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Chapter One

Introduction and Preliminaries

1.1 Preliminaries

A brief overview of some definitions and results of exact homology sequences from [5], [18] , [23], and [25] is given in this section. [12], [16], and [18] are used for the background on rings and modules.

Definition 1.1.1:[18]

consider a sequence (finite or infinite) of abelian group and homomorphism

$$\dots D1 \xrightarrow{\varphi_1} D2 \xrightarrow{\varphi_2} D_3 \rightarrow \dots$$

This sequence is said to be *exact* at D_2 if and only if Im (φ_1) =Ker(φ_2). If it is everywhere exact, then it is said to be "an *exact sequence*".

Theorem 1.1.2: [18]

(1) $D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} 0$, is exact sequence if and only if φ_1 is epimorphism.

(2) $0 \xrightarrow{\varphi_1} D_1 \xrightarrow{\varphi_2} D_2$ is exact sequence if and only if φ_2 is monomorphism.

Definition 1 .1 .3: [16]

An exact sequence of the form

$$0 \rightarrow D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} D_3 \xrightarrow{\varphi_3} 0$$

is called a *short exact sequence*. A diagram of modules

$$\begin{array}{ccc} D_1 & \overset{\varphi_1}{\rightarrow} & D_2 \\ \Psi_1 \downarrow & & \downarrow \Psi_2 \\ D_3 & \underset{\varphi_2}{\longrightarrow} & D_4 \end{array} \quad .$$

and homomorphisms is said to be *commutative* if and only if $\Psi_2\, \varphi_1 = \varphi_2\, \Psi_1$.

Theorem 1.1.4:[18]

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & D_2 & \overset{\emptyset}{\longrightarrow} & D_1 & \overset{\Psi}{\longrightarrow} & D_0 & \rightarrow & 0 \\ & & \delta_2 \downarrow & & \downarrow \delta_1 & & \downarrow \delta_0 & & \\ 0 & \rightarrow & M_2 & \overset{\emptyset'}{\longrightarrow} & M_1 & \overset{\Psi'}{\longrightarrow} & M_0 & \rightarrow & 0 \end{array}$$

with exact rows. If any two of the three homomorphisms δ_0, δ_1 and δ_2 are isomorphism, then the third is an isomorphism too.

Lemma 1.1.5: [16]

Suppose φ : D→M is epimorphism with kernel K, then the sequence

$0 \rightarrow K \overset{\iota}{\longrightarrow} D \overset{\varphi}{\longrightarrow} M \overset{\Psi}{\longrightarrow} 0$ is exact where ι is the inclusion map.

Theorem 1.1.6:[16]

Suppose that the sequence $D_1 \overset{\varphi_1}{\rightarrow} D_2 \overset{\varphi_2}{\rightarrow} D_3 \overset{\varphi_3}{\rightarrow} D_4$ is exact, then the following are equivalent:

1- φ_1 is epimorphism.

2. φ_2 is the zero homomorphism.

3. φ_3 is monomorphism.

Definition 1.1 .7: [2]

Consider the sequences

$$\dots \to D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} \dots$$

$$\dots \to M_1 \xrightarrow{\Psi_1} M_2 \xrightarrow{\Psi_2} \dots.$$

A homomorphism from the sequence $\dots \to D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} \dots$ into the sequence $\dots \to M_1 \xrightarrow{\Psi_1} M_2 \xrightarrow{\Psi_2} \dots$ is a family of homomorphism of $\alpha_\iota : D_\iota \to M_\iota$ such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
\cdots \to D_{-1} & \xrightarrow{\emptyset-1} & D_0 & \xrightarrow{\emptyset_0} & D_1 & \to \cdots & \to D_i & \xrightarrow{\emptyset_1} & D_{i+1} & \to \cdots \\
\alpha_{-1} \downarrow & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \downarrow \alpha_i & & \downarrow \alpha_{i+1} & \\
\cdots \to M_{-1} & \xrightarrow{\Psi-1'} & M_0 & \xrightarrow{\Psi_0'} & M_1 & \to \cdots & \to M_i & \xrightarrow{\Psi_1} & M_{i+1} & \to \cdots
\end{array}$$

Definition 1.1.8: [22]

