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CHAPTER ONE

FUNDAMENTAL CONCEPTS WITH SOME RESULTS

1.1 Introduction

In this chapter, we have given a basic information, which are useful in our work. We defend the left (right) ideal on semi-group K this will lead us for the smallest ideal $M(K)$, which is the union of minimal left (or right) ideal. We explore the concepts of a filter and ultra-filters, which will be used to define the set $\beta\mathbb{N}$ for more information you can see [27]. In addition, several proofs.

1.2 Some Basic definitions and properties

Definition 1.2.1 [15]: A semi-group is a pair $(K,*)$ where K is non-empty set and $*$ is a associative binary operation on K.

Formally a binary operation on K is a function $*: K \times K \longrightarrow K$ such that the operation is associative iff $(p * q) * r = p * (q * r)$ for all p, q and r in K. Also K is closed under $*$ if $p * q \in K$ for any $p, q \in K$.

Example 1.2.2: Each of the following is a semi-group

1- The set of natural numbers $\mathbb{N}$ under multiplication or addition is a semi-group.

2- $(K,*)$ where K is a non-empty set where $x * y \ = x$ or y for all $x, y \in K$ is a semi-group.

3- $(\mathbb{N},\vee)$ such that $p \vee q = \max\{p, q\}$, where $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

Definition 1.2.3 [14]: Let K be a semi-group, and let $\mathcal{F}, \mathcal{R}$ and I be a non-empty subset of K then:

1- $\mathcal{F}$ is a **left ideal** of K if and only if $\emptyset \neq \mathcal{F} \subseteq K$ and $K\mathcal{F} \subseteq \mathcal{F}$.

2- $\mathcal{R}$ is a **right ideal** of K if and only if $\emptyset \neq \mathcal{R} \subseteq K$ and $\mathcal{R}K \subseteq \mathcal{R}$.

3- I is an **ideal** of K if and only if I is a left ideal and right ideal of K.

Example 1.2.4 [20]:

1- Let K be a semi-group. If z is a zero in K then $\{z\}$ is an ideal in K.

2- In the commutative semi-group $(\mathbb{N}, +)$, the ideals are the sets $[a, \infty) = \{n \in \mathbb{N} : a \leq n\}$, where a is an arbitrary element of $\mathbb{N}$.

Definition 1.2.5 [14]: Let K be a semi-group, $\mathcal{R}$ is a right ideal of K, and $\mathcal{F}$ left ideal of K. Then

1- $\mathcal{F}$ is a **minimal left ideal** of K if and only if $\mathcal{F}$ is a left ideal of K and whenever J is a left ideal of K and $J \subseteq \mathcal{F}$ one has $J = \mathcal{F}$.

2- $\mathcal{R}$ is a **minimal right ideal** of K if and only if $\mathcal{R}$ is a right ideal of K and whenever J is a right ideal of K and $J \subseteq \mathcal{R}$ one has $J = \mathcal{R}$.

Example 1.2.6 [20]:

1- A Semi-groups with a zero has only one minimal left (right two-sided) ideal of K namely the trivial one $\{0\}$.

2- The integer numbers with addition $(Z, +)$ has no trivial minimal ideal.

The next Lemma can, be found in [11] as a problem, and will be given a proof for that.

Lemma 1.2.7 [11]: Let K be a semi-group.

1- Let $\mathcal{F}$ and L be left ideals of K. Then $\mathcal{F} \cap L$ is a left ideal of K if and only if $\mathcal{F} \cap L \neq \emptyset$.

2- Let $\mathcal{R}$ be a right ideal of K and let $\mathcal{F}$ be a left ideal of K. Then $\mathcal{R} \cap \mathcal{F} \neq \emptyset$.

Proof:

1- Suppose that $\mathcal{F} \cap L$ is a left ideal immediately by definition of left ideal we get $\mathcal{F} \cap L \neq \emptyset$. Conversely, suppose $\mathcal{F} \cap L \neq \emptyset$.

