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من قبل

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Chapter One

Some Basic Concepts for Bornological Structures

1.1 Introduction

In this chapter, we recall some basic concepts for bornological structures to solve the problem of boundedness for a set and a group. It is a natural to study fundamental construction for this structure such as bornological subset, product bornological set. Also, the most important part in this chapter is the practical applications of bornological structure; see [2].

1.2 Bornological Set

In this section, we recall some basic concepts for the bornological set with some examples and fundamental constructions for this structure.

Definition 1-2-1 [5]:

Let X be a nonempty set. A ***bornology*** β on X is a collection of subsets of X such that:

i. β forms a cover of X;
ii. β inclusion under hereditary, i.e. if $B \in \beta, \exists\, A \subseteq B$ then $A \in \beta$;
iii. β inclusion under finite union, i.e. if $\forall\, B_1, B_2 \in \beta$ then $B_1 \cup B_2 \in \beta$.

A pair (X, β) consisting of a set X and a bornology β on X is called ***a bornological set***, and the elements of β are called bounded subsets of X.

We can satisfy the first condition in different ways, if the whole set X belong to the bornology or $\forall\, x \in X, \{x\} \in \beta$ or $X = \bigcup_{B \in \beta} B$.

Definition 1-2-2 [5]:

Let (X, β) be a bornological set. A ***base*** β_0 is a sub collection of bornology β, and each element of the bornology is contained in an element of the base.

Example 1-2-3:

Let $X = \{1,3,5\}$, $\beta = \{\emptyset, X, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}\}$.

To satisfy that β is a bornology on X.

i. Since $X \in \beta$, then X is covering itself.

ii. If $B \in \beta, A \subseteq B$, then $A \in \beta$

Since β is the set of all subsets of X, i.e. $\beta = P(X) = 2^X$.

Then β inclusion under hereditary.

iii. β inclusion under finite union, i.e. $\bigcup_{i=1}^{n} B_i \in \beta, \forall B_1, B_2, \dots, B_n \in \beta$.

Since

$\{1\} \cup \{3\} = \{1,3\}$, $\{1,3\} \cup \{1,5\} = X$, $\emptyset \cup \{1,3\} = \{1,3\}$,

$\{1\} \cup \{5\} = \{1,5\}$, $\{1,3\} \cup \{3,5\} = X$, $\emptyset \cup \{3,5\} = \{3,5\}$,

$\{3\} \cup \{5\} = \{3,5\}$, $\{1,5\} \cup \{3,5\} = X$, $\emptyset \cup \{1\} = \{1\}$.

Then β is bornology on X. We can define only one bornology on finite set it is called discrete bornology.

Now, to find the base for bornology

$\beta_0 = \{\{1,3\}, \{1,5\}, \{3,5\}, X\}$, or $\beta_0 = \{X\}$.

Example 1-2-4 [12]:

Let $\mathbb{R}$ be the set of real numbers (with Euclidean norm (absolute value)).
We want to define a usual bornology on $\mathbb{R}$ that mean β is the collection of all usual bounded sets on $\mathbb{R}$, in fact, (a subset B of $\mathbb{R}$ is bounded if and only if there exist bounded interval such that $B \subseteq (a, b)$).

So, $\beta = \{B: B \subseteq (a, b) : \forall a, b \in \mathbb{R}\}$.

i. Since $\forall x \in (a, b)$ and $\{x\} \subset (a, b)$ (the bounded intervals with respect to absolute value (w.r.t | |) where the absolute value divided $\mathbb{R}$ into bounded interval), (every subset of bounded interval is bounded). Implies that, $\forall x \in \mathbb{R}, \{x\} \in \beta$. Then β covers $\mathbb{R}$.

ii. If $B \in \beta$ and $A \subseteq B$, there is bounded interval such that $A \subseteq B \subseteq (a, b)$.

Therefore $A \in \beta$, and β stable under hereditary.

iii. If $B_1, B_2, \dots, B_n \in \beta$, then there is $L_1, L_2, \dots, L_n$ least upper bound and $g_1, g_2, \dots, g_n$ greater lower bound, such that

$\bigcup_{i=1}^{n} B_i$ has finite union of upper and lower bounds.

