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Chapter One

Basic Concepts

1.1 Introduction

In this chapter, we recall some basic concepts for bornological structures to solve the problem of boundedness for a set and a group. It is a natural to study fundamental construction for this structure such as bornological subset, product bornological sets.

1.2 Definition of Bounded Set of Some Spaces

1- What is the usual definition of bounded set in real number?

A set B of real numbers is called bounded from above if there exists some real number k (not necessarily in B) such that $k > b$ for all b in B. The number k is called an upper bound of B. The terms bounded from below and lower bound are similarly defined.

A set B is bounded if it has both upper and lower bounds. Therefore, a set of real numbers is bounded if it is contained in a finite interval.

2- What is the usual definition of bounded set in matric space?

A subset B of a matric space (M , d) is bounded if there exists $r > 0$ such that for all b and t in B, we have $d(b, t) < r$. The matric space (M, d) is a bounded matric space (or d is a bounded matric) if M is bounded as a subset of itself.

3- What is the bounded set in normed space?

Let V be a vector space with norm which it is map from V to $\mathbb{R}^{+}$.

Then $(V, \|.\|)$ is normed space.

That mean B is bounded if it is contained in a finite interval such that,

$B = \{b: |b - 0| < r \}$, r is a positive real number.

4- What is the bounded set in topological vector space?

The set B is bounded if it is absorbent by every neighbourhood of zero. i.e.,

N absorbs B. (neighbourhood N containing B). $i.e.,\exists\ \kappa > 0$, such that

$\kappa\ N \supseteq B$.

1.3 Bornological Set

In this section, we recall some basic concepts for the bornological set with some examples and fundamental constructions for this structure.

Definition 1-3-1 [5]:

Let X be a nonempty set. A bornology β on X is a collection β of a subsets of Xsuch that:

1. β forms a cover of X;
2. β is hereditary under inclusion, i.e. if $A \subseteq B$ and $B \in \beta$ then $A \in \beta$;
3. β is stable under a finite union, i.e. if $\forall\ B_1, B_2 \in \beta$ then $B_1 \cup B_2 \in \beta$.

A pair (X, β) is called a ***bornological set*** on a set X, and the elements of β are called bounded sets.

we can satisfy the first condition in different ways, sometimes if the whole set X belong to the bornology β or $\forall\ x \in X$, $\{x\} \in \beta$ or X=$\bigcup_{B\in\beta} B$.

Types of a bornology:

1. The discrete bornology is the collection of all subsets on X, i.e. discrete bornology $= 2^X$.
2. The usual bornology (canonical bornology) is the collection of all usual bounded subsets on X. i.e. usual bornology $= \{B \subseteq X: B\ is\ usual\ bounded\}$.
3. The finite bornology is the collection of all finite bounded subsets on X.

Definition 1-3-2 [5]:

A ***base*** $\boldsymbol{\beta_o}$ ***for a bornology*** β on X is any sub collection of β such that every element of bornology is contained in an element of the base β_o.

Example 1-3-3:

Let $X = \{1,3,5\}$.

$\beta \ = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{3,5\}, \{1,5\}, X\}$.

To satisfy that β is a bornology of X.

We must satisfy the three conditions:

1. Since $X \in \beta$, then β a covers X .
2. If $B \in \ \beta, A \subseteq B$, then $A \in \beta$.

   Since β is the set of all subsets of X, i.e. $\beta = p(X) = 2^X$.

   Then β is stable under hereditary.
3. β is stable under finite union, $\bigcup_{i=1}^{n} \beta_i \in \beta \ \forall \ B_1, B_2, \ldots B_n \in \beta$.

Then β is a bornology on X which we can define only one bornology on finite set it is called discrete bornology.

Now to find a base β_o for β. Let $\beta_o = \{X\}$ or $\beta_o = \{ \{1,3\}, \{1,5\}, \{3,5\}, X\}$

Example 1-3-4 [13]:

let $\mathbb{R}$ be the set of real numbers (with Euclidean norm (absolute value)), we want to define usual bornology on $\mathbb{R}$ (canonical bornology), i.e. β is the collection of all usual bounded sets on $\mathbb{R}$.

In fact, (a subset B of $\mathbb{R}$ is bounded if there exists bounded interval (a, b) such that $B \subseteq (a, b)$) see [8].

