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Chapter One

Fundamental Introduction and some new results

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1.1 Introduction

One of the powerful tools used on topology and set theory is the notion of filter which is simply a collection of subsets from any set say $\mathcal{M}$. However,filters are known to be applicable in a wide variety of mathematical areas like algebraic,functional topology and dynamical topology. From example in functional analysis , its use as a method of convergence, identically for the notion of nets. As an example of an appearance in algebra, the notion of filters is the dual of the ideals Boolean rings on these sets. Some important notions and theorems about filter and ultra-filter will be explored in this chapter. Al so we will try to give some properties of filter and ultra-filter this will be our principal important tools for the structure of the stone cech compactification. In addition, we will discuss some types of filters and give proof of some results.

1.2 Some Basic definitions and properties:

In this section, we introduce the basic definitions and properties of filter and ultra-filter and provide some results.

Definition 1-2-1[15]: A semi-group is a pair $(\mathcal{M},*)$ where $\mathcal{M}$ is nonempty set and $*$ is a binary associative operation on $\mathcal{M}$.

Examples 1-2-2::

$(\mathbb{N}, +)$ is semi-group .

Definition 1-2-3[5]: Let $\mathcal{M}$ be a set and $D \subseteq \rho(\mathcal{M})$ be the collection of subsets of a set $\mathcal{M}$. We say that D has a finite intersection property (FIP) if the finite intersection for any specific subset of D is not empty.

Definition 1-2-4[20]: Let $\mathcal{M}$ be any set , a filter on a set $\mathcal{M}$ is a non-empty set $\mathcal{F}$ with the following properties:

1- $\phi \notin \mathcal{F}$.

2- If p and q $\in \mathcal{F}$ then $p \cap q \in \mathcal{F}$.

3- If $p \in \mathcal{F}$, and $p \subseteq q \subseteq \mathcal{M}$, then $q \in \mathcal{F}$.

Example 1-2-5: Consider the set $\mathcal{F}$ to be a neighborhood of a point b in a topological space X. Then $\mathcal{F}$ is a filter.

1- It's clearly that for any neighborhood of a point b say $p = (b-\in, b+\in)$ we have $\phi \notin \mathcal{F}$

2- Let $p = \left[b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right], q = \left[b - \frac{\epsilon}{4}, b + \frac{\epsilon}{4}\right] \in \mathcal{F}$

then $p \cap q = \left[b - \frac{\epsilon}{4}, b + \frac{\epsilon}{4}\right] \in \mathcal{F}$

3- Take $p = \left[b - \frac{\epsilon}{4}, b + \frac{\epsilon}{4}\right] \in \mathcal{F}$, and $q = \left[c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}\right]$ be a

neighborhood for some point c, s. t $\left[b - \frac{\epsilon}{4}, b + \frac{\epsilon}{4}\right] \subseteq \left[c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}\right] \subseteq \mathcal{M}$ then $\left[b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right] \in \mathcal{F}$.

Remarks 1-2-6[9]: Let $(\mathcal{F}_i)_{i \in J}$ be a family of a non-empty set of filters on a set $\mathcal{M}$(which mean is non-empty) then the set:

$$\mathcal{F} = \bigcap_{i \in J} \mathcal{F}_i$$

Satisfies conditions (1),(2) and (3) and is implies $\mathcal{F}$ is a filter that is the intersection sets of filters.

Example l-2-7: Let $\mathcal{M}$ be an infinite set, the set $\mathrm{T} = \{\, \mathrm{A} \subseteq \mathcal{M} \colon \mathrm{s.t}\, \mathcal{M} / \mathrm{A}\ \text{is finite} \,\}$ the set of all cofinite subset of $\mathcal{M}$ is afilter which is called a cofinite filter or frechet filter on $\mathcal{M}$ and is denoted by FR.

Example 1-2-8: Let A={1,2,3}and

let $\mathcal{F}_1$={A}; $\mathcal{F}_2$={{1,2},A};$\mathcal{F}_3$={{2,3},A};$\mathcal{F}_4$={{3,1},A}; $\mathcal{F}_5$={{1},{1,2},{1,3} ,A};$\mathcal{F}_6$={{3},{1,3},{2,3},A}; $\mathcal{F}_7$={{3},{3,1},{2,3},A}: $\mathcal{F}_8$={{1,2},{2,3},A} ; $\mathcal{F}_9$={{1},{2},{1,2},A}; $\mathcal{F}_{10}$={∅ {1},{1,2},A}.