Assume that $L = \max\{L_i\}$, and $g = \min\{g_i\}$

$\bigcup_{i=1}^{n} B_i$ has least upper bound L and greater lower bound g.

That mean $\bigcup_{i=1}^{n} B_i$ is bounded subset of $\mathbb{R}$.

That mean $\bigcup_{i=1}^{n} B_i \in \beta$. Then β is bornology on $\mathbb{R}$.

And the base of this bornology is:

$\beta_0 = \{B_r(x): r \in \mathbb{R}, x \in \mathbb{R}\} = \{(x - r, x + r): r \in \mathbb{R}, x \in \mathbb{R}\}$.

It is clear that every element of that bornology β is contained in an element of the base β_0.

Example 1-2-5:

Let $X = \mathbb{R}^2$ with Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{2} |x_i|^2\right)^{1/2}$$

where $x = (x_1, x_2)$.

$$D_r(b) = \{x \in \mathbb{R}^2 : \|x - b\| \le r, for\ r > 0\}$$

be a disk of the radius r with center at $b = (b_1, b_2)$.

A subset B of $\mathbb{R}^2$ is bounded if there exists a disk with center x_0 $D_r(x_0)$ such that $B \subseteq D_r(x_0)$, $x_0 \in B, r \geq 0$.

Let β_u be a family of all bounded subset of $\mathbb{R}^2$.

$\beta_u = \{B \subseteq \mathbb{R}^2 : B \text{ is usual bounded subset } of\ \mathbb{R}^2\}$.

Then $(\mathbb{R}^2, \beta_u)$ is a bornological set and β_u is called usual bornology on $\mathbb{R}^2$.

i. It is clear that every close disk in $\mathbb{R}^2$ is bounded set that means, $B \in \beta_u$ since $\mathbb{R}^2$ is covered by the family of all disks. Then β_u covers $\mathbb{R}^2$.
ii. If $B \in \beta_u$ and $A \subseteq B$, then there is close disk such that $A \subseteq B \subseteq D_r$. Therefore $A \in \beta_u$.
iii. If $B_1, B_2 \in \beta_u$ then D_{r_1}, D_{r_2} are close disks such that $B_1 \in D_{r_1}, B_2 \in D_{r_2}$.

And $r = Max_{i=1,2}\{r_i\}$. Then $B_1 \cup B_2 \subseteq D_{r_2}$.

Thus, the finite union subsets of $\mathbb{R}^2$. Then β_u is a bornology on $\mathbb{R}^2$.

A base of bornology on $\mathbb{R}^2$ is any sub family such that $\beta_0 = \{D_r(x_0) : r > 0\}$.

Definition 1-2-6 [12]:

Let (X, β) be a bornological set, a family $\mathbb{S}$ of members of a bornology β is said to be ***subbase*** for β if the family of all finite unions of members of $\mathbb{S}$ is a base for β.

Example 1-2-7:

Let $X = \{1, 2, 3\}$, $\beta = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Let $S = \{\{1\}, \{2\}, \{3\}\}$.

Then S is subbase for β where $\beta_0 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$, or $\beta_0 = \{X\}$ are bases for β.

Remarks 1-2-8:

1. Not every family of subsets of a set X will form a base for a bornology on X.

For example $X = \{1,3,5\}$, $\beta_0 = \{\{1\}, \{3\}, \{5\}\}$ then β_0 cannot be a base for any bornology because the base must be contain a large subsets of X see [12].

2. Let $(X, \beta), (Y, \beta')$ be bornological sets and $\beta_0, {\beta_0}'$ are bases for β and β', respectively. If $\beta_0 \subseteq {\beta_0}'$ then, $\beta \subseteq \beta'$.

