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Chapter One

On Bornological Structures and It's Practical Applications

1.1 INTRODUCTION

Bornological structure is a very important structure because it has many practical applications to solve the problems of identification and bounded in our daily lives [1][2] [3], for example, determining the location and identity of a person through an eye print or finger print, and because this structure lacks theoretical sources that facilitate its study. So, the main purpose of this chapter is to explain and clarified the bornological structures. Furthermore, we presented some practical application of this structures in our daily life as well [7] [8].

Additionally, we constructed new bornological structures in different ways. For example, our bornological structures from the base and sub base and these structures coinciding with the bornological structures associated with stable operator. Also, we presented many results and characteristics in bornological structures and show many fundamental constructers of bornological structure such as subspace and product bornology. Finally, we recall the definition of bornological group with some detailed examples [11] [21].

1.2 BORNOLOGICAL SETS

Definition (1.2.1.)(Bornological set) [10]**:**

Let X be a set. A bornology on X is a family $\beta \subset \mathcal{P}(X)$ such that:

(i) β covers X, i.e. $X = \bigcup_{B\in\beta} B$.

(ii) β is hereditary under inclution, i.e. if $A \subset B$ and $B \in \beta$ then $A \subset \beta$.

(iii) β is stable under a finite union, i.e. if $B_1, B_2 \in \beta$, then $B_1 \cup B_2 \in \beta$.

A pair (X, β) consisting of a set X and a bornology β on X is called a bornology set, and the elements of β are called bounded sets.

Any sub collection of bornology construct a base for bornology if every element of the bornology is contained in an element of the base.

Example (1.2.2):

let R be the set of real numbers with Euclidean norm (absolute value) we want to define usual bornology on R (canonical bornology), i.e. β is the collection of all usual bounded sets on R.

In fact, a subset B of R is bounded iff there exist a bounded interval. Such that

$$B \subseteq (a, b).$$

i- Since $\forall\, x \in (a, b)$ (the bounded intervals w.r.t. $|\,|$) , and $\{x\} \subset (a, b)$ (every subset of bounded interval is bounded).

This implies that $\forall\, x \in R, \{x\} \in \beta$, then β is a covering R.

ii- If $A \in \beta$ and $B \subseteq A$, then $\exists$ bonded interval s.t. $B \subseteq A \subseteq (a, b)$, therefore $B \in \beta$.

iii- If $B_1, \ldots, B_n \in \beta$, then $\exists$ $L_1, \ldots, L_n$ lest upper bounded and $g_1, \ldots, g_n$ greater lower

bounded,
such that $\bigcup_{i=1}^{n} B_i$ has finite union of upper and lower bounded.

Put $L = \min_{1 \le i \le n} \{L_i\}$ and $g = \max_{1 \le i \le n} \{g_i\}$, i.e.

$\bigcup_{i=1}^{n} B_i$ has least upper bounded L and greater lower bounded g, i.e.

$\bigcup_{i=1}^{n} B_i$ is bounded subset of R, i.e.

$\bigcup_{i=1}^{n} B_i \in \beta$, then β is a bornology on R.

The base of this bornology is

$$\beta_0 = \{B_r(x) : r \in R , x \in R\} = \{(x - r, x + r) : r \in R, x \in R\}.$$

Example (1.2.3):

Let $X = R^n$ with Euclidean norm, i.e.

$$||X|| = (\sum_{i=1}^{n} |X|^2)^{1/2} \qquad for \quad x = (x_1, \dots \dots , x_n),$$

and

$$D_r\ (a) = \{x \in R^n : \| x - a \| \leq r, \text{ for } r \geq 0\}.$$

The center is a = $(a_1, a_2, \dots, a_n)$.

A subset B of R^n is bounded if there exists unit disk with center zero $D_r(0)$, such that

$$B \subseteq D_r(0).$$

Let $\beta_{\|\cdot\|}$ be a family of all bounded subsets of R^n.