Now, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, and $\mathcal{F}_7$ are filters on A. Whereas, $\mathcal{F}_8$, $\mathcal{F}_9$, and $\mathcal{F}_{10}$ are not filters.

Remarks 1-2-9[9]:

1-The union of two filters on a set need not be a filter. Form the above example $\mathcal{F}_2$, $\mathcal{F}_3$ are filters but $\mathcal{F}_2 \cup \mathcal{F}_3$ is not a filter.

2- In above example, The intersection of all filters which is the filter $\{A\}$,this will be the weakest filter on A.

Theorem 1-2-10[20]: Let $\mathcal{F}$ be a filter on the set $\mathcal{M}$ and let $p \subseteq \mathcal{M}$ either:

a- There is some $q \in \mathcal{F}$ s. t $p \cap q = \emptyset$. Or

b- $\{c \subseteq \mathcal{M}$: there is some $q \in \mathcal{F}$, $p \cap q \subseteq c\}$ is a filter on $\mathcal{M}$.

Proof :

Suppose for all $q \in \mathcal{F}$, $p \cap q \neq \phi$, we need to show the set $k = \{c \subseteq \mathcal{M}, \exists$ some $q \in \mathcal{F}$ with $p \cap q \subseteq c\}$ is actually a filter

1- If $p_1, p_2 \in k$, then there exist $q_1, q_2 \in \mathcal{F}$ s. t $p \cap q_1 \subseteq p_1$, $p \cap q_2 \subseteq p_2$. Then $p \cap (q_1 \cap q_2) \subseteq p_1 \cap p_2 \in k$.

2- If $p' \in k$ take $q \in \mathcal{M}$ such that $p' \subseteq q \subseteq \mathcal{M}$ to show that that $q \in k$.

Now $p' \in k$ implies that $\exists q' \in \mathcal{F}$ such that $p \cap q' \subseteq p'$, hence $p \cap q' \subseteq q$ and then $q \in k$.

3- By the negative condition of (a) then $\phi \notin k$.

In The next following definition, we will introduce another important type of filter.

Definition 1-2-11[16]: A filter$\mathcal{F}$ on $\mathcal{M}$ is called an ultra-filter if it is not properly contained in any other filter.

Among filters, ultra-filters are uncommon, we can see it in general for trivial examples.

Example 1-2-12: The trivial filter $\{\mathcal{M}\}$ on $\mathcal{M}$ is not ultra-filter unless $\mathcal{M}$ is a singleton. Also, the frechet filter is not ultra-filter if $\mathcal{M}$ is infinite, since there are infinite cofinite subsets in $\mathcal{M}$. For example, if $\mathcal{M} = \mathbb{Z}$, then neither the set of positive integer numbers neither its complement is contained in $\mathcal{M}$ which is not ultrafilter according to the next lemma .

The next lemma is in [20] which is a nother characteristic for the ultra-filter but we proved in different way.

Lemma 1-2-13[20]: Let $\mathcal{M}$ be a non-empty set and $\mathcal{F}$ be a filter on $\mathcal{M}$. Then $\mathcal{F}$ is an ultra-filter iff for every $\mathrm{p} \subseteq \mathcal{M}$ either $\mathrm{p} \in \mathcal{F}$ or $\mathcal{M}\backslash\mathrm{p}$ is an element on $\mathcal{F}$.

Proof**:**

For the first direction, it's direct proof by condition filter (2)

Conversely, let $\mathcal{F}$ be an ultra-filter in $\mathcal{M}$. Assume that for $\mathrm{p} \subseteq \mathcal{M}$ neither p nor its complement $\mathcal{M}\backslash\mathrm{p}$ belongs to $\mathcal{F}$.

Case (1): p and $\mathcal{M}\backslash\mathrm{p} \in \mathcal{F}$ implies by definition of filter $\mathrm{p} \cap \mathcal{M}\backslash\mathrm{p} = \emptyset \in \mathcal{F}$, which is a contradiction.

Case (2): we have p and $\mathcal{M}\backslash\mathrm{p} \notin \mathcal{F}$. Note that $\mathcal{F} \cup \mathrm{p}$ and $\mathcal{F} \cup \mathcal{M}\backslash\mathrm{p}$ are filters and $\mathcal{F} \subseteq \mathcal{F} \cup \mathrm{p}$ and $\mathcal{F} \subseteq \mathcal{F} \cup \mathcal{M}\backslash\mathrm{p}$ which is a contradiction . Therefore p or

$\mathcal{M}\backslash p \in \mathcal{F}$.