For example: $X = \{1, 2, 3\}, Y = \{1, 2, 3, 4\}$

$\beta = \left\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1,3\}, \{2, 3\}\right\}$,

$\beta' = \{\emptyset, Y, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\},$

$\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

$\beta_0 = \{\{1, 2\}, \{1,3\}, \{2, 3\}, X\}$,

${\beta_0}' = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, Y, \{1,2\}, \{1,3\}, \{2,3\}, X\}$.

It is clear that $\beta_0 \subseteq {\beta_0}'$ then, $\beta \subseteq \beta'$.

3. Types of a bornology:
   a. The discrete bornology β_{dis} is the collection of all subsets of X, i.e. discrete bornology $P(X) = 2^X$.
   b. The usual bornology β_u is the collection of all usual bounded subsets of X, i.e. usual bornology $= \{B \subseteq X: B \text{ is usual bounded}\}$.
   c. The finite bornology β_{fin} is the collection of all finite bounded subsets of X.

Definition 1-2-9 [5]:

Let (X,β) and (Y,β') be bornological sets. A map $\psi:(X,\beta)\to(Y,\beta')$ is called ***bounded map*** if the image for every bounded set in (X,β) is bounded set in (Y,β'). That means, $\forall B\in\beta \Rightarrow \psi(B)\in\beta'$.

Notice that, the composition of two bounded maps is bounded map.

Definition 1-2-10 [12]:

A map ψ between two bornological sets (X,β) and (Y,β') is called a ***bornological isomorphism*** if ψ is bijective also, ψ , ψ^{-1} are bounded maps.

Consequently, for every $B\in\beta$ there is $B'\in\beta'$, such that

$$B'=\psi(B), B=\psi^{-1}(B').$$

Example 1-2-11:

Consider $X=\{10,11\}, Y=\{5,6\}$ with finite bornology on X, Y.

$$\beta=\{\emptyset, X,\{10\},\{11\}\}, \beta'=\{\emptyset, Y,\{5\},\{6\}\}.$$

Define a function $\psi: X\longrightarrow Y$ such as, $\psi(10)=5, \psi(11)=6$.

It is clear that ψ is bijective and ψ and ψ^{-1} bounded map.

$$\psi(\{10\})=\{5\} \Rightarrow \psi^{-1}(\{5\})=\{10\}$$

$$\psi(\{11\})=\{6\} \Rightarrow \psi^{-1}(\{6\})=\{11\}$$

$$\psi^{-1}(Y)=X, \psi^{-1}(\emptyset)=\emptyset.$$

Proposition 1-2-12:

If $\psi:(X,\beta)\longrightarrow(Y,\beta')$ and $\phi:(Y,\beta')\longrightarrow(Z,\beta'')$ are both bornological isomorphism, then the composition $\phi\circ\psi:(X,\beta)\longrightarrow(Z,\beta'')$ is bornological isomorphism.

Proof:

Since ψ and ϕ are one to one and onto, then $\phi \circ \psi$ is one to one and onto (composition of any two one to one, onto maps is one to one, onto map respectively). Since ψ and ϕ bounded maps, then $\phi \circ \psi$ is bounded map.

Since ψ and ϕ are bornological isomorphism, then ψ^{-1} and ϕ^{-1} are bounded also $\psi^{-1} \circ \phi^{-1}$ is bounded, but $\psi^{-1} \circ \phi^{-1} = (\phi \circ \psi)^{-1}$ is bounded.

Therefore, $\phi \circ \psi$ is bornological isomorphism.

Definition 1-2-13 [5]:

Let (X, β) be a bornological set and let $Y \subseteq X$. Then the collection $\beta_Y = \{B \cap Y:$

$B \in \beta\}$ is a bornology on Y. The bornological set (Y, β_Y) is called bornological subset of a bornological set (X, β) and β_Y is called ***relative bornology*** on Y.

Example 1-2-14:

$X = \{2,4,6\}$, with discrete bornology on X.

$\beta = \{\emptyset, X, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}\}$.

And the base of bornology is β_0= $\{\{2,4\}, \{2,6\}, \{4,6\}, X\}$, and let $Y = \{2,4\}$.