Then $(R^n, \beta_{\|\cdot\|})$ is a bornological set and $\beta_{\|\cdot\|}$ is called canonical (sometimes usual) bornology on R^n.

A base of a bornology β on R^n is any subfamily such that

$$\beta_0 = \{ D_r(0) : r > 0\}.$$

Indeed,

i-It is clear that every closed disk in R^n is bounded set, i.e. $D \in \beta$, since R^n is covered by the family of all closed disk.
Then β covers R^n.

ii- If $B \in \beta$ and $B_1 \subseteq B$, then there a closed disk such that $B_1 \subseteq B \subseteq D$. Therefore

$$B_1 \in \beta.$$

iii- If

$$B_1, \dots , B_n \in \beta.$$

Then there exist

$$D_{r1}, \dots, D_{rn} \text{ are closed disks.}$$

Such that

$$A_1 \subseteq D_{r1}, \dots , An \subseteq D_{rn} ,$$

and

$$r = \max r_i .$$

Then

$$\bigcup_{i=1}^{n} Bi \subseteq D_r.$$

Thus the finite union is bounded subset of R^n.

Then β is a bornology on R^n.

Example (1.2.4):

Let X be a set and (β_{fin}) be the collection of all finite subsets of X.

Then (X, β_{fin}) is a bornological set and β_{fin} is called finite bornology.

Indeed,

i-It is clear that for every $x \in X, \{x\}$ is finite set, then for all $x \in X, \{x\} \in \beta$.

Thus X covered by the family of all singleton set.

Then β_{fin} covers X.

ii-Let $B \in \beta_{fin}$ $s.t. B$ is finite set and $A \subset B$.

Since every subset of finite set is finite set.

Then

$$A \in \beta.$$

iii-Let

$$B_1, \ldots, B_n \in \beta_{fin}\ .$$

Since the finite union of finite sets is finite set.
Then

$$\bigcup_{i=1}^{n} B_i \ \ is\ finite\ set.$$

Thus the finite union is bounded subset of X.

Then β_{fin} is a bornology on X and it is called a finite bornology.

<u>Definition (1.2.5)(Bounded Map)</u> [10]**:**
A map between two bornological sets is called bounded if the image of every bounded set is bounded set.

<u>Definition (1.2.6)</u> [10] [22]**:**
An isomorphism between two bornological sets (X, β_x) and (Y, β_y) is one-to-one correspondence such that f, f^{-1} are bounded.
Consequently, for every $B \in \beta_x$ there is $B_1 \in B_y$
such that

$$B_1 = f(B).$$

$$B = f^{-1}(B_1).$$

1.3 SOME PRACTICAL APPLICATION OF BORNOLOGICAL STRUCTURES

What is the motivation and effect to study bornology? Why they construed bornological structure exactly with three conditions? What are the practical applications for this structure? In other words, how they apply this structure to solve some problems in our live.
In this research, we answer all these questions and explain many examples for this structure.
The most important practical application for bornology, in spying programs (KPJ).

Exactly, when they want to determine person location or the identity of person from his finger print or from an eye print, let we explain how.
First of all, let assume that the person is original point, to determine his signed (status).

We start to study the behavior of objects within his domain by introducing open unit ball, as shown in figure [1].

$$B_1 = \{x \in V, Ƥ(x) < 1\}$$

or closed unite ball

$$B_1 = \{x \in V, Ƥ(x) \leq 1\}.$$

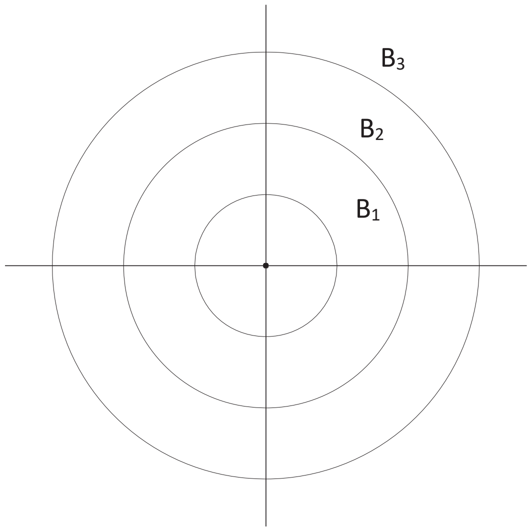

FIGURE 1.1. The behavior of objects within his domain by introducing open unit ball .

