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قسم الرياضيات

طريقة التكرار المتغيرة
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إشراف

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Chapter One
Basic
Concepts

Introduction

The general concepts of fuzzy set theory are introduced, including Zadeh‘s original definition of fuzzy sets and representations of membership functions are given. Also, this section presents fundamental concepts in fuzzy set theory which collect as a bridge between ordinary sets and fuzzy sets and will be used later in the solution of fuzzy differential equations which is the $\alpha -$level sets and fuzzy number. In addition, this section presents the extension principle which is used to generalize nonfuzzy concepts to fuzzy set theory . Finally, fuzzy relation , fuzzy functions and their kinds and fuzzy differentiation and integration are presented in this section.

1.1 Fuzzy Set Theory:

In the present section, some of the fundamental and necessary concepts in fuzzy set theory are given. We start with Zadeh‘s original definition of the subject.

Definition (1.1), [11],[12]:

Let X be a nonempty set of objects , then a fuzzy set $\tilde{A}$ in X with generic element denoted by x can be described by its membership function $\mu_{\tilde{A}}(x)$ as follows:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$$

where $\mu_{\tilde{A}}(x): X \to [0,1]$ is interpreted as the degree of compatibility or degree of membership (also degree of truth or the grade of membership) of x in $\tilde{A}$.

(when the membership space contains only the two points 0 and 1, then $\mu_{\tilde{A}}(x)$ is identical to the characteristic function of a nonfuzzy set and $\tilde{A}$ is nonfuzzy).

1. 2 Methods for Representing Fuzzy Sets, [11]:

There are several approaches to represent fuzzy sets in literatures, which can be divided into three categories:

(i) An ordered set of pairs is used to represent a fuzzy set, where the first element represents the element and the second represents the degree of membership.

(ii) One of the ways to represent a fuzzy set is to state its membership function.

(iii) A fuzzy set represented as :

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + \cdots = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i$$

$$\text{or } \int_x \mu_{\tilde{A}}(x)/x$$

1.3 Representations of Membership Functions, [13]:

A membership function may be used to define each fuzzy set. We will illustrate the most methods used to represent membership function, these types may be classified into three main categories:

I. Graphical Representation:

This method is based on the selection of graphic form of curves or straight or zigzag lines to represent the membership function of fuzzy concepts.

II. Tabular Representation, [14]:

Tables can always be used to express membership functions. All of the items in the universal set are listed in the table that depicts a fuzzy set, along with the accompanying membership grades. By employing this technique, we may characterize a fuzzy set as a list where each member is connected according to the degree to which they belong to the set .

III. Functional Representation,[13]:

This representation is one of the most common methods of representation, when a universal set is a subset of the real line, It is difficult to mention all the elements together with their membership grads. A functional form is often used to represented this type of fuzzy set, which known as a fuzzy number, this form is specifies the shape of the fuzzy number.

The three primary membership functions that make up the functional membership function, which is frequently utilized in practice, are as follows:

(i) Triangular Membership Function

Three parameters, a, b, and c, describe the general mathematical formula for this function as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & c \leq x \end{cases}$$