Then we have

$\beta_Y = \{\emptyset \cap Y, X \cap Y, \{2\} \cap Y, \{4\} \cap Y, \{6\} \cap Y, \{2,4\} \cap Y, \{2,6\} \cap Y, \{4,6\} \cap Y\}$

$\beta_Y = \{\emptyset, Y, \{2\}, \{4\}\}$. Then the subset of bornology is β_Y.

Definition 1-2-15 [10]:

Let (X, β), and (Y, β') be two bornological sets, $\beta_0 = \{$The family of all $B \times B'$ where $B \in \beta, B' \in \beta'\}$.We say that β_0 is a base, and is called ***product bornology***.

If we defines a bornological structure on $X \times Y$, then the product set $X \times Y$ with this bornological structure is called a bornological product sets of (X, β), and (Y, β').

Example 1-2-16:

Suppose $X = \{a, b\}, Y = \{1, 2\}$.

We defined a discrete bornology on X, Y.

$$\beta = \{\emptyset, X, \{a\}, \{b\}\}$$

$$\beta' = \{\emptyset, Y, \{1\}, \{2\}\}.$$

$$\beta \times \beta' = \{\emptyset, X \times \emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times \emptyset, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}, \{b\} \times \emptyset, \{b\} \times Y, \{b\} \times \{1\}, \{b\} \times \{2\}\}$$

$$\beta \times \beta' = \{\emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}, \{b\} \times Y, \{b\} \times \{1\}, \{b\} \times \{2\}\}$$

$\beta \times \beta' = \{\emptyset, X \times Y, \{(a, 1), (b, 1)\}, \{(a, 2), (b, 2)\}, \{(a, 1), (a, 2)\}, \{(a, 1)\},$

$\{(a, 2)\}, \{(b, 1), (b, 2)\}, \{(b, 1)\}, \{(b, 2)\}\}$

$\beta_0 = \beta \times \beta'$ is a base, and is called product bornology. And if

$X \times Y = \{(x, y): x \in X, y \in Y\}$

$= \{(a, 1), (a, 2), (b, 1), (b, 2)\}$, define a discrete bornology on $X \times Y$.

$\beta_{X \times Y} = \{\emptyset, X \times Y, \{(a, 1)\}, \{(a, 2)\}, \{(b, 1)\}, \{(b, 2)\}, \{(a, 1), (b, 1)\},$

$\{(a, 1), (b, 2)\}, \{(a, 2), (b, 1)\}, \{(a, 2), (b, 2)\}, \{(a, 1), (a, 2)\}, \{(b, 1), (b, 2)\},$

$\{(a, 1), (a, 2), (b, 1)\}, \{(a, 1), (a, 2), (b, 2)\}, \{(a, 1), (b, 1), (b, 2)\},$

$\{(a, 2), (b, 1), (b, 2)\}\}$. $(X \times Y, \beta_{X \times Y})$ is called a bornological product sets.

1.3 Bornological Group

In this section, we recall concept of bornological group to solve the problems of boundedness for group.

Definition 1-3-1 [18]:

A bornological group (G, β) is a set with two structures:

i. $(G,*)$ is a group;
ii. β is a bornology on G.

Such that the product map $\psi: (G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$ is bounded and $\psi^{-1}: (G, \beta) \longrightarrow (G, \beta)$ is bounded.

In the other words, a bornological group is a group G together with a bornology on G such that the group binary operation and the group inverse maps are bounded with respect to the bornology.

Let G be a bornological group and B_1, B_2 be two bounded subsets of G.

We denote for the image of $B_1 \times B_2$ under product map

$$(G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$$

by $B_1 * B_2 = \{b_1 * b_2 : b_1 \in B_1, b_2 \in B_2\}$, $*$ is the binary operation defined on the group which we want to bornologies it, and $B^{-1} = \{b^{-1} \in B\}$.

Example 1-3-2:

Let $G = (\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}_{2\times 2}, \ \cdot\ )$ be a group $a, b \in \mathbb{R}, a \neq 0$, which is the set of matrix elements and multiplication operation.

We can define a finite bornology β on this group, which it is a collection of all finite subsets of G.

