conditions of the two fixed factors. The second aspect is (a mean bias reducing method) for estimating the variance in the repeated measures model, by adjusting the score function of the maximum likelihood method. The final part of the study focuses on the theoretical aspect of investigating the asymptotic normality of the simulation method in order to identify composite estimators with reduced bias in variance.

2. DESCRIPTION OF THE MODEL

In this section, we define the one-way repeated measures model by (1). Our model includes two random effects and two fixed effects (between units and within units).

$$y_{iqd} = \gamma + A_q + B_q + (AB)_{qd} + \nu_{1i[q]} + \nu_{2i[d]} + \varepsilon_{iqd} \tag{1}$$

Model parameters can be clarified as follows: $i = 1,2,3,\dots,a$ refers to the index of experimentation units, $q = 1,2,3,\dots,b$ refers to the index of the group effect (between units), $d = 1,2,3,\dots,c$ refers to the index of the time effect (within-unit), y_{iqd} refers to measuring observations in the experiment, ε_{iqd} refers to the vector of random error and γ refers to the vector of the overall mean.
To classify the model's effects, each effect according to its type and conditions added to it, note Table (1).

Table 1. Describe the effects of our study model

The effect	The type	The classification	The condition
$\boldsymbol{A_j}$	Fixed vector	(Group) Between-unit effect	$\sum_{q=1}^{b} A_q = 0$
$\boldsymbol{B_d}$	Fixed vector	(Time) Within-unit effect	$\sum_{d=1}^{c} B_d = 0$
$\boldsymbol{(AB)_{qd}}$	Fixed vector	Effect (between X within	$\sum_{q=1}^{b} A_q B_d = 0$ $\sum_{d=1}^{c} A_q B_d = 0$
$\boldsymbol{\nu_{1i(q)}}$	Random vector	The effect of group (q) on unit i	$\nu_{1i(q)} \sim N(0,\sigma_{\nu_1}^2)$
$\boldsymbol{\nu_{2i(d)}}$	Random vector	The effect of time (d) on unit i	$\nu_{2i(d)} \sim N(0,\sigma_{\nu_2}^2)$
$\boldsymbol{\varepsilon_{iqd}}$	Random vector	The random vector of error	$\varepsilon_{iqd} \sim N(0,\sigma_{\varepsilon}^2)$

We write the study model in the matrices form, so there are many transformations, as the following [9], [10], [11]. First, we write (1) as in (2):

$$y_{iq} = \alpha_j + 1_c\psi_{i(q)} + 1_b\eta_i + e_{iq} \tag{2}$$

Where $y_{iq} = [y_{iq1,}, \dots, y_{iqC}]'$ is the vector of responses, $\alpha_q = [\alpha_{q1,}, \dots, \alpha_{qC}]'$ is the vector effects of the fixed parameters, $\psi_{i(q)} = \nu_{1i(q)}$ is the vector effect of the random parameter for unit i within the group (q), $\eta_i = [\eta_{i(1),}, \dots, \eta_{i(C)}]'$ is the vector effect of the random parameter for unit i within time ($\nu_{2i(k)}$) and $\varepsilon_{ij} = [\varepsilon_{ij1,}, \dots, \varepsilon_{ijC}]'$ is the vector of random error.
Note that (1) is the unit vector and (I) is the identity matrix and ⊗ is the Kronecker product [12], [13]. Let:

$$\xi_{ij} = \begin{cases} 1, & unit\ i\ is\ from\ btween-unit\ \ q \\ 0, & o.w. \end{cases}$$

By setting $\Omega_i' = [\alpha_{i1,}, \dots, \alpha_{in_2}]'$, we can rewrite (2) as in (3) such that:

$$Y_i = X_i\,\alpha + Z_iV + \varepsilon_i \tag{3}$$

Where:

$$Y_i' = [Y_{i1}, \cdots, Y_{iC}]$$

$$X_I = I_C \otimes \xi$$

$$\alpha' = [\alpha_{11}, \dots, \alpha_{bc}]$$

$$Z_i = 1_c \otimes 1_b$$

$$V = \left[\nu_{1i(j)}, \nu_{2i(k)}\right]$$

$$\varepsilon_i' = [\varepsilon_{i1}, \cdots, \varepsilon_{iC}]$$

Set α is Vec $\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1c} \\ \vdots & \cdots & \vdots \\ \alpha_{b1} & \cdots & \alpha_{bc} \end{bmatrix}$, (4) is the last form for our model as shown in the following:

$$Y = X\alpha + ZV \tag{4}$$

Where:

$$Y' = [Y_1', \dots, Y_a']$$

$$X' = [X_1, \dots, X_a]'$$

$$V = \left[\nu_{1i(q)}, \nu_{2i(d)}, \varepsilon_i\right]$$

$$Z = [z_1', \dots, z_a']$$

The basic idea of this work in a one-way repeated measures model is finding the estimators for the variance of random parameters. Therefore, in (5) we display the form of the variance matrix Σ of the model.

$$\Sigma = \begin{bmatrix} \sum_{\nu_1} + \sum_{\nu_2} + \sum_{\varepsilon} & 0 & \cdots & 0 \\ 0 & \sum_{\nu_1} + \sum_{\nu_2} + \sum_{\varepsilon} & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sum_{\nu_1} + \sum_{\nu_2} + \sum_{\varepsilon} \end{bmatrix} \tag{5}$$

Where J is a matrix of ones., $\sum_{\boldsymbol{\nu}_1} = \sigma^2_{\boldsymbol{\nu}_1} [I_a \otimes I_b \otimes J_c]$, $\sum_{\boldsymbol{\nu}_2} = \sigma^2_{\boldsymbol{\nu}_2}[I_a \otimes J_b \otimes I_c]$ and

$$\sum_{\varepsilon} = \sigma^2_{\varepsilon} [I_a \otimes I_b \otimes I_c].$$

3. MEAN BIAS REDACTING METHOD

In this section, the maximum likelihood method is used to estimate the variance components of our model (4). For the responses $Y \sim N(X\,\alpha, \sum)$ derivative of the maximum likelihood function F with respect to

$$\sigma = \left(\sigma^2_{\boldsymbol{\nu}_1}, \sigma^2_{\boldsymbol{\nu}_2}, \sigma^2_e\right) = \sigma_r, r = 0,1,2.$$

We solve (6) using the numerical method (Newton-Raphson) to obtain the estimators' variance [14], [15].

$$F_{\sigma_r} = \begin{bmatrix} \frac{1}{2}\left[(Y - X\,\alpha)'\,\Sigma^{-1}\,\Sigma_{\sigma_0}\,\Sigma^{-1}(Y - X\,\alpha) - T_r\left(\Sigma^{-1}\Sigma_{\sigma_0}\right)\right] \\ \frac{1}{2}\left[(Y - X\,\alpha)'\,\Sigma^{-1}\,\Sigma_{\sigma_1}\,\Sigma^{-1}(Y - X\,\alpha) - T_r\left(\Sigma^{-1}\Sigma_{\sigma_1}\right)\right] \\ \frac{1}{2}\left[(Y - X\,\alpha)'\,\Sigma^{-1}\,\Sigma_{\sigma_2}\,\Sigma^{-1}(Y - X\,\alpha) - T_r\left(\Sigma^{-1}\Sigma_{\sigma_2}\right)\right] \end{bmatrix} = \begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} \tag{6}$$

We noticed that the estimations generated by this technique have a bias. To reduce bias, we adjusted the score function in the maximum likelihood method by subtracting the average bias from F, resulting in a new score function (7).

$$F_{\tilde{\sigma}} = \begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} + \frac{1}{2} Tr \left[\begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1} \left[\frac{1}{2} Tr \left[\Sigma^{-1} \begin{bmatrix} \frac{\partial^2 \Sigma}{\partial\sigma_0 \partial\sigma_0} & \frac{\partial^2 \Sigma}{\partial\sigma_0 \partial\sigma_1} & \frac{\partial^2 \Sigma}{\partial\sigma_0 \partial\sigma_2} \\ \frac{\partial^2 \Sigma}{\partial\sigma_1 \partial\sigma_0} & \frac{\partial^2 \Sigma}{\partial\sigma_1 \partial\sigma_1} & \frac{\partial^2 \Sigma}{\partial\sigma_1 \partial\sigma_2} \\ \frac{\partial^2 \Sigma}{\partial\sigma_2 \partial\sigma_0} & \frac{\partial^2 \Sigma}{\partial\sigma_2 \partial\sigma_1} & \frac{\partial^2 \Sigma}{\partial\sigma_2 \partial\sigma_2} \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \frac{\partial \Sigma}{\partial \sigma_0} \\ \frac{\partial \Sigma}{\partial \sigma_1} \\ \frac{\partial \Sigma}{\partial \sigma_2} \end{bmatrix} \right] \right] \right] \tag{7}$$

