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CHAPTER ONE

Bornological Structure and it's Applications

1.1 Introduction

Since bornological structure is a very important structure because it has many practical applications to solve the problems of identification and bounded in our daily life, for example, determining the location and identity of a person through an eye print or finger print, and because this structure lacks theoretical sources that facilitate its study. So, the main purpose of this chapter is to explain and clarified the bornological structures. Furthermore, we presented some practical application of this structure in our daily life as well. In addition, we constructed new bornological structures in different ways. For example, our bornological structures from the base and sub base and these structures coinciding with the bornological structures associated with stable operator. Also, we presented many results and characteristics in bornological structures

1.2 Bornological Sets

In this section we explain and clarify bornological structure with some detailed examples.

Definition(1.2.1)(Bornological set) [2]:

A bornological on a set X is a family $\beta \subset \mathcal{P}(X)$ such that:

(i) β covers X, i.e. $X = \bigcup_{B \in \beta} B$;

we can satisfy the first condition in different ways, sometime if the whole set belong to the bornology or $\forall$ x $\in$ X, {x} $\in$ β or X= $\bigcup_{B \in \beta} B$.

(ii) β is hereditary under inclusion, i.e. if $A \subset B$ and $B \in \beta$ then $A \in \beta$;

(iii) β stable under a finite union, i.e., if $B_1, B_2 \in \beta$, then $B_1 \cup B_2 \in \beta$.

A pair (X, β) consisting of a set X and a bornology β on X is called ***a bornological set***, and the elements are called bounded sets.

Definition (1.2.1) (Base of Bornology)[6]: Any sub collection of a bornology collection construct ***a base for bornology*** if every element of the bornology is contain in an element of the base. And we denote it by β_0.

Example (1.2.2):

Let X= {1,2,3}.

β = {{1}, {2}, {3}, {1,2}, {2,3}, {1,3}, X}.

To satisfy that β is a bornology of X, we must satisfy three conditions:

(i) Since $X \in \beta$, then X is covering itself;

(ii) If $A \in \beta$, $B \subseteq A$, then $B \in \beta$;

Since β is the set of all subsets of X, i.e. $\beta = p(X) = 2^x$, then β is stable under hereditary.

(iii) β is stable under finite union, i.e. $\bigcup_{i=1}^{n} B_i \in \beta,\ \forall\, B_1,\ldots,B_n \in \beta$.

Since

$\{1\} \cup \{2\} = \{1,2\}$ $\{1,2\} \cup \{2,3\} = X$.

$\{1\} \cup \{3\} = \{1,3\}$ $\{1,3\} \cup \{2,3\} = X$.

$\{2\} \cup \{3\} = \{2,3\}$ $\{1,2\} \cup \{1,3\} = X$.

Then β is a bornology on X.

Now to find a base for β.

Let $\beta_0=\{X\}$

or $\beta_0 = \{\{1,2\}, \{1,3\}, \{2,3\}, X\}$

It is clear that every element of that bornology is contain in an element of the base.

Example(1.2.3):

let R be the set of real numbers with Euclidean norm (absolute value) we want to define usual bornology on R (canonical bornology), i.e. β is the collection of all usual bounded sets on R.

In fact, a subset B of R is bounded iff there exist bounded interval such that $B \subseteq (a,b)$.

i) Since $\forall\, x \in (a,b)$ (the bounded intervals with respect to absolute value (w.r.t | |) where the absolute value divided R into bounded interval), and $\{x\} \subset (a,b)$ (every subset of bounded interval is bounded), implies that $\forall\, x \in R, \{x\} \in$ β.

Then β is a covers R;

ii) If $B \in$ β and $A \subseteq B$, then ∃ bonded interval s.t $A \subseteq B \subseteq (a,b)$.

Therefore $A \in$ β, and β stable under hereditary;

iii) If $B_1, \ldots, B_n \in$ β, then ∃ $L_1, \ldots, L_n$ least upper bounded and $g_1, \ldots, g_n$ greater lower bounded, such that $\bigcup_{i=1}^{n} B$ has finite union of upper and lower bounded.

Assume that $L = \min\limits_{1 \le i \le n} \{L_i\}$ and $g = \max\limits_{1 \le i \le n}\{g_i\}$, i.e.

$\bigcup_{i=1}^{n} B_i$ has least upper bounded L and greater lower bounded g, i.e.

$\bigcup_{i=1}^{n} B_i$ is bounded subset of R, i.e.