To show $\mathcal{F} \cap L$ left ideal we need to show $K(\mathcal{F} \cap L) \subseteq \mathcal{F} \cap L$. Let $x \in K(\mathcal{F} \cap L)$, so $x = ky$ where $k \in K$ and $y \in \mathcal{F} \cap L$. But $\mathcal{F}$ and L are left ideal then $ky \in \mathcal{F}$ and $ky \in L$, but $ky = x$ hance $x \in \mathcal{F} \cap L$.

2- Let $x \in \mathcal{R}$ and $y \in \mathcal{F}$ then $xy \in \mathcal{R}$ and $xy \in L$ by definition (1.2.3).

Lemma 1.2.8 [11]: Let K be a semi-group and let $t \in K$. Then Kt is a left ideal, tK is a right ideal and KtK is an ideal.

Proof: See the proof of Lemma (1.30 part a) in [11].

Theorem 1.2.9 [20]: Let K be a semi-group.

1- If $\mathcal{F}$ is a left ideal of K and $x \in \mathcal{F}$, then $Kx \subseteq \mathcal{F}$.

2- Let $\emptyset \neq \mathcal{F} \subseteq K$. Then $\mathcal{F}$ is a minimal left ideal of K if and only if for each $x \in \mathcal{F}$ implies $Kx = \mathcal{F}$.

Proof:

For part 1: This follows immediately from the definition of left ideal.

For part 2: Assume that $\mathcal{F}$ is a minimal left ideal of K and $x \in \mathcal{F}$. By Lemma (1.2.8) Kx is a left ideal and $Kx \subseteq \mathcal{F}$ by part (1) above. Since $\mathcal{F}$ is minimal left ideal, hence $Kx = \mathcal{F}$.

Conversely, Suppose $\mathcal{F}$ is a left ideal. Let L be a left ideal of K with $L \subseteq \mathcal{F}$. Pick $x \in L$. Then by part (1) above, $Kx \subseteq L$ and so $L \subseteq \mathcal{F} = Kx \subseteq L$. Hence $\mathcal{F}$ is a minimal.

Theorem 1.2.10 [11]: Let $\mathcal{F}$ be a minimal left ideal of the semi-group K, and let $J \subseteq K$. Then J is a minimal left ideal of K if and only if there is some $t \in K$ such that $J = \mathcal{F}t$.

Proof:

Assume J is a minimal left ideal of K and pick $t \in J$. Since $K\mathcal{F}t \subseteq \mathcal{F}t$ and $\mathcal{F}t \subseteq KJ \subseteq J$ then $\mathcal{F}t$ is left ideal of K in J. But, J is minimal and so $\mathcal{F}t = J$. Conversely, let $t \in K$ clearly $\mathcal{F}t \subseteq \mathcal{F}$ and a left ideal of K. But $\mathcal{F}$ is a minimal left ideal, so $\mathcal{F}t = \mathcal{F}$. Thus, $\mathcal{F}t$ is a minimal left ideal implies J is a minimal left ideal.

Corollary 1.2.11 [20]: Let K be a semi-group. If K has a minimal left ideal, then every left ideal of K contains a minimal left ideal.

Proof:

Let $\mathcal{F}$ be a minimal left ideal of K and let L be a left ideal of K. Pick $x \in L$. Then by Theorem (1.2.10), $\mathcal{F}x$ is a minimal left ideal which is contained in L.

Remark 1.2.12: We will denote of the **smallest ideal** set in the semi-group K by $M(K)$ which is the set contained in every ideal in K.

Theorem 1.2.13 [11]: Let K be a semi-group. If K has a minimal left ideal, then $M(K)$ exists and $M(K) = \cup \{\mathcal{F} : \mathcal{F}$ is a minimal left ideal of $K\}$.