Definition 1-2-14[20]: A filter $\mathcal{F}$ on $\mathcal{M}$ is a maximal filter if for any $p \subseteq \mathcal{M}$ and $p \notin \mathcal{F}$, then $\mathcal{F} \cup \{p\}$ is not a filter.

Proposition 1-2-15[10]: A filter $\mathcal{F}$ on a non-empty set $\mathcal{M}$ is an ultra-filter if only if it is a maximal filter.

Proof:

For the first direction, we will show that $\mathcal{F}$ is a max filter. Let us extend an ultra-filter $\mathcal{F}$ by combine $p \subseteq \mathcal{M}$ to get $\mathcal{F} \cup \{p\}$. suppose $p \notin \mathcal{F}$, which implies $p^c \in \mathcal{F}$. Assume $\mathcal{F} \cup \{p\}$ is a filter. Note $\mathcal{F} \subseteq \mathcal{F} \cup \{p\}$ and $\mathcal{F}$ is ultra-filter, this is a contraction. So $\mathcal{F}$ is a max filter.

Conversely, to show every maximal filter $\mathcal{F}$ is an ultra-filter. Suppose $\mathcal{F}$ is not ultra-filter, then $\exists\, k \subset \mathcal{M}$ such that k and $k^c \notin \mathcal{F}$. We claim either k or k^c should intersect with every set in $\mathcal{F}$. If not then $\exists$ a set $C \in \mathcal{F}$ such that $C \subseteq k$ or $C \subseteq k^c$ which follows that k or $k^c \in \mathcal{F}$ which is a contradiction with our hypothesis. Without losing the generality, let $C \cap A \neq \emptyset, \forall A \in \mathcal{F}$. Define $D = \{B \in p(\mathcal{M}): k \cap A \subseteq B$ for some $A \in \mathcal{F}\}$. We claim D is a filter and $\mathcal{F} \subseteq D$. Since $k \cap A \subseteq A$ for all $A \in \mathcal{F}$ so $\mathcal{F} \subseteq D$. Obviously, $\emptyset \notin D$. Let $B_1 \in D$ and $B_2 \in \mathcal{M}$ s. t $B_1 \subseteq B_2$.
Since $k \cap A \subseteq B_1 \subseteq B_2$, then $B_2 \in D$. For $A_1, A_2 \in D$. $k \cap A \subseteq A_1$ and $A_1 \cap A_2 \in D$. Hence D is a filter but our claim will be a contradiction with maximality. Therefore every maximal filter is an ultra-filter.

The next theorem provides us more about the filter and ultra-filter properties. Before to go to the theorem we need to define the following set.For given a set B on a non- empty set $\mathcal{M}$ we define the set:

$\rho_f(B) = \{D: \emptyset \neq D \subseteq B \text{ and } D \text{ is finite}\}$.

Theorem 1-2-16[20]: Let $\mathcal{M}$ be a non-empty set, and let $\mathcal{F} \subseteq p(\mathcal{M})$, the following statements are equivalent:

a. $\mathcal{F}$ is an ultra-filter on $\mathcal{M}$.

b. $\mathcal{F}$ has the finite intersection property, and for each $p \in \rho(\mathcal{M})/\mathcal{F}$, there is some $q \in \mathcal{F}$, s. t $p \cap q = \emptyset$.

c. $\mathcal{F}$ is a maximal set corresponding to the set $\{C \subseteq p(\mathcal{M})$: C satisfy the finite intersection property$\}$ (That is, $\mathcal{F}$ with respect to the finite intersection property is maximal set).

d. $\mathcal{F}$ is a filter on $\mathcal{M}$ and for all $F \in \rho_f(\rho(\mathcal{M}))$ if $\cup F \in \mathcal{F}$, then $F \cap \mathcal{F} \neq \emptyset$.

e. $\mathcal{F}$ is a filter on $\mathcal{M}$ and for all $p \subseteq \mathcal{M}$, either $p \in \mathcal{F}$ or $\mathcal{M}/p \in \mathcal{F}$.