So, $\beta = \{B: B \subseteq (a, b): \forall a, b \in \mathbb{R}\}$

1. Since $\forall\ x \in (a, b)$, we have and $\{x\} \subset (a, b)$ (the bounded intervals with respect to absolute value where the absolute value divided $\mathbb{R}$ into bounded interval, and $\{x\} \subset (a, b)$ (every subset of bounded interval is bounded).

   Implies that $\forall\ x \in \mathbb{R}$, $\{x\} \in \beta$.

   Then β covers $\mathbb{R}$.

2. If $B \in \beta$and $A \subseteq B$, there is bounded interval such that $A \subseteq B \subseteq (a, b)$.

   Therefore $A \in \beta$, and β stable under hereditary.

3. If $B_1, \dots, B_n \in \beta$,

   $B_1 \subseteq (a_1, b_1)$.

   $B_2 \subseteq (a_2, b_2)$.

   .
   .
   .

   $B_n \subseteq (a_n, b_n)$.

   Then $\exists\ L_1, \dots, L_n$ minimum upper bound and

   $g_1, \dots, g_n$ maximum lower bound, such that

$\bigcup_{i=1}^{n} B_i$ has finite union of upper and lower bounds.

Take $\quad g = \min_{1 \le i \le n} \{a_1, a_2, \dots, a_n\}$

$$L = \max_{1 \le i \le n} \{b_1, b_2, \dots, b_n\}$$

So, $\bigcup_{i=1}^{n} B_i \subseteq (g_i, L_i)$, i.e.

$\bigcup_{i=1}^{n} B_i$ is bounded subset of $\mathbb{R}$, i.e. $\bigcup_{i=1}^{n} B_i \in \beta$.

Then β is a bornology on $\mathbb{R}$ and it is called usual bornology.

And the base of this bornology is:

$\beta_o = \{B_r(x) : r \in \mathbb{R}, x \in \mathbb{R}\} = \{(x - r, x + r) : r \in \mathbb{R}, x \in \mathbb{R}\}$.

Example 1-3-5:

Let $X = \mathbb{R}^2$ with Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{2} |x_i|^2\right)^{1/2}$$

where $x = (x_1, x_2)$.

$$D_r(b) = \{x \in \mathbb{R}^2 : \|x - b\| \leq r, \text{for } r > 0\}$$

be a disk of the radius r with center at $b = (b_1, b_2)$.

A subset B of $\mathbb{R}^2$ is bounded if there exists a disk with center x_0 $D_r(x_0)$ such that $B \subseteq D_r(x_0)$, $x_0 \in B, r \geq 0$, see [13].

Let β_u be a family of all bounded subset of $\mathbb{R}^2$.

$\beta_u = \{B \subseteq \mathbb{R}^2 : B \text{ is usual bounded subset of } \mathbb{R}^2\}$.

Then $(\mathbb{R}^2, \beta_u)$ is a bornological set and β_u is called usual bornology on $\mathbb{R}^2$.

Such that:

1. It is clear that every closed disk in $\mathbb{R}^2$ is bounded set that means, $B \in \beta_u$ since $\mathbb{R}^2$ is covered by the family of all disks.

   Then, β_u covers $\mathbb{R}^2$.

2. If $B \in \beta_u$ and $A \subseteq B$, then there is closed disk such that $A \subseteq B \subseteq D_r$. Therefore $A \in \beta_u$.
3. If $B_1, B_2 \in \beta_u$ then D_{r_1}, D_{r_2} are closed disks such that $B_1 \in D_{r_1}, B_2 \in D_{r_2}$.

And $r = Max_{i=1,2}\{r_i\}$. Then $B_1 \cup B_2 \subseteq D_r$.

Thus, the finite union subsets of $\mathbb{R}^2$. Then β_u is a bornology on $\mathbb{R}^2$.

A base of bornology on $\mathbb{R}^2$ is any sub family such that

$$\beta_0 = \{D_r(x_0): r > 0\}.$$

Remarks 1-3-6 [13]:

1- Every bornology forms a base for itself and have more than one base on X.

2- Not every family of subsets of a set X will form a base for a bornology on X. For example $X = \{1,3,5\}$, $\beta_0 = \{\{1\},\{3\},\{5\}\}$ then β_0 cannot be a base for any bornology because the base must be contain large subsets of X.

Definition 1-3-7 [5]:

Let (X, β) and (Y, β') be bornological sets a map $f: (X, \beta) \to (Y, \beta')$ is called ***bounded map*** if the image for every bounded set in (X, β) is bounded set in (Y, β'). That means, $\forall B \in \beta \Rightarrow f(B) \in \beta'$.