Let C = (C_p , σ_p) and C' = (C'_p , σ'_p) be a chain complexes. A chain map φ: C→ C' is a collection of homomorphism φ_P : $C_p \to C'_p$ such that $\sigma'_p \circ \varphi_P$ = $\varphi_{P-1} \circ \sigma_p$, for all p (i.e., the following diagram commutes)

$$\begin{array}{ccccc}
\cdots \to C_{P+1} & \xrightarrow{\sigma_{P+1}} & C_P & \xrightarrow{\sigma_P} & C_{P-1} \to \cdots \\
\downarrow \varphi_{P+1} & & \downarrow \varphi_P & & \downarrow \varphi_{P-1} \\
\cdots \to C'_{P+1} & \xrightarrow{\sigma'_{P+1}} & C'_P & \xrightarrow{\sigma'_P} & C'_{P-1} \to \cdots
\end{array}$$

Lemma 1.1.9 : [29]

A chain map $\varphi : C \rightarrow C'$ induces a homomorphism

$(\varphi_*)_p : H_p(C) \rightarrow H_p(C')$, for all p given by:

$$(\varphi_*)_p(x + im(\sigma_{p+1})) = \varphi_p(x) + Im(\sigma'_{p+1})$$

Lemma 1. 1.10: [29]

a- The identity map $i : C \rightarrow C$ is a chain map and $(i_*)_p : H_p(C) \rightarrow H_p(C)$ is the *identity homomorphism*.

b- if $\varphi : C \rightarrow C'$ *and* $\Psi : C' \rightarrow \acute{C}'$ are chain maps, then $\Psi o\, \varphi : C \rightarrow \acute{C}'$ is a chain map and $(\Psi \circ \varphi)_* = \Psi_* \circ \varphi_*$

proof.

a- clear by Lemma (1.1.9).

b-consider the following diagram

$$\begin{array}{ccccccc}
\cdots \rightarrow & C_{P+1} & \xrightarrow{\sigma_{P+1}} & C_P & \xrightarrow{\sigma_P} & C_{P-1} & \rightarrow \cdots \\
& \varphi_{p+1}\downarrow & & \varphi_p\downarrow & & \varphi_{p-1}\downarrow & \\
\cdots \rightarrow & C'_{P+1} & \xrightarrow{\sigma'_{P+1}} & C'_P & \xrightarrow{\sigma'_P} & C'_{P-1} & \rightarrow \cdots \\
& \Psi_{p+1}\downarrow & & \Psi_p\downarrow & & \Psi_{p-1}\downarrow & \\
\cdots \rightarrow & C''_{P+1} & \xrightarrow{\sigma''_{P+1}} & C''_P & \xrightarrow{\sigma''_P} & C''_{P-1} & \rightarrow \cdots
\end{array}$$

since the diagram commutes we have $\varphi_{p-1} \circ \sigma_p = \sigma'_p \circ \varphi_p$ and so

$\Psi_{p-1}(\varphi_{p-1} \circ \sigma_p) = \Psi_{p-1}(\sigma'_p \circ \varphi_p)$. Similarly, we have

$\Psi_{p-1} \circ \sigma'_p = \sigma''_p \circ \Psi_p$, and so $\Psi_{p-1} \circ \sigma'_p \circ \varphi_p = \sigma''_p \circ \Psi_p \, o \, \varphi_p$. Therefore, $\Psi_{p-1} \circ \varphi_{p-1} \circ \sigma_p = \sigma''_p \circ \Psi_p \, o \, \varphi_p$.

By definition $(\varphi_*)_p : H_p(C) \to H_p(C')$ is given by

$(\varphi_*)_p(x + im\, \sigma_{p+1}) = \varphi(x) + im\, \sigma'_{p+1}$ and

$(\Psi_*)_p : H_p(C') \to H_p(C'')$ is given by $(\Psi_*)_p(\varphi(x) + im(\sigma'_{p+1})) = \Psi(\varphi(x)) + im(\sigma''_{p+1})$.

Now, $((\Psi \circ \varphi)_*)_p : H_p(C) \to H_p(C'')$ is given by:

$((\Psi \circ \varphi)_*)_p(x + im\, \sigma_{p+1}) = (\Psi \circ \varphi)(x) + im\, \sigma'_{p+1}$.

So, $((\Psi \circ \varphi)_*)_p(x + im\, \sigma_{p+1}) = (\Psi \circ \varphi)(x) + im\, \sigma''_{p+1}$.

$$= \Psi(\varphi(x)) + im\, \sigma''_{p+1}.$$

$$= (\Psi_*)_p(\varphi(x)) + im(\sigma''_{p+1}).$$

Hence $((\Psi \circ \varphi)_*)_p = (\Psi_*)_p \circ (\varphi_*)_p$.

1.2 Graded Rings

In the following section, the concept of graded rings and some of their properties are presented.