Then we need the length of this object (by norm) but B_1 is absorbent disk, i.e. absorbs every subset of V which consist singleton element (point).
But, we want to study the behaver for another object.
Since B_1 it is in vector space, then it is allowed to multiple B_1 by scaler and make B_1 bigger to get new open unit ball B_2, In the end, we get a collection of unit balls covers the place and the finite union which it is the bigger, bounded set should also be within the place not outside, i.e. inside the collection β.

Furthermore, if the father belongs for this family, then the son also belongs for this family. So, the hereditary property is satisfying.

Another application of bornology is to determine the person from his stamp (finger print) or his eye print, (see figure 2, a and b).

FIGURE 1.2 a: Application of bornology is to determine the person from his eye print.

FIGURE 1.3 b: Application of bornology is to determine the person from his stamp (finger print).

Also, there are many other applications of bornology. For example, to equip a building with the internet service. We take the center point on the surface, and the internet waves are the open interval therefore the collection of these open interval which covers the surface stable under finite union and hereditary property, (see figure 3). Add to that, there is an important of application of bornology, in satellite broadcast system when they want to determine the limits of the broadcast area.

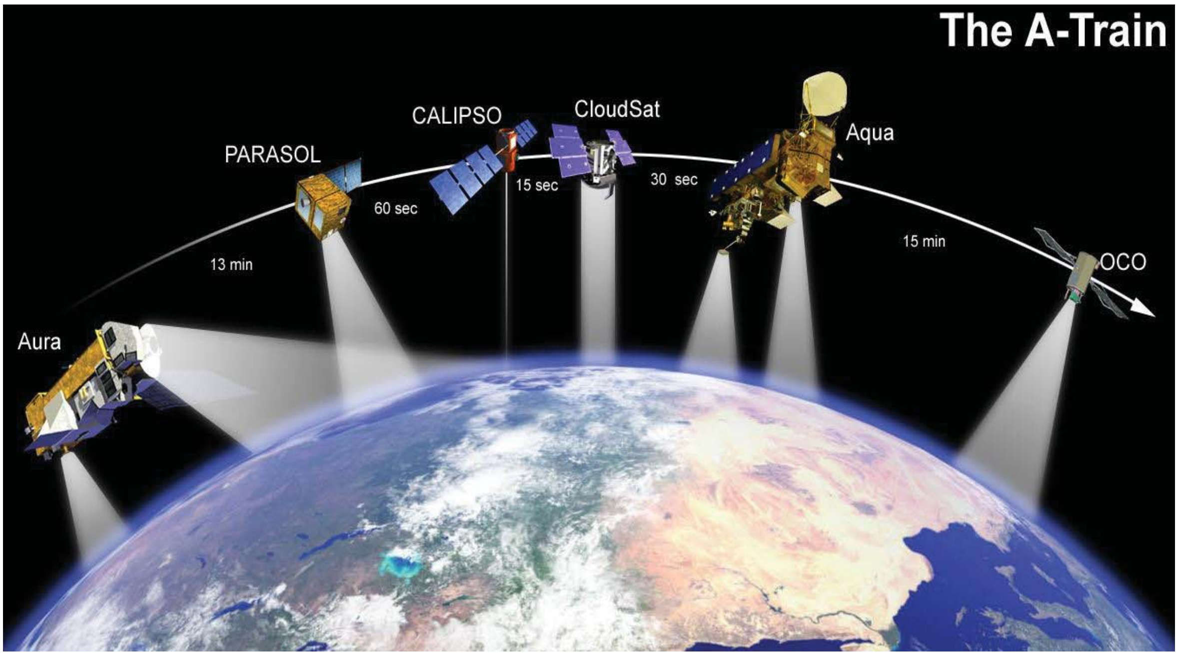

FIGURE 1.4 satellite broadcast system when they want to determine the limits of the broadcast area.