$\bigcup_{i=1}^{n} B_i \in$ β.

Then β is a bornology on R.

And the base of this bornology is:

$$\beta_0 = \{B_r(x) : r \in R , x \in R\} = \{(x - r, x + r) : r \in R, x \in R\}.$$

Example(1.2.4):

Let $X = R^n$ with Euclidean norm

$\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{½}$ for $x_i = (x_1,\ldots,x_n)$

and

$D_r(a) = \{x \in R^n : \|x - a\| \leq r$ for $r \geq 0\}$

be the disk of the radius r with the center at $a = (a_1,\ldots, a_n)$.

A subset B of R^n is bounded if there exists a unit disk with center zero $D_r(0)$ such that

$B \subseteq D_r(0)$.

Let β_u be a family of all bounded subsets of R^n.

$\beta_u = \{B \subseteq R^n : B$ is usual bounded subset of $R^n \}$.

Then (R^n, β_u) is a bornological set and β_u is called canonical (usual) bornology on R^n.

Such that:

i) It is clear that every closed disk in R^n is bounded set, i.e. $D \in \beta$, since R^n is covered by the family of all disks, then β covers R^n;
ii) If $B \in \beta$ and $K \subseteq B$, then $\exists$ a closed disk such that $K \subseteq B \subseteq D$. Therefore $K \in \beta$;
iii) If $B_1, \ldots , B_n \in \beta$, then $\exists$ $D_{r1}, \ldots , D_{rn}$ are closed disks s.t. $B_1 \subseteq D_{r1}, \ldots , B_n \subseteq D_{rn}$.

And $r = \max_{1 < i < n} \{r_i\}$.

Then $\bigcup_{i=1}^{n} B_i \subseteq D_r$.

Thus, the finite union subset of R^n.

Then β_u is a bornology on R^n .

A base of a bornology β on R^n is any subfamily such that

$\beta_0 = \{ D_r(0) : r > 0\}$.

Example(1.2.5):

Let X be a set and β_{finite} be the collection of all finite subsets of X. Then (X, β_{fin}) is a bornological set and β_{fin} is called finite bornology.

Indeed,

i) It is clear that for every $x \in X, \{x\}$ is finite set. Then for all $x \in X, \{x\} \in \beta$.

Thus X is covered by the family of all singleton set. Then β covers X.

ii) Let $B \in \beta$ and $A \subset B$.

Since every subset of finite set is finite set, then $A \in \beta$.

iii) Let $B_1, \ldots, B_n \in \beta$, since the finite union of finite sets is finite set.

Then $\bigcup_{i=1}^{n} B_i$, $\forall\, i = 1, \ldots, n$ is finite set. Thus the finite union is bounded subset of X.

Then β is a bornology on X and it is called finite bornology.

Definition (1.2.6) (bounded map)[6]:

A map between two bornological sets is called ***bounded*** if the image of every bounded set is bounded set.

Definition (1.2.7) (bornological isomorphism) [6]:

A map between two bornological sets (X, β_x) and (Y, β_y) is called ***a bornological isomorphism*** if it is bijective map and f, f^{-1} are bounded.

Consequently, for every B ∈ β_x there is B_1 ∈ B_y, such that:

$$B_1 = f(B)$$

$$B = f^{-1}(B_1).$$

Definition(1.2.8) (subspace) [2]:

Let (X, β) be a bornological space and let $Y \subseteq X$. Then the collection

$$\beta_Y = \{V \cap Y : V \in \beta\}$$

is a bornology on Y.

The bornological space (Y, β_Y) is called ***a subspace*** of (X, β), and β_Y is called relative bornology on Y.

Proposition (1.2.9):

Let (X, β) be a bornological space and let $Y \subseteq X$. Then the collection

$\beta_Y = \{V \cap Y : V \in \beta\}$ is a bornology on Y.

Proof:

(i) Let $B = V \cap Y$ and $V \in \beta$, to prove that β_Y is covering of Y, i.e. $Y = \bigcup_{B \in \beta y} B$

$$\bigcup_{B \in \beta y} B = \bigcup_{B \in \beta y} (V \cap Y)$$

$$= \left(\bigcup_{v \in \beta} V \right) \cap Y.$$

Where

$$\bigcup_{v \in \beta} V = X \text{ and } X \cap Y = Y$$

Then $Y = \bigcup_{B \in \beta y} B$, then β_Y is covering of Y.