Proof:

$a \Longrightarrow b$**:** By (1) and (3) of filter definition, implies $\mathcal{F}$ has finite intersection property. Again by (1) and (3), we know p, p^c can not both elements of $\mathcal{F}$. If $p \in p(\mathcal{M})\backslash\mathcal{F}$ then there is $q, p^c \in \mathcal{F}$, s. t $p \cap p^c = \emptyset$.

$b \Longrightarrow c$**:** Suppose $\mathcal{F} \subseteq Z \subseteq p(\mathcal{M})$. Pick $p \in Z\backslash\mathcal{F}$, and $q \in \mathcal{F}$, such that $p \cap q = \emptyset$. Hence $p, q \in Z$, implies Z does not have the finite intersection property.

$c \Longrightarrow d$**:** Suppose that $\mathcal{F}$ is a maximal with relative to finite intersection property through a subset of $p(\mathcal{M})$, immediately $\mathcal{F} \neq \emptyset$ subsets of $\mathcal{M}$.

Since $\mathcal{F} \cup \{\mathcal{M}\}$ has (F. I. P), and $\mathcal{F} \subseteq \mathcal{F} \cup \{\mathcal{M}\}$, then by maximality of $\mathcal{F}$ get $\mathcal{F} = \mathcal{F} \cup \{\mathcal{M}\}$, that is mean $\mathcal{M} \in \mathcal{F}$. We need to show $\mathcal{F}$ is a filter:

1- Pick $p, q \in \mathcal{F}$, then $\mathcal{F} \cup \{p \cap q\}$ has finite intersection property. Since

$\mathcal{F} = \mathcal{F} \cup \{p \cap q\}$ then $p \cap q \in \mathcal{F}$, So $p \cap q \in \mathcal{F}$.

2- Given p, q with $p \in \mathcal{F}$ such that $p \subseteq q \subseteq \mathcal{M}$, $\mathcal{F} \cup \{q\}$ has finite intersection property then $q \in \mathcal{F}$.

3- By definition of finite intersection property $\emptyset \notin \mathcal{F}$. Therefore $\mathcal{F}$ is a filter.

We need only to show for all $F \in \rho_f(\rho(\mathcal{M}))$,$\cup F \in \mathcal{F}$, then $F \cap \mathcal{F} \neq \emptyset$. Let $F \in \rho_f(p(\mathcal{M})$, and $\cup F \in \mathcal{F}$, then that for each $p \in F, p \notin \mathcal{F}$, then $\mathcal{F} \subsetneq \mathcal{F} \cup \{p\}$ implies $\mathcal{F} \cup \{p\}$ does not have the finite intersection property Since of maximality of $\mathcal{F}$. Therefore $\exists\, g_p \in \rho\, f(\mathcal{F})$ s.t $p \cap g_p = \emptyset$, Let $H = f_{p \in F} g_p$, then $H \cup \{\cup F\} \subseteq \mathcal{F}$ while $(\cup F) \cap \cap H = \emptyset$. That is a contradiction .

d $\Longrightarrow$ e**:** Let $F = \{p, \mathcal{M} \backslash p\} \in \rho f(p(\mathcal{M})$, implies that $\cup F = \mathcal{M} \in \mathcal{F}$. Then

$\mathcal{F} \cap F \neq \emptyset$. Hence since $\mathcal{F}$ is a filter then $\exists$ either $p \in \mathcal{F}$ or $p^c \in \mathcal{F}$.

e $\Longrightarrow$ a**:** Assume that $\mathcal{F}$ is a filter on $\mathcal{M}$ and all $p \subseteq \mathcal{M}$ either $p \in \mathcal{F}$ or $\mathcal{M} \backslash p \in \mathcal{F}$. Let V be a filter, and $\mathcal{F} \subseteq v$, and suppose that $\mathcal{F} \neq v$, Pick $A \in v \backslash \mathcal{F}$, then $\mathcal{M} \backslash A \in \mathcal{F} \subseteq v$ while $A \cap (\mathcal{M} \backslash A) = \emptyset$. That is a contradiction.