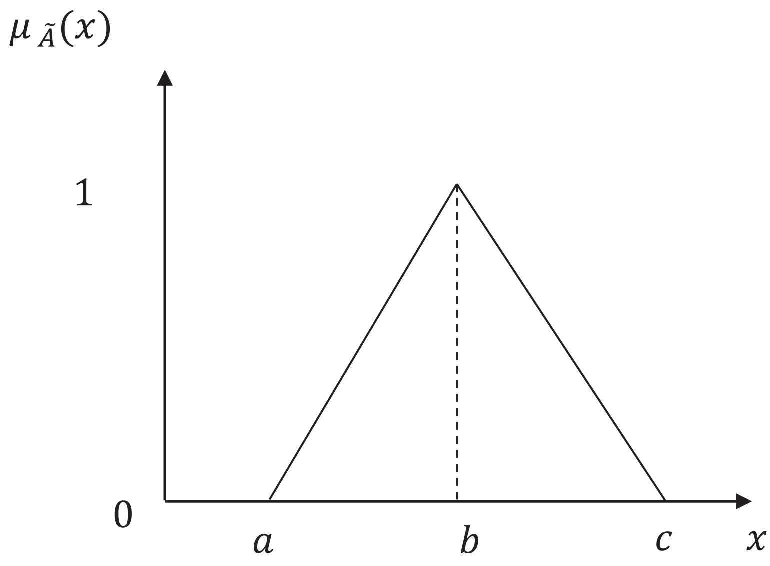

Figure (1.1) Triangular membership functions

(ii) ***Trapezoidal Membership Function***

Four parameters, a, b, c, and d, make up the general mathematical formula for this function, which is as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{d-x}{d-c}, & c \leq x \leq d \\ 0, & c \leq x \end{cases}$$

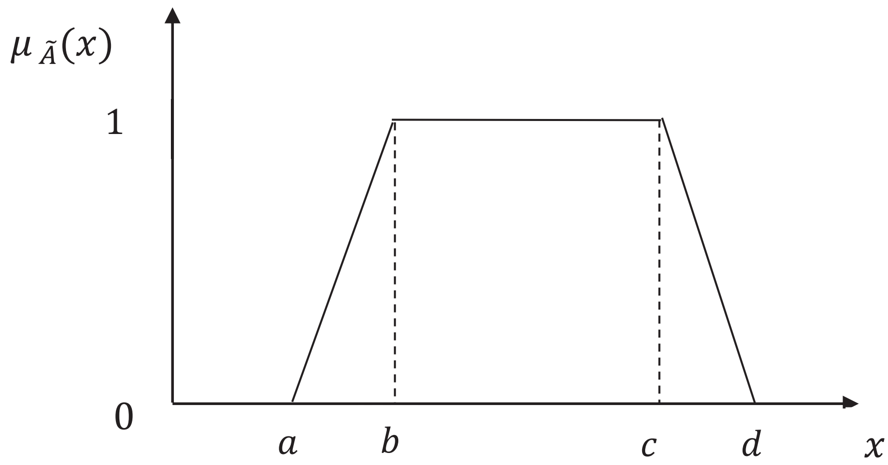

Figure (1.2) Trapezoidal membership functions

(iii) ***Gaussian Membership Function,[15]:***

Two parameters, a, and b, define the general mathematical formula for this function as follows:

$$\mu_{\tilde{A}}(x) = e^{\frac{-(x-a)^2}{b^2}}$$

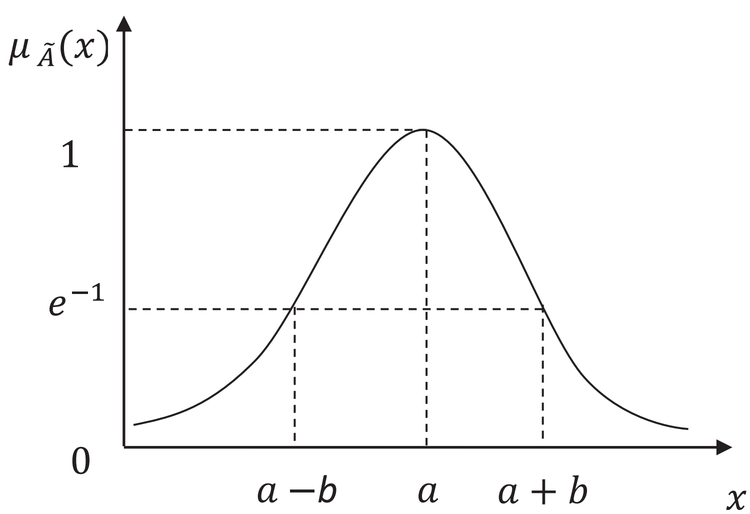

Figure (1.3) Gaussian membership functions

1.4 Algebraic Operations on Fuzzy Sets, [11],[16],[17]:

Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy subsets of a universal set X, then the notions below can be defined in relation to $\widetilde{A}$ and $\widetilde{B}$ as fuzzy subsets of X :