(ii) Let $B \in \beta_Y$, i.e. $B = V \cap Y$, $V \in \beta$ and $A \subseteq Y$, $A \subseteq B \longrightarrow A \subseteq V \cap Y$.

To prove $A \in \beta_Y$ we must satisfy that $A = U \cap Y$, where $U \in \beta$

$\because A \subseteq V \cap Y \longrightarrow A \subseteq V \cap A \subseteq Y$.

$\because V \in \beta$, then $A \in \beta$.

Take $A = U$

$\therefore A = U \cap$Y and $U \in \beta$.

Then $A \in \beta_Y$.

(iii) Let $\{ B_i \}_{i=1,\ldots n}$, be a finite element of β_Y, to prove β_Y is stable under finite union .

Since $\forall$ i = 1, 2 ,…, n $\exists\, V_i \in \beta$ such that $B_i = V_i \cap Y$

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (V_i \cap \mathrm{Y}) = \left(\bigcup_{i=1}^{n} B_i \right) \cap Y = V \cap Y \text{ (since } \beta \text{ is stable under finite union) .}$$

Then $V \cap Y \in \beta_Y$, i.e. $\bigcup_{i=1}^{n} B_i \in \beta_Y$.

Remark(1.2.10)

It is clear that every bounded subset of Y is also a bounded subset of X, i.e. $\beta_Y \subseteq \beta$.

Definition (1.2.11) (Product Bornology) [2]:

Let (X_1, β_1) and (X_2, β_2) be two bornological sets. The family of all $B_1 \times B_2$, where $B_1 \in \beta_1$ and $B_2 \in \beta_2$, defines a bornological structure in $X_1 \times X_2$. The product set $X_1 \times X_2$ with this bornological structure is called ***a bornological product space*** of (X_1, β_1) and (X_2, β_2).

1.3 Some Practical Applications of Bornological Spaces

In this section we show some of the practical applications for bornological structure. In other words, how to apply a bornology to solve many problems in our live. As we know, the effect of bornology is to determine the boundedness for sets, vector spaces, groups, any space in general way not just by norm or usual definition of bounded set for all these structures. That means, this structure is general solution to solve the bounded problems.

The most important practical application for bornology, in spyware program KPJ.

Exactly, when they want to determine person location or the identity of person from his finger print or an eye print, let we explain how.

First of all, let assume that the person is original point, to determine his signed (status). We start to study the behavior of objects within his domain. That means, we study the frequency of these objects by introducing open unit ball.

$B_1 = \{ x \in V, P(x) < 1\}$ or closed unit ball $B_1 = \{ x \in V, P(x) \leq 1\}$.

Then we need the length of one of these objects B (the distance between the object and the person whose location is to be determined, by the norm given that the person is the original point).

But, we want to study the behaver for another object, by find the length of B_1(see figure 1).

But B_1 is absorbent disk (A set A ***absorbs*** B i f there exists $a \in R$, a > 0 , such that $\delta A > B$ whenever $|\delta| \geq a$.

A set A is ***absorbent*** in a vector space E if A absorbs every subset of E consisting of a single point), i.e. absorbs every subset of V (where V the space in which the person to be located is located) which consist singleton element (point).

Since B_1 it is in vector space, then it is allowed to multiple B_1 by scaler and make B_1 bigger, we get open unit ball B_2. In the end, we get collection of open balls covers the place and the finite union which it is the bigger bounded set should also be within the place not outside, i.e. inside the collection β. Furthermore, if the father belong for this family, then the son also belong for this family. So, the hereditary property is satisfy.(see figure 1).