Proof:

Let $\mathcal{H} = \cup \{\mathcal{F}: \mathcal{F}$ is a minimal left ideal of $K\}$. First we need to show that $\mathcal{H}$ is a minimal ideal. Let $\mathcal{F} \in \mathcal{H}$ be a minimal left ideal and let I be any ideal of K. By Lemma (1.2.7) part (2), $\mathcal{F} \cap I \neq \emptyset$. Let $s \in \mathcal{F} \cap I$ and $t \in K$, implies $ts \in \mathcal{F} \cap I$. So $\mathcal{F} \cap I$ is a left ideal and a subset of minimal left ideal $\mathcal{F}$. Therefore $\mathcal{F} \cap I = \mathcal{F}$. Since $\mathcal{F} \subseteq I$ hence $\mathcal{H} \subseteq I$, which implies that $\mathcal{H}$ is the smallest. It suffices to show that $\mathcal{H}$ is an ideal of K. Note that $\mathcal{H} \neq \emptyset$ by assumption. Let $s \in \mathcal{H}$ and pick a minimal left ideal $\mathcal{F}$ such that $s \in \mathcal{F}$. Then $ts \in \mathcal{F} \subseteq \mathcal{H}$, for all $t \in K$. Hence $\mathcal{H}$ is left ideal. By Theorem (1.2.10), $\mathcal{F}t$ is a minimal left ideal of K, so $\mathcal{F}t \subseteq \mathcal{H}$ while $st \in \mathcal{F}t$.

The next Lemma can, be found in [11] as a problem, and will be given a proof for that.

Lemma 1.2.14 [11]: Let K be a semi-group.

1- Let $\mathcal{F}$ be a left ideal of K. Then $\mathcal{F}$ is minimal if and only if $\mathcal{F}t = \mathcal{F}$ for every $t \in \mathcal{F}$.

2- Let J be an ideal of K. Then J is the smallest ideal if and only if $JtJ = J$ for each $t \in J$.

Proof:

1- If $\mathcal{F}$ is a minimal and $t \in \mathcal{F}$, then $\mathcal{F}t$ is a left ideal of K and $\mathcal{F}t \subseteq \mathcal{F}$, so $\mathcal{F}t = F$. Now assume $\mathcal{F}t = \mathcal{F}$ for every $t \in \mathcal{F}$ and let L be a left ideal of K with $L \subseteq \mathcal{F}$. Pick $t \in L$. Then $\mathcal{F} = \mathcal{F}t \subseteq \mathcal{F}L \subseteq L \subseteq \mathcal{F}$.

2- Suppose J is smallest ideal then $J = \cup$ {minimal left ideals}. Since J is an ideal, $t \in J$, then $JtJ \subseteq J$ by definition of ideal. Since J is smallest ideal then $J \subseteq JtJ$, hence $JtJ = J$. Conversely, suppose that $JtJ = J$, for each $t \in J$, to show J is the smallest ideal. Let I be an ideal of K such that $I \subseteq J$. Let $t \in I$, then is $t \in J$ implies $J = JtJ \subseteq JIJ \subseteq I \subseteq J$, and hence $J = JtJ$ is a minimal ideal. To show J is smallest ideal. Let I be an ideal of K to show $J \subseteq I$. Note that $I \cap J \neq \emptyset$, let $a \in I, b \in J$, this is $ab \in I$,and $ab \in J$. To show $J \cap I$ is an ideal, let $x \in J \cap I$,$t \in K$

$\Rightarrow tx \in J$ and $tx \in I$, hence $J \cap I$ is left ideal.

Similarly, $J \cap I$ is right ideal. This leads $J \cap I \subseteq J, I$

$$\Rightarrow J \cap I = J$$

$$\Rightarrow J \subseteq I$$

Hence J is the smallest ideal.

Theorem 1.2.15 [11]: Let K be a semigroup. If $\mathcal{F}$ is a minimal left ideal of K and $\mathcal{R}$ is a minimal right ideal of K, then $M(K) = \mathcal{FR}$.

Proof:

Clearly $\mathcal{FR}$ is an ideal of K. By Lemma (1.2.14 part 2), we need to show that $M(K) = \mathcal{FR}$. So, let $y \in \mathcal{FR}$. Then $\mathcal{FR}y\mathcal{F}$ is a left ideal of K which is contained in $\mathcal{F}$. So $\mathcal{FR}y\mathcal{F} = \mathcal{F}$ and hence $\mathcal{FR}y\mathcal{FR} = \mathcal{FR}$ since $\mathcal{F}$ is minimal left ideal.