1. An ***empty*** fuzzy set has a zero membership value and is denoted by $\widetilde{\emptyset}(x)$, $\forall x \in X$.

2. Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy sets then $\tilde{A}$ is a ***subset*** of $\tilde{B}$ if and only if

$$\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall x \in X.$$

3. $\tilde{A} = \tilde{B}$ if and only if

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \forall x \in X.$$

4. The ***height*** of a fuzzy set $\tilde{A}$ is represent largest membership value of its membership function, i.e.;

$$height(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x_i)$$

If a fuzzy set has a height of 1 , it is considered ***normal*** fuzzy set ; otherwise, it is ***subnormal***, and if it is subnormal, it can be normalized by letting

$$\mu_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{\sup \mu_{\tilde{A}}(x)}$$

5. The crisp set of all $x \in X$ is called the ***support*** of a fuzzy set $\tilde{A}$, denoted by $S(\tilde{A})$, if $\mu_{\tilde{A}}(x) > 0$.

6. If $\tilde{A}$ is a fuzzy set, then $\alpha - level$ set (or $\alpha - Cut$) of $\tilde{A}$ is represent the crisp set of elements that belong to $\tilde{A}$ at least to the degree α, i.e.;

$$A_\alpha = \{ x \in X | \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1]\},$$

and the *strong* α *−level* set (or *strong* α *−Cut)* is defined by:

$$A'_\alpha = \{ x \in X | \mu_{\tilde{A}}(x) > \alpha, \alpha \in [0,1]\}$$

7. If $\tilde{A}$ is a fuzzy set, then $\tilde{A}$ is ***convex*** if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \forall x_1, x_2 \in X,\ \lambda \in [0,1].$$

8. If $\tilde{A}$ is a finite fuzzy set, then the ***cardinality*** for $\tilde{A}$ denoted by $|\tilde{A}|$ is defined as:

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$$

and the ***relative cardinality*** of $\tilde{A}$ is $\|\tilde{A}\| = \frac{|\tilde{A}|}{|X|}$, where $|X|$ is the number of the elements of X. For infinite X, the cardinality is defined by $|\tilde{A}| = \int_x \mu_{\tilde{A}}(x)$.

9. If $\tilde{A}$ is a fuzzy set, then the ***complement*** $\tilde{A}^c$ of $\tilde{A}$ is a fuzzy set with the following membership function

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x), \forall x \in X.$$

10. The ***intersection*** $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function

$$\mu_{\tilde{C}}(x) = min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad x \in X.$$

11. The ***union*** $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with the membership function

$$\mu_{\tilde{D}}(x) = max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad x \in X.$$

Remark (1.1), [11]:

The symbols $\wedge$ and $\vee$ One can using for " and " (= intersection) and for " or " (= union) , respectively, such that if ***S*** and ***T*** are two statements which the truth value for ***S*** is μ_S and for ***T*** is μ_T, where, $\mu_S, \mu_T \in [0,1]$ then

$$\mu_{S \wedge T} = min(\mu_S, \mu_T) \quad \text{and} \quad \mu_{S \vee T} = max(\mu_S, \mu_T)$$

1. 5 Fuzzy Number, [18]:

Fuzzy number is described as a fuzzy set defining a fuzzy interval in the real numbers $\mathbb{R}$. This interval is likewise a fuzzy set because its boundary is ambiguous. A fuzzy interval is typically expressed as $[a_1, a_2, a_3]$ with two end points, a_1 and a_3, and a peak point, a_2.

Definition (1.2), [11]:

A fuzzy set $\widetilde{M}$ which is defined in the real numbers $\mathbb{R}$, called a fuzzy number if $\widetilde{M}$ satisfied :

1. $\widetilde{M}$ is convex
2. $\mu_{\widetilde{M}}(x)$ is piecewise continuous.
3. $\widetilde{M}$ is normalized, i.e.; $\exists\ x_0 \in \mathbb{R}$ with $\mu_{\widetilde{M}}(x_0) = 1$ where x_0 is called the mean value of $\widetilde{M}$.