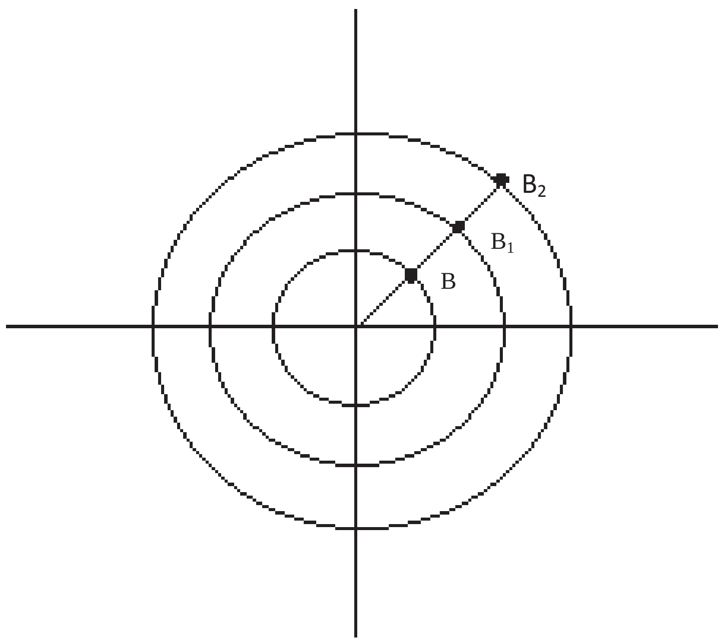

FIGURE 1. The behavior of objects within his domain by introducing open unit ball.

Another application of bornology is to determine the person from his stamp (finger print) or his eye print. (see figures 2 and 3).

FIGURE 2. Application of bornology is to determine the person from his eye print

FIGURE 3. b: Application of bornology is to determine the person from his stamp (finger print).

Also, there are many other applications of bornology, for example to equip a building with the internet service. We take the center point on the surface, and the internet waves are the bounded interval and therefor the collection of these open interval which it is covers the surface, also stable under finite union and hereditary property can determine the place or the building .

Add to that, there is an important application of bornology. Such that, in satellite broadcast system when they want to determine the limits of the broadcast area. (see figure 4).

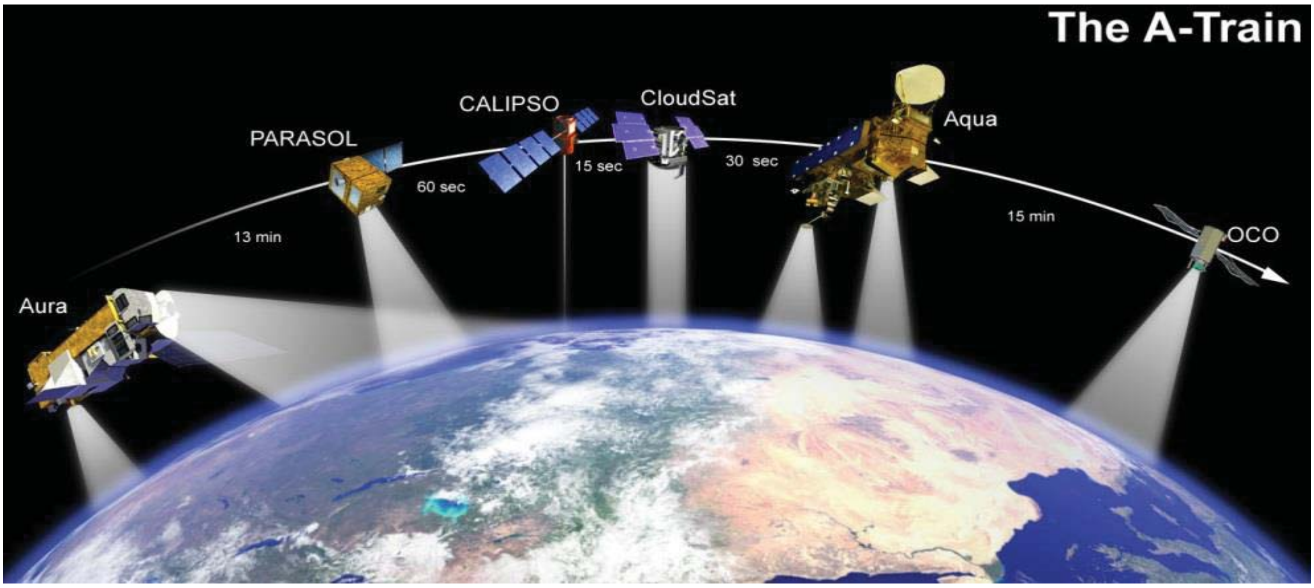

FIGURE 4. satellite broadcast system when they want to determine the limits of the broadcast area

1.4 Induced New Bornological Structures

In this section we construct new bornological structures in different ways. First of all, we recall the concept of base of bornology from section one which states. Any sub collection of a bornology collection construct a base for bornology if every element of the bornology is contain in an element of the base [2].We denote it by β_0.