Definition 1.2.1: [2□]

Let G be a group with identity e. Then a ring R is said to be *G-graded ring* if there exists an additive subgroup R_g of R such that

$R = \oplus \sum_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ *for all* $g, h \in G$.

We denote the G-graded ring R by (R, G), and we define the support of the graded ring (R, G) by

$$Supp(R,G) = \{g \in G : R_g \neq 0\}.$$

The elements of R_g are called *homogenous* of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g. In addition, we write $h(R) = \bigcup_{g \in G} R_g$.

Definition 1.2.2 [12]

Let A be a subset of R, for $\gamma \in G$ we write A_γ *for* $A \cap R_\gamma$. A subset A is called *subset* of R if A □ $\sum_{\gamma \in G} A_\gamma$. Let I be an ideal of R, we say that I is a graded ideal of (R, G) if $I = \oplus \sum_{g \in G} (R_g \cap I)$.

Remark 1.2.3 : [1□]

Obviously, $\oplus \sum_{g \in G} (R_g \cap I) \subseteq I$ and hence I is graded of (R, G) if

$I \subseteq \oplus \sum_{g \in G} (R_g \cap I)$. Also, □ □$\sum_{g \in G} (R_g \cap I)$ is the largest ideal of R which is contained in I.

Example 1.2.4 :[16]

Let G be any group, then R is a G −graded ring with :$R_e = R$ *and* $R_g = 0$ for all $g \in G - \{e\}$. This grading is called the trivial grading of R by G.

Example 1.2.5: [18]

The polynomial ring S $= R[x_1, x_2, \cdots, x_n]$ in n variables over the commutative ring R is an example of a graded ring. Here $S_0 = R$ and the homogenous component of degree k is the subgroup of all R-linear combinations of monomials of degree k, i.e.,

$$S_d = \{\sum_{m \in N} r_m X^m \,|\, r_m \in R \text{ and } m_1 + \cdots + m_n = d\}.$$

This is called the standard grad and the polynomial $R[x_1, x_2, \cdots, x_n]$. The ideal I generated by $x_1, x_2, \cdots, x_n$ is a graded ideal: every polynomial with a zero constant term may be written uniquely as a sum of homogenous polynomials of degree $k > 1$, and each of these has a zero constant term, hence lies in I. More generally, an ideal is a graded ideal if and only if it can be generated by homogenous polynomials (see theorem 1.1.4 for the proof).

Example 1.2.6 [2]:

Let K be a field, and $R = K[x]$ be the polynomial over K in one variable x. Let $G = \mathbb{Z}_3$, then R is a G-graded ring with :

$$R_j = \left(kx^{3r+j} : k \in K, r = 0, 1, 2, \cdots\right), for\ j \in \mathbb{Z}_3.$$

Example 1.2.7: [2]

Let $R = \mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$ (the Gaussian integers), and $G = \mathbb{Z}_2$, then R is a G-graded ring with :$R_0 = \mathbb{Z}$, and $R_1 = i\mathbb{Z}$.

The following example shows that an ideal of a G-graded ring need not be graded ideal in general.

Example 1.2.8

Let $R = \mathbb{Z}[i]$, and let $G = \mathbb{Z}_2$, then R is a G-graded ring with $R_0 = \mathbb{Z}$, and $R_1 = i\mathbb{Z}$. Let I $= < 1 + i >, x_0 = 1$ and $x_1 = i$. Clearly $x_0 \notin I$ because if $x_0 \in I$ then there is $a + ib \in \mathbb{Z}[i]$ such that $1 = (a + ib)(1 + i)$ which implies $a - b = 1$ and $a + b = 0$. Hence $2a = 1$, contradiction. Thus, I is not a graded ideal of (R, G).

The following lemma is an exercise in a source [16], and we have proven it.

Lemma 1.2.9 :[16]

An ideal is a graded ideal if and only if it can be generated by homogenous polynomials.

Proof.

Let R be a graded ring such that $R = \bigoplus \sum_{g \in G} R_g$, where the R_g are additive abelian groups such that $R_g R_h \subseteq R_{g+h}$ $for\ all\ g, h \geq 1. if\ I \subset K[x]$ is graded, the homogenous parts of the generators of I obviously generate I . in opposite, let I be an ideal generated by homogenous polynomials $f_i, i = 1, \cdots, r$.