1.4 INDUCED NEW BORNOLOGICAL STRUCTURES

In this section, we constructed a new bornological structures. First of all, we may ask when a family of subsets of X forms a base for the bornology on X, what is the form of the bornology generated by this base?
By next result we get that any base β_0 has two properties.
First, β_0 covers X. Second, β_0 stable under finite union.
Conversely, we can generate a uniqe bornology from the base and this is the benifit from the base.

Theorem (1.4.1):
Let (X, β) be a bornological space and is β_0 the base of β, then the following holds:

i- $X = \bigcup_{B_0 \in \beta_0} B_0$.

ii- Finite union of elements of β_0 is contained in an element of β_0.

Iff $X \neq \emptyset$ and β_0 is a family of subsets of X satisfying : (i) and (ii) then $\exists$ a bornology β on X generated from the base s.t.

$$\beta = \{B\colon \exists\, B_0 \in \beta_0,\, B \subset B_0\}.$$

Proof.

i-Since β_0 formed a base of a bornology β on a set X.

Then every element of β is contained in an element of β_0, i.e. $\forall B \in \beta, \exists\, B_0 \in \beta_0$

Such that

$$B \subseteq \beta_0$$

$$\bigcup_{B \in \beta} B \subseteq \bigcup_{B_0 \in \beta_0} B_0\,.$$

$$X = \bigcup_{B \in \beta} B.$$

Then
$$X \subseteq \bigcup_{B_0 \in \beta_0} B_0.$$

i.e. β_0 covers X.

ii- let $B_i \in B_0$ $\quad \forall$ i = 1, 2, · · ·, n.

To prove $\bigcup_{i=1}^{n} B_i$ is contained in an element of β_0.

Since $\beta_0 \subseteq \beta$, then $B_i \in \beta$ $\quad \forall$ i = 1, 2, · · ·, n (by definition of bornology).

Then $\bigcup_{i=1}^{n} B_i \in \beta$.

From definition of a base ($\forall\, B \in \beta, \exists\, B_0 \in \beta_0$ such that $B \subseteq \beta_0$).

We have $\bigcup_{i=1}^{n} B_i \in \beta$ is contained in an element of β_0.

$\Leftarrow$ Conversely

If $\beta = \{B \subseteq X, \exists\, B_0 \in \beta_0, B \subseteq B_0\}$.

Then to show that β is a bornology on X, we must satisfy the following conditions:

i- $\forall B_0 \in \beta_0 \longrightarrow B_0 \subseteq X \longrightarrow B_0 \subseteq B_0 \longrightarrow B_0 \in \beta$ (by definition of β).

$$\rightarrow \beta_0 \subseteq \beta.$$

Since β_0 covers X.

Then

$$\beta \text{ is covering to } X.$$

ii-$if\ B \in \beta$, $A \subseteq X$, $A \subseteq B$, A contained in an element of β_0. (by definition of β).

Then

$$A \in \beta.$$

iii-Now $if\ B_1, B_2 \in \beta$.

Then

$$\exists\, B_0, B_0' \in \beta_0.$$

Such that

$$B_1 \subseteq B_0.$$

$$B_2 \subseteq B_0' \qquad (by\ condition\ ii).$$

Then

$B_1 \cup B_2 \subseteq\ B_0 \cup B_0'$ for some $B_0 \in \beta_0$ (by definition of β).

Then β is a bornology on X, and by definition or the base β_0 is a base

for it, we call β a bornology generated by the basis.