The identity map between two bornological sets is bounded map see [5].

Example 1-3-8:

Let $X = Y = \{1,2,3\}$ with two discrete bornological set.

$\beta = \beta' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$.

We define $f: (X, \beta\ ) \to (Y, \beta')$.

As follows

$$f(B) = B', \quad \forall B \in \beta, \ B' \in \beta' .$$

As $\quad f(\emptyset) = \emptyset \ , \ f(\{1\}) = \{1\} \ , f(\{2\}) = \{2\}, \ f(\{3\}) = \{3\}.$

It is clear that f is bounded map.

Example 1-3-9:

Let $(\mathbb{R}, \beta)$ be the canonical bornological set.

Then, the map $f: \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined as $f\ (x) = \frac{1}{x}$ is not bounded map.

Indeed, $B = (0,1]$ bounded set in $\mathbb{R}\backslash\{0\}$, and the image of this bounded set under f is $[1,\infty)$ that is not bounded in $\mathbb{R}$.

Therefore f is not bounded map.

Proposition 1-3-10:

Let $f: (X, \beta) \to (Y, \beta')$ and $h: (Y, \beta') \to (Z, \beta'')$ be two bounded maps. Then the composition $f \circ h : (X, \beta\ ) \to (Z, \beta'')$ is bounded map.

Proof:

Suppose $B \in \beta$ since $f: (X, \beta) \to (Y, \beta')$ is bounded map then $f(B)$ is bounded set and $f(B) \in \beta'$.

Again since $h: (Y, \beta') \to (Z, \beta'')$ is bounded map.

It follows that $f(h(B)) \in \beta''$.

So, the composition $f \circ h: (X, \beta'\ ) \to (Z, \beta'')$ is bounded map.

Definition 1-3-11 [8]:

A map f between two bornological sets (X, β_X) and (Y, β_Y) is called a ***bornological isomorphism*** if f it is bijective and f, f^{-1} are bounded maps.

Consequently, for every $B \in \beta_X$ there is $B' \in \beta_Y$, such that:

$B' = f(B),\ B = f^{-1}(B')$.

Example 1-3-12:

Consider $X = (-3,3)$ and $Y = (-5,7)$ with usual bornology

and $f: \mathbb{R} \to \mathbb{R}$, defined as $f(x) = 2x + 1$

Clearly that f is bijective and bounded map.

Consider $X = (-3, 3)$, which is bounded subset of $\mathbb{R}$.

Since the image of this bounded set $(-3, 3)$ under f is $(-5, 7)$ and it is bounded set.

Then, f bounded map.

Furthermore f^{-1} exists and bounded map.

Then, f is bornological isomorphism.

Proposition 1-3-13:

If f is bornological isomorphism, then f^{-1} is also bornological isomorphism.

Proof:

Let f is bijective, then f^{-1} is bijective.

Since f is bornological isomorphism then f^{-1} is bounded also

$f = (f^{-1})^{-1}$ is bounded.

Therefore, f^{-1} is bornological isomorphism.

Definition 1-3-14 [5]:

Let (X, β) be a bornological set and let $Y \subseteq X$. Then the collection $\beta_Y = \{B \cap Y : B \in \beta\}$ is a bornology on Y. The bornological set (Y, β_Y) is called a ***bornological subset*** of (X, β), and β_Y is called ***relative bornology*** on Y.

Example 1-3-15:

Let $X = \{a, b, c\}$.

$\beta = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ and let $Y = \{a, b\}$ then

Then, the relative bornology on Y, $\beta_Y = \{\emptyset, Y, \{a\}, \{b\}\}$.

Definition 1-3-16 [8]

Let (X, β), and (Y, β') be two bornological sets, $\beta_0 = \{ B \times B'$ where $B \in \beta, B' \in \beta'\}$. We say that β_0 is a base, and is called ***product bornology***.

Remark 1-3-17:

If we defines a bornological structure on $X \times Y$, then the product set $X \times Y$ with this bornological structures is called a ***bornological product sets*** of (X, β), and (Y, β').

Example 1-3-18:

Let $X = \{a, b\}$, $Y = \{1,2\}$.