Suppose that $v \in I$, i.e., $v = \sum_{i=1}^{r} a_i f_i, a_i \in K[x]$. Note that each homogenous part $(a_i)_{[j]} f_i$ of $a_i$is in I, because I is an ideal. Since this holds for any $g \in I$, we have that $\oplus_{i\geq 1}(I \cap R_g) \subseteq I \subseteq \oplus_{i\geq 1}(I \cap R_g)$. This means both are equal and I is a graded ideal.

The following proposition is an exercise in a source [1□], and we have proven it.

Proposition 1.2.10: [1□]

Let R be a graded ring, and let I be a graded ideal in R. Let

$I_k = I \cap R_k$ for all $k \geq 0$. Then R/I is a naturally a graded ring whose homogeneous component of degree k is isomorphic to R_k/I_k.

Proof .

First, we will show that $R_pI_q \subseteq I_{p+q}$. Let $x \in R_pI_q$ then $x = r_pa_q$ where $r_p \in R_p$ and $a_q \in I_q$. So $x \in R_pI_q$ implies that $r_pa_q \in R_pI_q$ implies that $r_pa_q \in R_pR_q \cap I$ (since $R_q \cap I = I_q$) implies that $r_pa_q \in R_{p+q} \cap I$ (since $R_pR_q \subseteq R_{p+q}$) which implies that $r_pa_q \in I_{p+q}$ (since $R_{p+q} \cap I = I_{p+q}$). Thus $R_pI_q \subseteq I_{p+q}$.

Second, we need to show that the multiplication $(R_p/I_p)(R_q/I_q) \subseteq R_{p+q}/I_{p+q}$. is well defined. We will show that:

$$(r_p + I_p)(r_q + I_q) = r_pr_q + I_{p+q}$$

where $r_p + I_p \in R_p/I_p$ and $r_q + I_q \in R_q/I_q$.

Let $r_p + I_p = r_p' + I_p$ *and* $r_q + I_q = r_q' + I_q$. We want to show that:

$$(r_p + I_p)(r_q + I_q) = (r_p' + I_p)(r_q' + I_q)$$

i.e., we want to show $r_p r_q + I_{p+q} = r'_p r'_q + I_{p+q}$. So if we show that

□$r_p r_q - r'_p r'_q) \in I_{p+q}$ then we are done. □ote that $r_p + I_p = r'_p + I_p$ implies that $r_p - r'_p \in I_p$ implies that $(r_p - r'_p) r_q \in I_{p,}$ □by multiply both sides by r_q□.

So, $r_p r_q - r'_p r_q \in I_q$ □because I_p is an ideal □. Similarly, $r_q + I_q = r'_q + I_q$ implies that $r_q - r'_p \in I_q$ implies that $r'_p (r_p - r'_p) \in I_q$ □by multiply both sides by r'_p □.□ence, $r'_p r_q - r'_p r'_q \in I_q$. □herefore, $(r_p r_q - r'_p r_q) + (r'_p r_q - r'_p r'_q) \in I_p$ □$I_q \subset I$, which implies that $(r_p r_q - r'_p r'_q) \in I_p$□$I_q \subset I$. □ut, $r_p r_q \in R_p R_q \subset R_{p+q}$. So $r_p r_q \in R_{p+q}$ and $r'_p r'_q \in R_{p+q}$. □ence, $r_p r_q - r'_p r'_q \in I \cap R_{p+q} = I_{p+q}$.

□hree, we will prove that $R/I \cong \oplus_{k=0}^{\infty} R_k / I_k$ where $I_k = R_k \cap I$. □or each $r \in R, r = \sum_{i=0}^{n} r_i$ such that $r_i \in R_{i,}$we define $\alpha : R \to\cong \oplus_{k=0}^{\infty} R_k / I_k$ by :

$$\alpha(r) = \sum r_i + I_i$$

1- α is ring homomorphism for:

(a) if $r = \sum r_i$ and $s = \sum s_j \in R$ then,

$$\alpha(r+s) = \alpha(\sum r_i + \sum s_i) = \alpha(\sum r_i + s_i) = \sum_{k=i+j} (r_i + s_j) + I_k$$

$$= (\sum r_i + I_k) + (\sum s_j + I_k)\square\alpha(r) + \alpha(s).$$

(b) if $r = \sum r_i$ and $s = \sum s_j \in R$ then,

$$\alpha(r.s) = \alpha((\sum r_i).(\sum s_j)) = \alpha(\sum\sum r_i s_j) = \sum\sum r_i s_j + I_i$$

$= (\sum r_i + I_j)(\sum s_j + I_i)\square\alpha(r).\alpha(s)$. So α is ring homomorphism

2- α is onto for:

let $a \in \oplus_{k=0}^{\infty} R_k/I_k$ implies that $a = \sum_{i=0}^{n} r_i + I_i$ implies that there exists $b \in R, b = \sum r_i$ such that $\alpha(b) = \alpha(\sum r_i) = \sum r_i + I_i$. Thus α *is onto*.