Remark (1.4.2):

i- Every bornology forms a base for itself and have more than one base.

ii- Not every family of subsets of a set X will form a base for some bornology on X

For example, $X = \{a, b, c\}$ and the collection $\beta_0 = \{\{a\}, \{b\}, \{c\}\}$ cannot be a base for any bornology on X.

By the next example we clarify how we can construct bornology from the base.

Example (1.4.3)

i- $X = \{1, 2, 3\}$.

And

$\beta_0 = \{\{1, 2\}, \{2, 3\}, X\}$.

$\beta = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}\}$.

Theorem (1.4.4):

If X≠∅ , then a family S of subsets of X, such that X= $\bigcup_{s \in S} s_i$ forms subbase for a bornology on X.

Proof:

We must show that the family $\beta_0 = \{B \subseteq X : B = \bigcup_{finite} s\, , s \in S\}$ forms a base for a bornology on X.

Therefore, we must satisfy the following:

i- $\forall s \in S \rightarrow s \in X \rightarrow s = s \bigcup s$

$$\rightarrow s \in \beta_0$$

$$\rightarrow S \subseteq \beta_0$$

Since S covers X, then β_0 covers X.

ii- $B_1, B_2 \in \beta_0$

$$\longrightarrow B_1 = \bigcup_{finite} S_j$$

$$B_2 = \bigcup_{finite} S_i$$

الخلاصة

الهدف الرئيسي من هذا العمل هو شرح وتوضيح بنية الفضاء البرنولوجي. كذلك ، قمنا ببناء هياكل برنولوجيه جديدة بطرق مختلفة. على سبيل المثال ، أنشأنا برنولوجي من القاعدة والقاعدة الفرعية. أيضًا ، قمنا ببناء برنولوجي مرتبط بتطبيق خاص وهو (Stable Operator). أيضا ، قدمنا العديد من النتائج والخصائص التي تخص الفضاء البرنولوجي. والجزء الأكثر أهمية في عملنا أننا قدمنا بعض التطبيقات العملية لفضاء البرنولوجي. وكذلك ، تم وصف فكرة الزمر البرنولوجية والتي رمزنا لها بالرمز (BG) في هذا العمل مع بعض الأمثلة. بالإضافة إلى ذلك ، فإن الهدف الأساسي لهذا العمل هو تقديم دوال مقيده (Actions)، وهي دوال مقيده يتم تنفيذها على الفضاء البرنولوجي. يشار الآن إلى جميع الخواص المتعلقة بمجموعات G على أنها مجموعات -G bornological. ثم نعرض بعض الحقائق الأساسية التي هي صحيحة. بالنسبة للدوال المقيدة ، أظهرنا بشكل خاص ، أن حدود عمل الفضاء البرنولوجي يمكن اشتقاقها في موضوع الهوية ، تتم دراسة مجموعة علم البرنولوجي للمدارات ، ووجد أن حاصل القسمة X / G هو رمز على حاصل البرنولوجي. بالإضافة إلى ذلك ، يُشار إلى مجموعة حاصل البرنولوجي الخاصة بالدوال(Actions)على هذا النحو. أخيرًا ، نقوم ببناء chomology مقيد استنادًا إلى cochain المقيد. تتمثل إحدى السمات الرئيسية لمجموعات cochain المقيده في أنها تؤدي إلى تسلسلات دقيقة طويلة في علم الكهومولوجي ، وهي أول نتيجة رئيسية لنا. علاوة على ذلك ، قد يبدو أن الهدف النهائي هو العثور على تعريف لعلم الكهومولوجي للمجموعات البرنولوجيه التي ترث جميع الخصائص من الكروب الزمر البرنولوجية. لذلك ، قمنا بتطوير عائله من المجموعات البرنولوجيه باستخدام كوشاينات مقيده. وهو يعمل مع المجموعة البرنولوجيه (G، β) ووحدات G البرنولوجيه (M، β). والنتيجة المهمة الرئيسية هي أن نظرية cohomology للكوشاين المقيد تتشابه مع نظرية cohomology للكوشاين المتجانس.