Where:

$$E(\sigma) = \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}$$

$$E_{01} = -\frac{1}{2}\, Tr\,)(\Sigma^{-1}\Sigma_1\Sigma^{-1}\Sigma_0 - \Sigma^{-1}\Sigma_0\Sigma^{-1}\Sigma_1)$$

$$E_{02} = -\frac{1}{2}\, Tr\, (\Sigma^{-1}\Sigma_2\Sigma^{-1}\Sigma_0 - \Sigma^{-1}\Sigma_0\Sigma^{-1}\Sigma_2)$$

$$E_{10} = -\frac{1}{2}\, Tr\, (\Sigma^{-1}\Sigma_0\Sigma^{-1}\Sigma_1 - \Sigma^{-1}\Sigma_1\Sigma^{-1}\Sigma_0)$$

$$E_{12} = -\frac{1}{2}\, Tr\, (\Sigma^{-1}\Sigma_2\Sigma^{-1}\Sigma_1 - \Sigma^{-1}\Sigma_1\Sigma^{-1}\Sigma_2)$$

$$E_{20} = -\frac{1}{2}\, Tr\, (\Sigma^{-1}\Sigma_0\Sigma^{-1}\Sigma_2 - \Sigma^{-1}\Sigma_2\Sigma^{-1}\Sigma_0)$$

$$E_{21} = -\frac{1}{2}\, Tr\, (\Sigma^{-1}\Sigma_1\Sigma^{-1}\Sigma_2 - \Sigma^{-1}\Sigma_2\Sigma^{-1}\Sigma_1)$$

Reapplying the Newton-Raphson method to (7) in a repeated measures model, we determine, the estimator's variance $\tilde{\sigma}$. Equation (8) represents the bias in the mean bias reducing method:

$$M = Tr \left[\Sigma^{-1} \begin{bmatrix} \frac{\partial^2 \Sigma}{\partial\sigma_0 \partial\sigma_0} & \frac{\partial^2 \Sigma}{\partial\sigma_0 \partial\sigma_1} & \frac{\partial^2 \Sigma}{\partial\sigma_0 \partial\sigma_2} \\ \frac{\partial^2 \Sigma}{\partial\sigma_1 \partial\sigma_0} & \frac{\partial^2 \Sigma}{\partial\sigma_1 \partial\sigma_1} & \frac{\partial^2 \Sigma}{\partial\sigma_1 \partial\sigma_2} \\ \frac{\partial^2 \Sigma}{\partial\sigma_2 \partial\sigma_0} & \frac{\partial^2 \Sigma}{\partial\sigma_2 \partial\sigma_1} & \frac{\partial^2 \Sigma}{\partial\sigma_2 \partial\sigma_2} \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \frac{\partial \Sigma}{\partial \Theta_0} \\ \frac{\partial \Sigma}{\partial \Theta_1} \\ \frac{\partial \Sigma}{\partial \Theta_2} \end{bmatrix} \right] \tag{8}$$

4. SIMULATION METHOD BASED ON MEAN BIAS REDACTING

Our objective is to minimize the bias in the variance estimator when using the repeated measures model. By modifying the score function for the maximum likelihood method, we will investigate the simulation method for bias amount M. In order to achieve the same objective, we need to prove the asymptotic normality of the simulation method to be able to apply it, especially when analytical solutions are challenging in repeated-measures models. Thus, the simulation method becomes the alternative solution for finding parameter estimates. We defined:

$$M^* = \frac{\sum_{\varkappa=1}^{G} \tilde{\sigma}_r(X_\varkappa)}{G} - \sigma$$

The solution of $F_\sigma(\varphi_\varkappa) = 0$ is $\tilde{\sigma}(\varphi_\varkappa)$ and $\varphi_\varkappa$ represents responses that our model simulates using σ, substituted $M^*(\tilde{\sigma})$ in (7), we opened a new simulated score function $F_{\tilde{\sigma}}{}^*$ based on a mean bias reducing method.