We may ask when a family of subsets of X forms a base for the bornology on X, what is the form of the bornology generated by this base? The next theorem will be answer the question:

<u>Theorem(1.4.1)</u>

Let (X, β) be a bornological space and β_0 is the base of β, then the following holds:

i) $X = \bigcup_{B_0 \in \beta_0} B_0$.

ii) Finite union of elements of β_0 is contained in an element of β_0.

If and only if; If $X \neq \emptyset$ and β_0 is a family of subsets of X satisfying :

(i) and (ii) then $\exists$ a bornology β on X s.t $\beta=\{ B \subseteq X, \exists B_0 \in \beta_0 , B \subseteq B_0 \}$.

And, β_0 is a base for β.

Proof:

(i) Since β_0 formed a base of a bornology β on a set X.

Then every element of β is contained in an element of β_0 , i.e.

$$\forall B \in \beta, \exists B_0 \in \beta_0$$

الخلاصة:

إن الغرض من هذه الرسالة هو شرح وتوضيح البنى البرنولوجية لبعض البنى الجبرية. على وجه الخصوص ، المجاميع البرنولوجية و الزمر البرنولوجية و أشباه الزمر البرنولوجية.

إن الدافع لدراسة البنى البرنولوجية هو حل مشكلة التقييد للمجاميع والدوال والفضاءات بشكل عام وليس بالمعنى المعتاد للتقييد في هذه البنى.

في البداية، نذكر بعض الرموز والتعاريف الأساسية للهياكل البرنولوجية ، كما نقدم العديد من التطبيقات العملية المهمة لهذه الهياكل لحل العديد من المشاكل في حياتنا اليومية. ثم ندرس بعض الخصائص والنتائج ، ومن بعض هذه النتائج أننا انشأنا بنى او تراكيب برنولوجية جديدة بطرق مختلفة.

كما ذكرنا مفهوم الزمر البرنولوجية مع امثلة مفصلة عنها.النتيجة الأكثر أهمية في هذا العمل هي عبارة عن تركيب جديد يسمى زمرة شبه برنولوجية حيث أن الغرض من إنشاءها ودراستها هو لحل مشكلة تقييد الزمر التي لا يمكن تقييدها لأن دالة الضرب تكون غير محددة او غير مقيدة. الدافع وراء دراسة هذا الهيكل الجديد هو أن التحويلات اليسرى (اليمنى) تكون متماثلة لذلك فإن هذا التركيب الجديد سيكون متجانس على العكس من شبه الزمرة البرنولوجية التي تكون غير متجانسة.

تمت دراسة المزيد من الخصائص للزمرة شبه البرنولوجية لإعطاء الشرط الكافي لجعل أي زمرة شبه برنولوجية هي زمرة برنولوجية.

اخيرًا ، نناقش نظرية الفئة بالمعنى العام. على وجه الخصوص ، تمت مناقشة مفهوم الضرب المباشر والضرب الاساسي والمعادلات. و نظرًا لأن العلم الحديث يضع كل بناء (تركيب) جديد في فئة ، فمن الطبيعي أنه بعد إنشاء أي هيكل جديد ، يتم وضعه ضمن منطقة الفئة الخاصة به وهذا هو الدافع للفصل الثالث الذي يناقش نظرية التصنيف للزمر شبه البرنولوجية.

نذكر بعض النتائج المهمة لهذا العمل:

- التركيب البرنولوجي هو حل عام لمشكلة التقييد للمجموعات والدوال والفضاءات.
- هناك العديد من التطبيقات العملية الهامة للفضاء البرنولوجي في حياتنا اليومية.
- استحداث تراكيب برنولوجية جديدة بطرق مختلفة.
- مراجعة مفهوم الزمرة البرنولوجية واعطاء بعض الامثلة المفصلة لتوضيحه.
- إعطاء حل لمشكلة تقييد الزمر التي لا يمكن تقييدها لأن دالة الضرب تكون غير مقيدة وذلك من خلال تقديم هيكل جديد يسمى (الزمرة شبه البرنولوجية).
- كما تمت دراسة المزيد من الخصائص لهذا الهيكل الجديد و إعطاء الشرط الكافي لجعل اي زمرة شبه برنولوجية هي زمرة برنولوجية وهو ما يعتبر النتيجة الرئيسية لهذا العمل.
- كل تحويل أيسر (أيمن) يكون متماثلاً برنولوجياً. لذا ، فإن الزمرة شبه البرنولوجية متجانسة. لكن شبه الزمرة البرنولوجية ليست متجانسة.
- مناقشة نظرية الفئة لهذه البنية الجديدة.