3- $\ker(\alpha) = I$ *for*:

$a \in \ker(\alpha)\ iff\ \alpha(\sum_{i=0}^{n} a_i) = 0\ \ iff\ \alpha(\sum_{i=0}^{n} a_i) = \sum a_i + I_i = \sum_{i=0}^{n} I_i\ \ iff\ \sum a_i \in \sum I_i \cong \oplus_{k=0}^{\infty} I_i$ I . hence $R/I \cong \oplus_{k=0}^{\infty} R_k/I_k$

(by the first isomorphism theorem)

4- now we check the ring axioms.

1- $R/I \cong \oplus_{k=0}^{\infty} R_k/I_k$ is abelian group.

2- if $r_i + I_i, r_j + I_j$ *and* $r_n + I_n \in R/I$ then,

$[(r_i + I_i).(r_j + I_j)].(r_n + I_n) = (r_i r_j + I_{i+j})(r_n + I_n) = r_i r_j r_n + I_{i+j+n} = (r_i + I_i).(r_j r_n + I_{j+n}) = (r_i + I_i).[(r_j + I_j).(r_n + I_n)]$.

In addition, $(r_i + I_i).[(r_j + I_j) + (r_n + I_n)] = [(r_i + I_i).(r_j + I_j)] + [(r_i + I_i).(r_n + I_n)]$. Hence associative holds.

Proposition 1.2.11:[12]

Let R be a G −graded ring and $a, b \in R, g \in G$. then

1. $(a + b)_g = a_g + b_g$.

2. $(ab)_g = \sum_{\tau \in G} a_\tau b_{\tau^{-1}g}$.

Proposition 1.2.12: [30]

Let R be a G −graded ring then

1- R_e is a subring of R and $1 \in R_e$.

2- R_g and R are R_e −modules.

1.3 Graded Modules

A brief overview of some definitions and results of graded algebras, and differential graded modules over the graded polynomial ring $R = K[x_1, x_2, \cdots, x_n]$, following [3], [13], [2□], [2□] and [31], is given in this section.

Definition 1.3.1: [29]

A *graded* K*-algebra*, A, is a sequence of K-vector spaces $\{A_j\}_{j \in \mathbb{Z}}$, together with vector space homomorphisms:

$\rho: A_i \otimes_K A_j \to A_{i+j}$ for $i, j \in \mathbb{Z}$ and

$\mu: K \to A_0$, such that the following diagrams

$$\begin{array}{ccc} A_i \otimes A_j \otimes A_m & \xrightarrow{\rho \otimes 1} & A_{i+j} \otimes A_m \\ \downarrow {\scriptstyle 1 \otimes \rho} & & \downarrow {\scriptstyle \rho} \\ A_i \otimes A_{j+m} & \xrightarrow{\rho} & A_{i+j+m} \end{array}$$

$$\begin{array}{ccc} \mathrm{K} \otimes A_j = A_j \otimes K & \xrightarrow{1 \otimes \mu} & A_j \otimes A_0 \\ \downarrow {\scriptstyle \mu \otimes 1} & \searrow & \downarrow {\scriptstyle \rho} \\ A_0 \otimes A_j & \xrightarrow[\rho]{} & A_j \end{array}$$

commute for all $i, j, m \in \mathbb{Z}$

(المستخلص)

لتكن K مجال للخاصية الثانية و نفرض □ حلقة متعددة الحدود متدرجة بطريقة موجبة نفرض M مقاس مدرجة تفاضلي في الدرجة □. لقد انشئنا تصنيفًا لبعض أنواع المقاسات المتدرجة التفاضلية حيث P اكبر او يساوي 0 و n اكبر من واحد. يعطي هذا التصنيف خوارزمية جزئية لاختبار ما إذا كانت هذه المقاسات قابلة للحل بالنسبة للمقاسات اما اذا كان المقاسات خارج التصنيف لا يمكننا أن نقرر باستخدام طرقنا فيما اذا كانت قابلة للحل أم لا. ندرسنا فئة الوحدات المتدرجة التفاضلية على حلقة متدرجة تفاضلية. في الواقع ، لقد قدمنا وصفًا مع بعض الأمثلة للمقاس R ، والمقاسات المتدرجة النمطية R والمقاسات المتدرجة التفاضلية.