Theorem 1.2.16 [11]: Let K be a semi-group and assume that there is a minimal left ideal of K which has an idempotent. Then every minimal left ideal has an idempotent.

Proof: See the proof of Theorem (1.56) in [11].

Definition 1.2.17 [18]: Let K be a semi-group and $x, y \in K$ we define the **left (resp. right) translations** on a function $\lambda_x: K \longrightarrow K$ (resp. $\rho_x: K \longrightarrow K$) as follows: $\lambda_x(y) = xy$ (resp. $\rho_x(y) = yx$).

Definitions 1.2.18 [11]:

1- The triple $(K, ., \tau)$ is called **right topological semi-group** where $(K, \cdot)$ is a semi-group, and (K, τ) is a topological space, if for all $x \in K$, $\rho_x : K \rightarrow K$ is continuous.

2- The triple $(K, ., \tau)$ is called **left topological Semi-group** where $(K, \cdot)$ is a semi-group, and (K, τ) is a topological space, if for all $x \in K$, $\lambda_x : K \rightarrow K$ is continuous

3- If the triple $(K, ., \tau)$ is a right topological semi-group and a left topological semi-group then $(K, ., \tau)$ is **a semi topological semi-group**.

4- The triple $(K, ., \tau)$ is **topological semi-group** where $(K, \cdot)$ is a semi-group, and (K, τ) is a topological space, if $\cdot : K \times K \to K$ is continuous.

Definition 1.2.19 [11]: We define the **topological center** of the semi-group K denoted by $\Lambda(K)$ which is define as follows: $\Lambda(K) = \{x \in K: \lambda_x$ is continuous$\}$, where K be a right topological semi-group.

Note 1.2.20: The center $\Lambda(K)$ is itself a semi-subgroup of K. Moreover $\Lambda(K) = K$ if and only if K is semi-topological semi-group.

Definition 1.2.21 [18]: A topological space $X$is Hausdorff (T_2-spaces) if for every $x, y \in X$ with $x \neq y$, there exist disjoint open subsets U, V of X such that $x \in U$ and $y \in V$.

Zorn's Lemma 1.2.22 [5]: If $(K, \leq)$ is a partially ordered set such that any increasing chain $k_1 \leq \cdots \leq k_i \leq \cdots$ has a supremum in K, then K itself has a maximal element.

The next theorem it is a fundamental important theorem that is related the compact right topological semi-group corresponding with the idempotent.

Theorem 1.2.23 [11]: Let K be a Hausdorff compact right topological semi-group. Then K contains at least one idempotent.

Proof:

Define the set $\mathcal{W} = \{Y \subseteq K: Y \neq \emptyset, Y$ is compact and $Y \cdot Y \subseteq Y\}$ which is the set of compact sub semi-groups of K. Note that $K \in \mathcal{W}$, So $\mathcal{W} \neq \emptyset$. Let $\mathcal{J}$ be a chain in $\mathcal{W}$. Since K is a Hausdorff consequently $\mathcal{J}$ is a collection of closed subsets from the compact space K. Hence, it has finite intersection property. So $\cap \mathcal{J} \neq \emptyset$ which is trivially compact and semi-group.

Implies $\cap \mathcal{J} \in \mathcal{W}$. So by Zorn's Lemma $\mathcal{W}$ has a minimal member say B. We need to show B is one member of $\mathcal{W}$.

Let $A = Bb$ where $b \in B$ then $A \neq \emptyset$. Since $A = \rho_b[B]$, then A is the continuous image of a compact space, hence it is compact.