$$F_{\tilde{\sigma}}{}^* = \begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} + \frac{1}{2}\, Tr \left[\begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} G^{-1} \sum_{\varkappa=1}^{G} \tilde{\sigma}_0(\varphi_\varkappa) - \Theta \\ G^{-1} \sum_{\varkappa=1}^{G} \tilde{\sigma}_1(\varphi_\varkappa) - \Theta \\ G^{-1} \sum_{\varkappa=1}^{G} \tilde{\sigma}_2(\varphi_\varkappa) - \Theta \end{bmatrix} \right] \tag{9}$$

Theorem 4.1
The compact set $\omega \subset \Re^e$ of the parameters with the continuous functions F_σ and $E(\sigma)$ then $F_{\tilde{\sigma}}$ converges in probability to $F_{\tilde{\sigma}}{}^*$.

Proof:
Let

$$F_{\tilde{\sigma}}{}^* = G^{-1} \sum_{\varkappa=1}^{G} \left\{ \begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} - \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_0(\varphi_\varkappa) & \sigma_0 \\ \tilde{\sigma}_1(\varphi_\varkappa) - & \sigma_1 \\ \tilde{\sigma}_2(\varphi_\varkappa) & \sigma_2 \end{bmatrix} \right\}.$$

Set $\sigma \in \omega$ then

$$\begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} - \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_0(\varphi_\varkappa) & \sigma_0 \\ \tilde{\sigma}_1(\varphi_\varkappa) - & \sigma_1 \\ \tilde{\sigma}_2(\varphi_\varkappa) & \sigma_2 \end{bmatrix}$$

is continuous for every σ. Using a triangle inequality for

$$\mathcal{K} = \left\| \begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_0(\varphi_\varkappa) & \sigma_0 \\ \tilde{\sigma}_1(\varphi_\varkappa) - & \sigma_1 \\ \tilde{\sigma}_2(\varphi_\varkappa) & \sigma_2 \end{bmatrix} \right\|$$

such that:

$$\left\| \begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} - \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_0(\varphi_\varkappa) & \sigma_0 \\ \tilde{\sigma}_1(\varphi_\varkappa) - & \sigma_1 \\ \tilde{\sigma}_2(\varphi_\varkappa) & \sigma_2 \end{bmatrix} \right\| \leq \mathcal{K}.$$

Then

$$\begin{bmatrix} F_{\sigma_0} \\ F_{\sigma_1} \\ F_{\sigma_2} \end{bmatrix} - \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_0(\varphi_\varkappa) & \sigma_0 \\ \tilde{\sigma}_1(\varphi_\varkappa) - & \sigma_1 \\ \tilde{\sigma}_2(\varphi_\varkappa) & \sigma_2 \end{bmatrix}$$

bounded on ω. $\forall \acute{\delta} > 0, \exists \delta > 0$ and $\sigma_0, \sigma \subset \omega$, $|\sigma - \sigma_0| < \delta$ when $\dot{\omega}$ is partition of ω then

$$\|\dot{\omega}\| < \delta\, e^{-1/2}.$$

The function $\mathcal{K}$ is continuous in ω with σ_i, σ_{0i} in ω_i, then $\mathcal{K}(\sigma_i) = sup\, \mathcal{K}(\sigma)_{\sigma \in \omega_i}$ also $\mathcal{K}(\sigma_0) = inf\, \mathcal{K}(\sigma)_{\sigma \in \omega_i}$, since $\| \dot{\omega}\| < \delta e^{-1/2}$, $abc(\sigma_i - \sigma_{0i}) < \delta$ where $\sigma = \sigma_i$ and $\sigma_0 = \sigma_{0i}$.