Theorem 1-2-17[10]: Let $\mathcal{M}$ be a set, and let **A** be a subset of $p(\mathcal{M})$, which has finite intersection property, then there is an ultra-filter $\mathcal{F}$ on $\mathcal{M}$ such that **A** $\subseteq \mathcal{F}$.

proof**:**

Let $K = \{q \subseteq p(\mathcal{M})$: **A** $\subseteq q$ and q has F. T. P$\}$ then **A** $\in K$ and so $K \neq \emptyset$. Let $\mathcal{C}$ be a chain in k immediately have **A** $\subseteq \cup\, \mathcal{C}$. Given $\mathcal{F} \subseteq \rho_f(\cup_{\mathcal{C}}) = \{\mathcal{F} : \emptyset \neq \mathcal{F} \subseteq \cup_{\mathcal{C}}$, and $\mathcal{F}$ is finite$\}$. Implies, there is some $V \in \mathcal{C}$ with $\mathcal{F} \subseteq q$. Since q has finite intersection property and $\mathcal{F} \subseteq q$.So, $\cap \mathcal{F} \neq \emptyset$. Then by Zorn's Lemma[3].K has max number $\mathcal{F}$. Note $\mathcal{F}$ not only max in K, but $\mathcal{F}$ also is max with respect to finite intersection property. By the theorem $(1-2-16)(c \rightarrow a)$, then $\mathcal{F}$ is ultra- filter.

Corollary 1-2-18[20]: Let $\mathcal{F}$ be a filter on $\mathcal{M}$, and let $p \subseteq \mathcal{M}$, then $p \notin \mathcal{F}$, if and only if there is an ultra-filter T such that $\mathcal{F} \cup \{ p^c \} \subseteq T$.

Proof:

For sufficiency, since $\mathcal{F}$ can't have p and p^c at the same time so this direction will be trivial.

For the necessity: by using theorem {1-2-17 }, we need only to show $\mathcal{F} \cup \{ p^c \}$ has a finite intersection property. Suppose that there is some finite non-empty $Z \subseteq \mathcal{F}$ such that $\{ p^c \} \cap Z = \emptyset$, so $p \notin \mathcal{F}$ i.e. $p^c \in \mathcal{F}$ and $\cap Z \in \mathcal{F}$. Therefore $\cap Z \subseteq p$ and So $p \in \mathcal{F}$ which is a contradiction since $p \notin \mathcal{F}$.

In the following theory, we can prove that we had a non-empty set that contains the filter and ultra-filter, and so the filter is part of the ultra-filter within this set.

Theorem 1-2-19[10]: Every filter $\mathcal{F}''$ on anon-empty set $\mathcal{M}$ there exists an ultra-filter $\mathcal{F}$ *on* $\mathcal{M}$ such that $\mathcal{F}'' \subseteq \mathcal{F}$

Proof**:**

Let $S = \{ F : F$ is a filter and $\mathcal{F}'' \subseteq F \}$ and take the partially ordered set $(S, \subseteq)$. Now consider a chain $L \subseteq S$, the set of union $\cup L$ of this collection of filter indicted with $\subseteq$ is clearly a filter on $\mathcal{M}$ and containing $\mathcal{F}''$ which is an upper bound for L. Hence, $(S, \subseteq)$ satisfies the hypotheses of Zorn's lemma [3] implies has maximal element $\mathcal{F}$ which is a maximal element of $(S, \subseteq)$. We claim that

$\mathcal{F}$ is an ultra-filter. If not then there exists A $\subseteq \mathcal{M}$ such that A $\notin \mathcal{F}$ and $\mathcal{M}$ \A $\notin \mathcal{F}$. Consider the collection C = $\mathcal{F} \cup \{A\}$. We claim that set C has finite intersection property. Let $y_1, y_2, \ldots, y_n \in C$.

Case1: Suppose that $y_i \in \mathcal{F}$ for every $1 \leq i \leq n$. since $\mathcal{F}$ has finite intersection property then $y_1 \cap y_2 \cap, \ldots, \cap y_n \in F \subseteq C$.

Case 2: Suppose that $y_i \notin \mathcal{F}$ for some $1 \leq i \leq n$. By changing these sets without changing their intersection we can suppose without loss of generality that y_1 = A and $y_2, \ldots, y_n \in \mathcal{F}$. As $\mathcal{F}$ has finite intersection property, we have that $y_2 \cap \ldots \cap y_n \in \mathcal{F}$. It follows that any superset of $y_2 \cap \ldots \cap y_n$ is in $\mathcal{F}$. From the other side, $\mathcal{M}\backslash A \notin \mathcal{F}$ and hence $y_2 \cap, \ldots, \cap y_n \nsubseteq \mathcal{M}\backslash A$, that is $A \cap y_2 \cap, \ldots, \cap y_n \neq \emptyset$. Therefore C has finite intersection property and can be an extension to a filter, Hence $F \subseteq C \subseteq \mathcal{F}$ which is a contradiction by Zorn's lemma [3] $\mathcal{F}$ is maximal.