Also $AA = BbBb \subseteq BBBb \subseteq Bb = A$, thus $A \in \mathcal{W}$. Since $A = Bb \subseteq BB \subseteq B$ and B is minimal of $\mathcal{W}$, so $A = B$. Let $C = \{x \in B : xb = b\}$. Note that $b \in B = Bb$, then $C \neq \emptyset$. Also, since $C = B \cap \rho_b^{-1}[\{b\}]$, so C is closed and implies its compact. Now given $x, c \in C$ one get $xc \in BB \subseteq B$ and $xcb = xb = b$ so $xc \in C$. Thus $C \in \mathcal{W}$. Since $C \subseteq B$ and B is minimal, Then $C = B$, so $b \in C$ and so $bb = b$.

Corollary 1.2.24 [11]: If K be a compact right topological semi-group. Then K has a minimal left ideal. More generally all-minimal left ideals in K will be closed and have an idempotent.

Proof:

Suppose $\mathcal{F}$ be a left ideal of K and let $x \in \mathcal{F}$. Since we have Hausdorff space, then $Kx = \rho_x(K)$ is a closed compact left ideal in $\mathcal{F}$. It follows any minimal left ideal is closed. By using the proof of Theorem (1.2.23), we have that any minimal left ideal has an idempotent. To complete the proof, we need to show that this satisfying for any left ideal of K contains a minimal left ideal. Let $\mathcal{F}$ be a left ideal of K and consider a set $\mathcal{H} = \{Y : Y \text{ is a closed left ideal of } K \text{ and } Y \subseteq \mathcal{F}\}$ which is partially ordered by inclusion. Note $\mathcal{H} \neq \emptyset$ since at least we have a left ideal Kx. Applying Zorn's Lemma, $\mathcal{H}$ has a minimal left ideals L. Since L is a minimal among these left closed ideals in $\mathcal{F}$, also since every left ideal contains a closed left ideal. Therefore L is a minimal left ideal.

1.3 A Filter and an Ultra-filter

Definition 1.3.1 [26]: Let K be any set, a **filter** on a set K is a non-empty set μ with the following properties:

1- $\emptyset \notin \mu$.

2- If $\mathcal{P}, q \in \mu$ then $\mathcal{P} \cap q \in \mu$.

3- If $\mathcal{P} \in \mu$ and $\mathcal{P} \subseteq q \subseteq K$ then $q \in \mu$.

Example 1.3.2: Consider the set μ to be a neighborhood of a point a in a topological space X. Then μ is a filter.

1- It's clearly that for any neighborhood of a point a say $q = (a-\in, a+\in)$ we have $\emptyset \notin \mu$.

2- Let $q = \left[a - \frac{\in}{2}, a + \frac{\in}{2}\right], \mathcal{P} = \left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \in \mu$ then

$$q \cap \mathcal{P} = \left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \in \mu.$$

3- Take $q = \left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \in \mu$, and $\mathcal{P} = \left[a - \frac{\in}{2}, a + \frac{\in}{2}\right]$ be a neighborhood for some point b, such that $\left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \subseteq \left[a - \frac{\in}{2}, a + \frac{\in}{2}\right] \subseteq K$ then $\left[a - \frac{\in}{2}, a + \frac{\in}{2}\right] \in \mu$.

Remarks 1.3.3 [4]:

1- The union of two filters on a set need not be a filter, for the counter example see example (2.1.4) (ii) in [4].

2- The intersection of all filters on K is the filter $\{K\}$ which is the weakest filter on K.

In the next following definition, we will introduce another important type of filter.

Definition 1.3.4 [26]: A filter μ on a set K is called an **ultra-filter** if it is not properly contained in any other filter on K.

Note 1.3.5 [8]: A filter μ on K is an **ultra-filter** if and only if for every $A \subseteq K$ either $A \in \mu$ or $A^c \in \mu$.

We record immediately the following very simple but also very useful fact about ultra-filters**.**

Example 1.3.6: 5: Let $K = \{a, b, c\}$

$\mu_1 = \{K\}$, $\mu_2 = \{\{a, b\}, K\}$, $\mu_3 = \{\{b, c\}, K\}$, $\mu_4 = \{\{c, a\}, K\}$,

$\mu_5 = \{\{a\}, \{a, b\}, \{a, c\}, K\}$, $\mu_6 = \{\{a, b\}, \{b, c\}, K\}$,

$\mu_7 = \{\{b\}, \{a, b\}, \{b, c\}, K\}$, $\mu_8 = \big\{\{c\}, \{c, a\}, \{b, c\}, K\big\}$

The filter μ_5, μ_7 and μ_8 are an ultrafilter on $K = \{a, b, c\}$ since there are no filter on K stecictly fine than μ_5, μ_7 and μ_8.