To prove G with finite bornology β is bornological group (G, β).

We must prove that, the product map and the inverse map are bounded.

i. $\psi: (G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$.

Let $M_1, M_2 \in (G, \beta)$ be two bounded subsets, we must prove that

$\psi(M_1 \times M_2)$ is bounded.

$\psi(M_1 \times M_2)= M_1 \cdot M_2 = \{m_1 \cdot m_2 : m_1 \in G, m_2 \in G\}$

$$=\{m_1 \cdot m_2 : \ m_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & 1 \end{bmatrix}, m_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & 1 \end{bmatrix}, a_1, a_2, b_1, b_2 \in \mathbb{R}\}$$

$$=\{\begin{bmatrix} a_1 & b_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 \cdot a_2 & a_1 b_2 + b_1 \\ 0 & 1 \end{bmatrix}\} \subset (G, \beta)$$

the product map is bounded.

ii. $\psi^{-1}: (G, \beta) \longrightarrow (G, \beta)$.

Let $M \in (G, \beta), M = \{m: m = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\}$ which is finite set.

Then $M^{-1} = \{m^{-1}: m^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{a} \\ 0 & 1 \end{bmatrix}\} \subset (G, \beta)$.

So, ψ^{-1} is bounded. Then (G, β) is bornological group.

<u>Example 1-3-3:</u>

Let $(\mathbb{Z}, +)$ be an additive group and β be the finite bornology define on $\mathbb{Z}$.

To prove $\mathbb{Z}$ with finite bornology β is a bornological group $(\mathbb{Z}, \beta)$.

We must prove that, the product map and the inverse map are bounded.

i. $\psi: (\mathbb{Z}, \beta) \times (\mathbb{Z}, \beta) \longrightarrow (\mathbb{Z}, \beta)$

Let $A, B \in \beta$ be two bounded subsets, to prove that $\psi(A \times B)$ is bounded.

Thus, $\psi(A \times B) = A + B$ is bounded subset belong to $\mathbb{Z}$, since $\mathbb{Z}$ group.

Thus, the image for every two bounded sets A, B under ψ is bounded set.

ii. $\psi^{-1}: (G, \beta) \to (G, \beta)$.

Let $A \in \beta$ be a bounded set, thus $\psi^{-1}(A) = A^{-1} \subset \mathbb{Z}$, since $\mathbb{Z}$ is a group.

Then ψ^{-1} is bounded.

1.4 Some Practical Applications of Bornological Space

In this section, we recall some of the practical applications for bornological structures. In other words, how to apply the bornology to solve many problems in our live. As we know the effect of bornology is to determine the boundedness for sets, vector spaces and groups. That means, bornological structure is general solution to solve the bounded problems [2].

The most important practical application for bornology is in the spyware program KPJ. To explain how this structures, we used to solve the problems for boundedness in specific way or with more details for example in spyware program KPJ. Exactly, when they want to determine the person location, or the identity of the person from his print finger and his print eye [2].

First, assume that the person is the original point, to determine his signed or status. We start to study the (behaviour, vibrations, frequencies) of objects within his domain. That means, we study the frequency of these objects by introducing:

open unit disk B = $\{x \in V, \|x\| < 1\}$ or closed unit disk B = $\{ x \in V, \|x\| \leq 1\}$. It is an absorbent disk but we need to study the behaviour for another objects. Then, we have to define another disk B_1 that is bigger (since B in a vector space, then it is allowed for us to multiply by scalar in vector space and make B bigger), so we get another (open or closed) absorbent disk B_1 (see figure 1). Thus by the same operation we can translate to all objects. In the end, we get collection of (open or closed) disk covers the place and the finite union which it is the bigger bounded set should also be within the place not outside, that means inside the collection see [2]. (A set A is absorbent disk in a vector space V if A absorbs every subset of V consisting of a single point), i.e. absorbs every subset of V.