The next lemma it's easy to show, which is one of an important fact for the ultra-filter.

Lemma 1-2-20[20]: Let $\mathcal{M}$ be a set and $\mathcal{F}$ and L be ultra-filters on $\mathcal{M}$ then $\mathcal{F} =$ L if and only if $\mathcal{F} \subseteq$ L.

Proof:

For the first direction is clear.

Conversely, let $\mathcal{F} \subseteq$ L, $\mathcal{F}$ is ultra-filter. Then by definition of ultra-filter $\mathcal{F} =$ L.

Definition 1-2-21[20]: Let $\mathcal{M}$ be a set, and let $\mathcal{F}$ be a filter on $\mathcal{M}$, a family A is a filter base of $\mathcal{F}$ if and only if:

1- A$\subseteq \mathcal{F}$

2- For each $p \in \mathcal{F}$, there is some $S \in A$ such that $S \subseteq p$.

Theorem 1-2-22[20]: Let $\mathfrak{B}$ be a set of sets , then we can have some filter $\mathcal{F}$ on $\mathcal{M} = \cup \mathfrak{B}$, such that $\mathfrak{B}$ will be a filter base for $\mathcal{F}$ if and only if $\mathfrak{B} \neq \emptyset$ and $\emptyset \notin \mathfrak{B}$, also for every finite non-empty subset A of $\mathfrak{B}$, there is some $B \in \mathfrak{B}$, such that $B \subseteq \cap A$.

Proof:

Suppose there is some filter $\mathcal{F}$ on $\mathcal{M} = \cup \mathfrak{B}$ such that $\mathfrak{B}$ a filter base for $\mathcal{F}$. This means $\mathfrak{B} \subseteq \mathcal{F}$ and for each $C \in \mathcal{F}, \exists K \in \mathfrak{B}$, s. t $K \subseteq C$. By hypothesis $\mathcal{F}$ is a filter so $\emptyset \notin \mathcal{F}$. Therefore $\emptyset \notin \mathfrak{B}$ since if $\emptyset \in \mathfrak{B} \subseteq \mathcal{F}$ then $\emptyset \in \mathcal{F}$ which is a contradiction. Moreover, since $\mathcal{M} = \cup \mathfrak{B}$ then $\mathfrak{B} \neq \emptyset$. Now from the definition of filter and the fact $\mathfrak{B}$ is a filter base, $\mathfrak{B} \subseteq \mathcal{F}$ then $\emptyset \neq \cap A \in \mathcal{F}$ and $\exists B \in \mathfrak{B}$ such that $B \subseteq \cap A$.

Conversely, let M be a finite subset of $\mathfrak{B}$, and since $\emptyset \notin \mathfrak{B}$ then $\cap M \neq \emptyset$. Hence $\mathfrak{B}$ has the finite intersection property. By theorem [1-2-19] there is an ultra-filter $\mathcal{F}$ on $\mathcal{M}$ such that $\subseteq \mathcal{F}$. Define $\mathcal{F}$={C: $\cap A \subseteq C$ where A is a finite subset of $\mathfrak{B}$} which is easy to show it's a filter. Moreover, we have every finite non-empty subset A of $\mathfrak{B}$, there is some $B \in \mathfrak{B}$ such that $B \subseteq \cap A$ then $\mathfrak{B}$ a filter base for $\mathcal{F}$.

The next theory it's called the ultra-filter theorem , the content of this theory that the filter can be extended to ultra-filter.

Theorem 1-2-23[10]: Any filter $\mathcal{F}$ on a non-empty set $\mathcal{M}$ be an expansion to an ultra-filter.