$$\rightarrow |\mathcal{K}(\sigma) - \mathcal{K}(\sigma_0)| < \acute{\delta}$$

This proof of the function $\mathcal{K}$ is continuous in $\mathfrak{R}^e$.
From the (Heine Borel) theorem, we get ω is bounded and closed. In bounded integrable, ω is subset close. Known $F_{\tilde{\sigma}} = F_\sigma$ when $\tilde{\sigma} = \sigma$ with $M = 0$ then $F_{\tilde{\sigma}} \xrightarrow{P} F_\sigma$. Also $F_{\tilde{\sigma}}{}^* = F_\sigma$ as $G \to \infty$ then $F_{\tilde{\sigma}}{}^* \xrightarrow{P} F_\sigma$, such that:

$$norm\,(F_{\tilde{\sigma}} - F_{\tilde{\sigma}}{}^*) \le norm\,(F_{\tilde{\sigma}} - F_\sigma + F_\sigma - F_{\tilde{\sigma}}{}^*) \le norm(F_{\tilde{\sigma}} - F_\sigma) + norm(F_\sigma - F_{\tilde{\sigma}}{}^*)$$

$$Sup\,(norm(F_{\tilde{\sigma}} - F_{\tilde{\sigma}}{}^*)) \le sup\,(norm(F_{\tilde{\sigma}} - F_\sigma)) + sup\,(norm(F_{\tilde{\sigma}}{}^* - F_\sigma)) \to 0\; as\; G \to \infty\,.$$

Theorem 4.2
The compact set $\omega \subset \mathfrak{R}^e$ of the parameters with the continuous functions F_σ and $E(\sigma)$.

$\tilde{\sigma} \in \omega$ is a unique solution for $F_{\tilde{\sigma}}$ and $F_{\tilde{\sigma}}{}^* \xrightarrow{Uniform} F_{\tilde{\sigma}}$ where $G \to \infty$ then all $\tilde{\sigma}^* \in \omega \xrightarrow{P} \tilde{\sigma}$ if $F_{\tilde{\sigma}}{}^* = 0$.

Proof:
This proof is simple, by Theorem 4.1 we get $F_{\tilde{\sigma}}{}^* \xrightarrow{Uniform} F_{\tilde{\sigma}}$

Where $G \to \infty$. Not $\forall \varepsilon > 0, \exists\, \theta > 0$ such that $G > \theta$,

$$\varepsilon > sup_{\sigma \in \omega}\left(norm(F_{\tilde{\sigma}} - F_{\tilde{\sigma}}{}^*)\right) \ge \left(norm\left(F_{\tilde{\sigma}}(\tilde{\sigma}^*) - F_{\tilde{\sigma}}{}^*(\tilde{\sigma}^*)\right)\right) = (norm(F_{\tilde{\sigma}}{}^*(\tilde{\sigma}^*)))$$

Then $\tilde{\sigma}^*$ converge to unique $\tilde{\sigma}$.

5. AN ASYMPTOTIC FOR THE ESTIMATOR $\tilde{\sigma}^*$

One of the main aspects of statistics is asymptotic, which is used to understand the behavior of parameter estimates in mathematical models. In this section of the study, we theoretically demonstrate the asymptotic of the estimator variance for the repeated measures model. The estimator was derived using the mean bias reduction method to produce estimators with reduced variance.

Theorem 5.1
The compact set $\omega \subset \mathfrak{R}^e$ of the parameters with the continuous functions F_σ and $E(\sigma)$ at σ, $\frac{1}{N}\sum_{i=1}^{N} F_{\sigma\sigma} \xrightarrow{P} lim_{N\to\infty} \frac{1}{n}\sum_{i=1}^{N} E\,(\sigma)_i$, the matrix $\mathcal{Z} = -\frac{1}{N}\sum_{i=1}^{N} E(\sigma)_i$ is positive definite with $\frac{1}{N}\left|-\frac{\partial F_{\tilde{\sigma}}{}^*}{\partial \sigma_j}\right| \xrightarrow{P} \mathcal{Z}$ when $N = a.b.c$ and $j = 1,2,3 \ldots, p$ then $N^{-\frac{1}{2}}\,(\tilde{\sigma}^* - \sigma)$ is normal $(0, (1 + G - 1)\, E(\sigma) - 1)$.