Remark 1.3.7 [11]: Let K be a set and let μ and υ be two ultra-filters on K. Then $\mu = \upsilon$ if and only if $\mu \subseteq \upsilon$.

Remark 1.3.8: Let K be a non-empty set and μ be a filter on K then by definition for every $q \subseteq K$ either $q \in \mu$ or $K \backslash q \in \mu$.

Definition 1.3.9 [26]: A filter μ on a set K is **principal** if there is a non-empty set $X \subseteq K$, such that $\mu = \{A \subseteq K : X \subseteq A\}$. Otherwise, μ is a non-principal.

Remark 1.3.10 [11]: Every ultra-filter on a finite set K is principal. Moreover, no principal ultra-filter is any ultra-filter on infinite set.

1.4 Topological Space on set $\beta\mathbb{N}$

In this section, we will define a topology on the set of ultra-filters on a special case for the set of natural number $\mathbb{N}$ and establish some of the properties. This will lead us to define the stone-Čech compactification space on $\mathbb{N}$.

Definition 1.4.1: Let $\mathbb{N}$ be a discrete topological space of natural number $\mathbb{N}$. We define the set of $\beta\mathbb{N} = \{ q : q$ is an ultra-filter on $\mathbb{N}\}$ that is the set of all ultra-filters on a set $\mathbb{N}$.

We will define a topology on the set of $\beta\mathbb{N}$ by describing a base explicitly and we shall be thinking of the ultra-filters as a point in this topology space $\beta\mathbb{N}$.

Definition 1.4.2: Let $\mathbb{N}$ be a discrete topological space we define the set.

1- For any $\mathcal{M}$ subset of $\mathbb{N}$, $\widehat{\mathcal{M}} = \{q \in \beta\mathbb{N} : \mathcal{M} \in q\}$, where q is an ultra-filter on $\mathbb{N}$.

2- Let $m \in \mathbb{N}$, then $e(m) = \{\mathcal{M} \subseteq \mathbb{N}, m \in \mathcal{M}\}$.

الخلاصة

الهدف الرئيسي المراد في هذا العمل تم من خلال عرض التعقيدات الكاملة لمفهوم منطوي شبه الزمرة يمكن أن تظهر في نظام أصغري. ونرى أن الدراسة في هيكل المنطوي شبه الزمرة في لنظام الأصغري هو موضوع جدير بالاهتمام في دراسة بعض المشكلات ذات أهمية للباحثين في طبيعة النظرية البحتة لشبه الزمرة. تمت دراسة المثاليات اليسارية الاصغرية $\mathcal{F}$ في فضاء التراص شبه الزمرة الشامل βK لشبه الزمرة التبولوجية K. تم البرهنة على أن المنطوي لشبه الزمرة $\mathcal{F}$ متماثل الشكل إلى βK إذا وفقط إذا أعطيت $q \neq r$ في βK ، فيمكن ايجاد عنصر p في المثالي الاصغري في الفضاء βK يحقق ان $q \cdot p \neq r \cdot p$. نشتق العديد من الشروط ، بعضها تضمن حول النظام الاصغري والذي يمكننا القدرة على فصل النقاط q و r باستخدام هذه الشروط. ومن خلال أخذ حالة خاصة والتي يكون فيها شبة الزمرة $K = \mathbb{N}$، وفضاء التراص هو $\beta \mathbb{N}$ تم اعطاء تطبيقًا بأخذ مجموعة قابلة للعد لنظام تكراري من الدوال مكونة من عديدين مع نظام محدد بشروط متعددة وتم دراسة النظام الادنى وكيفية فصل النقاط q و r بهذه الطريقة التي تمت دراستها.