Proof :

For some filter $\mathcal{F}$, suppose that $A = \{F' \subseteq \mathcal{M}: F' \text{ is filter s.t } \mathcal{F} \subseteq F'\}$ Note that

A is nonempty since $\mathcal{F} \in$ A. Claim that for each chain $\{ \mathcal{F}_\alpha : \alpha \in I \}$ in A , their union $\bigcup_{\alpha \in I} \mathcal{F}_\alpha$ is still a filter in A. It is clear that $\emptyset \notin \bigcup_{\alpha \in I} \mathcal{F}_\alpha$. For an element B $\in \bigcup_{\alpha \in I} \mathcal{F}_\alpha$, B $\in \mathcal{F}_\alpha$ for some $\alpha \in$ I. Then for all C such that $\mathfrak{B} \subseteq$ C, C $\in \mathcal{F}_\alpha \subset \bigcup_{\alpha \in I} \mathcal{F}_\alpha$. Similarly for P, Q $\in \bigcup_{\alpha \in I} \mathcal{F}_\alpha$ P$\in \mathcal{F}_\alpha$ and C $\in \mathcal{F}_\beta$. Without loss of generality, assume $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$ thus, C $\in \mathcal{F}_\alpha$ and B $\cap$ C $\in \mathcal{F}_\alpha \subseteq \bigcup_{\alpha \in I} \mathcal{F}_\alpha$. Hence, by Zorn's lemma [3], there exists a maximal element in A, and by proposition {1-2-15} it is an ultra-filter.

Corollary 1-2-24[2]: For any infinite set $\mathcal{M}$ there exists a non-principle ultra-filters $\mathcal{F}'$on $\mathcal{M}$.

Proof:

By Theorem (1-2-23) we extend the frechet filter $\mathcal{F}'$ to an ultra-filter $\mathcal{F}$. Note that $\mathcal{F} \setminus \mathcal{F}'$ consists only of an infinite set. Suppose $\mathcal{F} \setminus \mathcal{F}'$ have a finite set M. Note that $\mathcal{F}'$ is a frechet filter, so $\mathcal{M} \setminus$M $\in \mathcal{F}'$. Now $\mathcal{M} \setminus$M and M are both in ultra-filter $\mathcal{F}$, which is a contradiction. So ultra-filter $\mathcal{F}$ has no finite set, therefore is non-principle.

Corollary 1-2-25[7]: Let $\mathcal{M}$ be a non-empty set then any non-principle ultra-filter $\mathcal{F}$ on $\mathcal{M}$ include a cofinite filter.

Proof:

Let p $\in \mathcal{M}$. Since $\mathcal{F}$ is an ultra-filter, then {p} or $\mathcal{M} \setminus \{p\} \in \mathcal{F}$. But $\mathcal{F}$ is non-principle , implies $\mathcal{M} \setminus \{p\} \in \mathcal{F}$. Now for all D $\subseteq \mathcal{M}$ and D be a finite ,
$\mathcal{M} \setminus D = \bigcap_{p \in D} \mathcal{M} \setminus \{p\} \in \mathcal{F}$.

Definition 1-2-26: Let $\emptyset \neq K \subseteq \mathcal{M}$, the set $\{ D \subset \mathcal{M} : K \subset D\}$ is called filter generated by a set K denoted by $< \{K\} >$. When K is a singleton subset, i.e

المستخلص

الغاية من رسالتي تطبيق التبلوجيا والعمليات الجبرية على الفضاء $\beta\mathbb{N}$. حيث تناولت البحوث والرسائل السابقة حول الفضاء βD بصورة عامة,حيث D مجموعة متقطعة. وكان تركيزنا وصلب اهتمامنا على المجموعة $\mathbb{N}$ (مجموعة الاعداد الطبيعية) لأهمية هذه المجموعة وتطبيقاتها الواسعة بمختلف المجالات من نظرية الاعداد والديناميكية التبلوجية. حيث قمنا بدراسة هذا الفضاء وشكل العناصر الموجودة فيه وتمثلت هذه النقاط ب الالترافلتر. تم في الفصل الأول دراسة موجزه حول هذه النقاط بأنواعها المختلفة وخصائصها وتم الحصول على بعض النتائج الجديدة فيه. أما في الفصل الثاني درسنا خصائص هذه النقاط داخل مجموعة الاعداد الطبيعية وتم تعريف التبلوجيا عليها ودراسة الخصائص التبلوجية ومن ثم توسيع الفضاء $\mathbb{N}$ الى $\beta\mathbb{N}$ الذي هو اكبر فضاء متراص *Hausdorff* متولد من مجموعة الأعداد الطبيعية. وتم ايضا دراسة البنية الجبرية للمجموعة $\beta\mathbb{N}$ من خلال توسيع عملية الجمع المعروفة على الأعداد الطبيعية والحصول على $(\beta\mathbb{N},+)$ من خلال تعريف جمع الالترافلتر ودراسة بعض الخواص الجبرية له.