Proof:
By Theorem 4.1, $\tilde{\sigma}^*$ is the consistent for σ. Using the theorem (Taylor) for $F_{\tilde{\sigma}}{}^*$ at $\tilde{\sigma}^*$ we get:

$$F_{\tilde{\sigma}}{}^*(\sigma) + \nabla F_{\tilde{\sigma}}{}^*(\dot{\sigma})(\tilde{\sigma}^* - \sigma) = 0$$

$$\text{where } \dot{\sigma} = \sigma + \rho\,(\tilde{\sigma}^* - \sigma), \{\rho: 0 < \rho < 1\}.$$

$$(\tilde{\sigma}^* - \sigma) = -F_{\tilde{\sigma}}{}^*\,(\sigma)\{\nabla F_{\tilde{\sigma}}{}^*(\dot{\sigma})\}^{-1}$$

$$\sqrt{N}\,(\tilde{\sigma}^* - \sigma) = \sqrt{N}\,F_{\tilde{\sigma}}{}^*\,(\sigma)\{-\nabla F_{\tilde{\sigma}}{}^*(\dot{\sigma})\}^{-1}$$

$$\sqrt{N}\,(\tilde{\sigma}^* - \sigma) = \left\{\frac{-\nabla F_{\tilde{\sigma}}{}^*(\sigma)}{N}\right\}^{-1} \left(\frac{1}{\sqrt{N}} F_{\tilde{\sigma}}{}^*\,(\sigma)\right.$$

$\frac{1}{\sqrt{N}} F_\sigma$ is normal $(0_p, E(\sigma))$ as $N \to \infty$ by central limit theorem. We have $\sqrt{N}\,(\tilde{\sigma}^* - \sigma)$ is normal $(0_e, E(\sigma)^{-1})$. When $N \to \infty$. If $\sqrt{N}\,(\tilde{\sigma}^*{}_{\varkappa} - \sigma)$ is independent at $\varkappa = 1, 2, 3, 4 \ldots, G$ we get the joint limit as follows:

$$\begin{bmatrix} \sqrt{N}\,\tilde{\sigma}^{*}{}_{1} & - & \sqrt{N}\sigma \\ \sqrt{N}\,\tilde{\sigma}^{*}{}_{2} & - & \sqrt{N}\sigma \\ \vdots & \vdots & \vdots \\ \sqrt{N}\,\tilde{\sigma}^{*}{}_{G} & - & \sqrt{N}\sigma \end{bmatrix}$$

Is normal $(0_e G, \Lambda)$, Λ are the diagonal blocks for $E(\sigma)^{-1}$. From continuous function theorem

$$\sqrt{N}\, M(\sigma)^{*}_{\varkappa} = G^{-1} \sum_{\varkappa=1}^{G} \sqrt{n}\, (\tilde{\sigma}^{*}{}_{\varkappa} - \sigma)$$

Is normal $(0_e, G^{-1}E(\sigma)^{-1})$. Because $\sqrt{N}\, \sum_{i=1}^{n} F_\sigma$ and $G^{-1} \sum_{\varkappa=1}^{G} \sqrt{N}\, (\tilde{\sigma}^{*}{}_{\varkappa} - \sigma)$ are independent,

then $\begin{bmatrix} \sqrt{N}\, \sum_{i=1}^{n} F_\sigma \\ \sqrt{Nn}\, M(\sigma)^{*}_{\varkappa} \end{bmatrix}$ is normal $\left(0_{2e}\ ,\ \begin{pmatrix} E(\sigma) & 0_{e\times e} \\ 0_{e\times e} & G^{-1}E(\sigma)^{-1} \end{pmatrix}\right)$ and we have

$$\frac{1}{\sqrt{N}} F_{\tilde{\sigma}}{}^{*} = \frac{1}{\sqrt{N}} F_\sigma - N^{-1}\, E(\sigma)\, \sqrt{n}\, M(\sigma)^{*}_{\varkappa}$$

Is normal $(0_e, (1 + G^{-1})E(\sigma))$. $\dot{\sigma}$ is consistent, by slutusky lemma we get $\frac{1}{\sqrt{N}} (\tilde{\sigma}^{*} - \sigma)$ is normal $(0, (1 + G - 1)\ E(\sigma) - 1)$.

6. CONCLUSION

Although the classical maximum likelihood method is efficient, it is not the optimal choice for estimating the variance of random parameters in a repeated-measures model due to its tendency to produce biased estimators.

The primary objective of this research is to implement a convergent method to reduce bias. We use the Mean Bias Reduction method, a modified version of the traditional method that involves subtracting the amount of the bias rate from the score function of the maximum likelihood method.

We adopted a different technique, the simulation method based on the modified method, to study reducing the bias in the modified method and obtain a new variance estimator with reduced bias compared to previous methods. Finally, we demonstrated using Asymptotic normality that as the number of simulations increases, the estimators of the variance components approach the normal distribution.

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