Fuzzy number always a fuzzy set, but a fuzzy set is not always a fuzzy number, for example:

$\tilde{A} = \{(1,0.2), (2,0.6), (0,0.8)\}$ is a fuzzy set but not fuzzy number because $\tilde{A}$ is not normalized.

Some authors used the trapezoidal shaped membership functions, for the sake of computational efficiency and ease of data acquisition.

Definition (1.3), [11] :

A fuzzy number $\widetilde{M}$ is said to be of LR *−type* if there exist two decreasing functions L (for left), R (for right), $L, R: \mathbb{R}^+ \to [0,1]$ with $L(0) = R(0) = 1$, $\lim_{x\to+\infty} L(x) = \lim_{x\to+\infty} R(x) = 0$ and scalars $a > 0, b > 0$ such that

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\frac{m-x}{a}\right), & for\ x \le m \\ R\left(\frac{x-m}{b}\right), & for\ x \ge m \end{cases}$$

where $m \in \mathbb{R}$, called the mean value of $\widetilde{M}$, a is called the left spread and b the right spread. Symbolically $\widetilde{\mathrm{M}}$ is denoted by $(\mathrm{m}, \mathrm{a}, \mathrm{b})_{\mathrm{LR}}$ as it is shown in Figure (1.4).

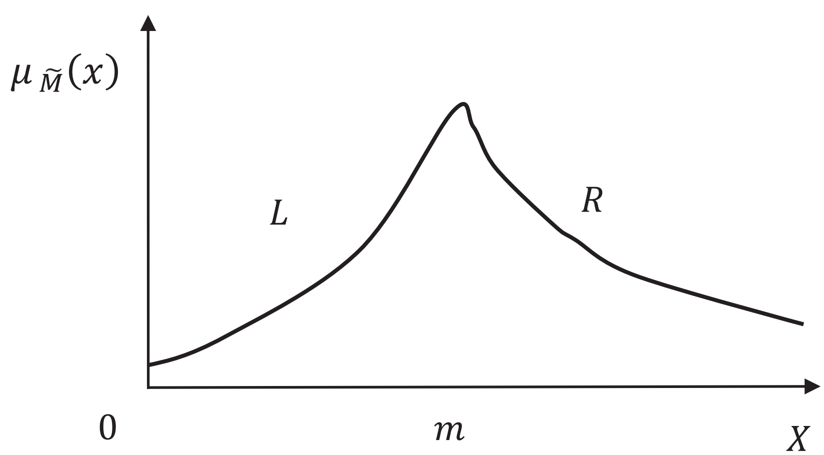

Figure (1.4) LR ***− representation of fuzzy numbers***

Example (1.1):

Let

$$L(x) = \frac{1}{1+4x^2}$$

$$R(x) = \frac{1}{1+3|x|}$$

and, $a = 1, b = 2, m = 3$

Then the fuzzy number $\tilde{3}$ has the membership function

$$\mu_{\tilde{3}}(x) = \begin{cases} L(3-x) = \dfrac{1}{1+4(3-x)^2} & for \quad x \leq 3 \\ R\left(\dfrac{x-3}{2}\right) = \dfrac{1}{1+3\left|\dfrac{x-3}{2}\right|} & for \quad x \geq 3 \end{cases}$$

If $m \notin \mathbb{R}$, but an interval $[\underline{m}, \overline{m}]$ then the fuzzy set $\widetilde{M}$ is not represented a fuzzy number but a fuzzy interval. Accordingly, we can define the fuzzy interval in LR −representation as follows:

Definition (1.4),[11],[17]:

If there exist shape functions L and R and four parameters $(\underline{m}, \overline{m})$, a, b, then the *fuzzy interval* $\widetilde{M}$ is of LR −type , and its membership function is defined by:

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\dfrac{\underline{m}-x}{a}\right) & for \quad x \leq \underline{m} \\ 1 & for \quad \underline{m} \leq x \leq \overline{m} \\ R\left(\dfrac{x-\overline{m}}{b}\right) & for \quad x \geq \overline{m} \end{cases}$$

The fuzzy interval is then represented by:

$$\tilde{M} = \left(\underline{m}, \overline{m}, a, b\right)_{LR}$$

If we supposed that $\tilde{M}$ is a real crisp number for $m \in \mathbb{R}$**,** then

$$\tilde{M} = (m, m, 0, 0)_{LR}\ , \forall L \ and \ R$$

if $\tilde{M}$ is a crisp interval,

$$\tilde{M} = (a, b, 0, 0)_{LR}\ , \forall L \ and \ R$$

1.6 The Extension Principle:

The extension principle is one of the most widely ideas in fuzzy set theory that can be used to apply crisp mathematical concepts to fuzzy sets. It was already implied in its most basic form in Zadeh's first contribution (Zadeh, 1965) **[12]**. Modifications have been suggested in the interim. Following (Dubois and Prade, 1980) **[19]**, the definition of the extension principle is given as follows:

***Definition (1.5), [11]*:**

Let $X = X_1 \times \ ... \times X_n$, be a Cartesian product of universes, and $\tilde{A}_1, \cdots, \tilde{A}_n$ be r fuzzy sets in $X_1,\ ... , X_n$, respectively. If f is a mapping from X to a universe Y, $y = f(x_1, \dots, x_n)$, we can then define a fuzzy set $\tilde{B}$ in Y by using the extension principle as follows:

$$\tilde{\mathrm{B}} = \left\{\left(y, \mu_{\tilde{B}}(y)\right) | y = f(x_1, \dots, x_n), (x_1, \dots, x_n) \in X\right\}$$

Where :

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\dots,x_n)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1),\cdots,\mu_{\tilde{A}_n}(x_n)\} & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

where f^{-1} is refers to the inverse of f.

If $n = 1$, then the extension principle, obviously, reduces to

$$\tilde{B} = f(\tilde{A}) = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in X\}$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_{\tilde{A}}(x) & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

Example (1.2):

Let $A = \{(-1, 0.7), (0, 0.9), (1, 1), (2, 0.6)\}$

and $f(x) = x^2$

Using the extension principle , we then get at

$B = f(A) = \{(0,0.9), (1, 1), (4,0.6)\}$.

1.7 Fuzzy Relations on Sets and Fuzzy Sets, [11]:

Fuzzy relations have many and significant applications. Numerous authors have studied them, most notably Zadeh (1965, 1971), Kaufmann (1975), and Rosenfeld (1975).

المستخلص

تسعى هذه الأطروحة إلى تعريف المجموعات الضيابية ودراسة بعض خصائصها التي تميز هذا الموضوع.

من ناحية أخرى تسعى هذه الأطروحة إلى دراسة المعادلات التفاضلية الضبابية وكيفية حلها باستخدام طريقة التكرار المتغيرة ، كما يتم ذكر وإثبات تحليل التقارب وتقدير الحد الأقصى للخطأ المطلق. بالإضافة إلى ذلك ، يتم حل بعض الأمثلة العددية التوضيحية باستخدام طريقة التكرار المتغيرة. تستخدم طريقة التكرار المتغيرة المعدلة بشكل فعال لتحسين نتائج طريقة التكرار المتغيرة وتسريع التقارب كما هو موضح في الأمثلة العددية.

وأخيراً، هو دراسة أساسيات المعادلات التفاضلية التباطئية وتطبيق طريقة التكرار المتغيرة لحل المعادلات التفاضلية الضبابية التباطئية. بالإضافة إلى ذلك ، إيجاد الحل العددي للمعادلات التفاضلية الضبابية التباطئية باستخدام طريقة التكرار المتغيرة ، الحسابات كتبت باستخدام البرنامج الرياضي MATHCAD 15 ، بينما يتم رسم النتائج العددية باستخدام Matlab20a .