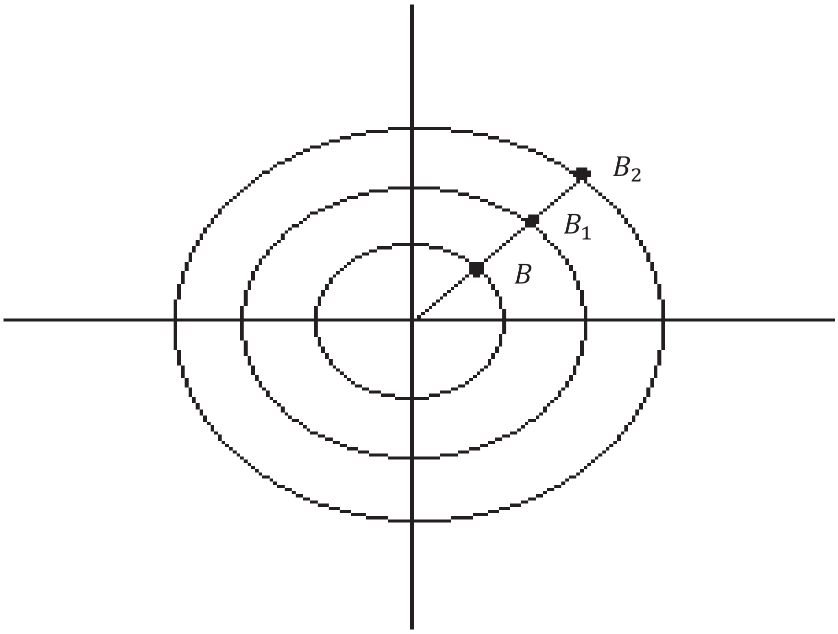

Figure1: The behaviour of objects within his domain by introducing open unit disk [2].

The fingerprint is one of the applications of the bornological space. Each person has his fingerprint that differs from the other. We start from the center, around the center there is a family of bounded sets that, the union of it, give the largest set covering the area of the thumb. Also, the hereditary property is available where a large set exists inside contains a smaller set, and so on. That means subset property is there is a transfer of all biological, formal, and life characteristics from the large set to the smaller set, and the finite union of these bounded sets gives a bounded set that belongs to the family; (see Figure 2).

المستخلص

الفضاء البرنلوجي هو بنية لحل مشاكل التقييد بالنسبة للمجموعات والزمر، والدوال بصورة عامة. الهدف الرئيسي من هذا العمل هو الدمج بين نظرية المجموعات اللينة والفضاء البرنلوجي لبناء هيكل جديد يسمى بنية برنلوجية لينة لحل مشاكل التقييد بالنسبة للمجموعات اللينة والزمر اللينة. أيضًا ، نقوم ببناء الاساس اللين والاساس الجزئي اللين لهذا الهيكل. من الطبيعي دراسة البنية الأساسية لهذا الهيكل الجديد مثل البرنلوجي الجزئي اللين ، وضرب البرنلوجي اللين. بالإضافة إلى ذلك، يتم إنشاء بنية جديدة تكون العناصر عبارة عن مجموعات لينة غير مقيدة. أخيرًا، ندرس إجراءً مقيد لين يعني أنه عندما تعمل زمرة برنلوجية لينة على مجموعة برنلوجية لينة، فإن هذه العملية تسمى فعل جماعي لين أو حركة ذات تقييد لين بحيث يكون تأثير الإجراء اللين هو تقسيم مجموعة البرنلوجي الى فئات مدارية لينة. النتائج الرئيسية المهمة: سوف نثبت أن عائلة من المجموعات البرنلوجية اللينة يمكن أن تكون مجموعة مرتبة جزئيا بواسطة علاقة ترتيب جزئي، تركيب دالتين مقيدة ولينة عبارة عن دالة مقيدة لينة، تقاطع مجموعات برنلوجية لينة هو مجموعة برنلوجية لينة، النقل من جهة اليسار أو جهة اليمين هو تماثل برنلوجي لين، ضرب الزمر البرنلوجية اللينة هي أيضًا زمرة برنلوجية لينة واجراء الزمرة البرنلوجية اللينة هو تماثل برنلوجي لين.