We defined a discrete bornological on X, Y.

$$\beta = \{\emptyset, X, \{a\}, \{b\}\},$$

$$\beta' = \{\emptyset, Y, \{1\}, \{2\}\}.$$

$$\beta \times \beta' = \{\emptyset, X \times \emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times \emptyset, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}, \{b\} \times \emptyset, \{b\} \times Y, \{b\} \times \{1\}, \{b\} \times \{2\}\}$$

$$\beta \times \beta' = \{\emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}, \{b\} \times Y, \{b\} \times \{1\}, \{b\} \times \{2\}\}$$

$$\beta \times \beta' = \{\emptyset, X \times Y, \{(a,1),(b,1)\}, \{(a,2),(b,2)\}, \{(a,1),(a,2)\}, \{(a,1)\}, \{(a,2)\}, \{(b,1),(b,2)\}, \{(b,1)\}, \{(b,2)\}\}$$

$\beta_0 = \beta \times \beta'$ is a base, and is called product bornology. And if

$X \times Y = \{(x,y): x \in X, y \in Y\}$

$= \{(a,1),(a,2),(b,1),(b,2)\}$, define a discrete bornology on $X \times Y$.

$$\beta_{X\times Y} = \{\emptyset, X \times Y, \{(a,1)\}, \{(a,2)\}, \{(b,1)\}, \{(b,2)\}, \{(a,1),(b,1)\}, \{(a,1),(b,2)\}, \{(a,2),(b,1)\}, \{(a,2),(b,2)\}, \{(a,1),(a,2)\}, \{(b,1),(b,2)\}, \{(a,1),(a,2),(b,1)\}, \{(a,1),(a,2),(b,2)\}, \{(a,1),(b,1),(b,2)\}, \{(a,2),(b,1),(b,2)\}\}$$

$(X \times Y, \beta_{X\times Y})$ is called a bornological product sets.

1.4 Bornological Group

In this section we recall the definition of bornological group and provided some examples in details.

Definition 1-4-1 [16]:

A ***bornological group*** (G, β) is a set G with two structures:

1. $(G,*)$ is a group;
2. β is bornology on G.

Such that, the product map $f : (G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$ is bounded and the inverse map $f^{-1}: (G, \beta) \longrightarrow (G, \beta)$ is bounded.

In the other words, a bornological group is a group G together with a bornology on G such that the group binary operation and the group inverse maps are bounded with respect to the bornology.

Let G be a bornological group and B_1, B_2 be any two bounded subsets of G. We denote for the image of $B_1 \times B_2$ under product map

$$(G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$$

by $B_1 * B_2 = \{b_1 * b_2 : b_1 \in B_1, b_2 \in B_2\}$, $*$ is the binary operation defined on the group, and $B^{-1} = \{b^{-1} : b^{-1} \in B\}$.

Example(1.4.2):

A bornology on general linear group.

Let $G = (\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}_{2\times 2}, \cdot\ )$ be a group, $a, b \in \mathbb{R}^+$, which is the set of matrix elements and multiplication operation.

We can define a finite bornology β on this group, which is the collection of all finite subsets of G.

To show G with finite bornology β is a bornological group (G, β).

We have to show that, the product map and inverse map are bounded.

1. $f : (G, \beta) \times (G, \beta) \rightarrow (G, \beta)$.

Let M_1, M_2 are two bounded sets (finite sets) in (G, β) we have to show that

$f\ (M_1 \times M_2)$ is bounded.

$$f(M_1 \times M_2) = M_1.M_2 = \{m_1 . m_2 : m_1 \in (G, \beta), m_2 \in (G, \beta)\}$$

$$= \{m_1.m_2 : m_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & 1 \end{bmatrix} \text{ and } m_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & 1 \end{bmatrix}, a_1, a_2, b_1, b_2 \in \mathbb{R}^+\}$$

$$= \{\begin{bmatrix} a_1 & b_1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} a_2 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 . a_2 & a_1 . b_2 + b_1 \\ 0 & 1 \end{bmatrix}\} \subset (G, \beta).$$

Thus, the product map is bounded.

2. $f^{-1} : (G, \beta) \rightarrow (G, \beta)$.

Let $M \in (G, \beta)$, $M = \{ m : m = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, a, b \in \mathbb{R}^+ \}$, which is finte set.

Then, $M^{-1} = \{m^{-1} : m^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{a} \\ 0 & 1 \end{bmatrix}\} \subset (G, \beta)$.

So, f^{-1} is bounded. Then, (G, β) is a bornological group.

Example 1-4-3:

Let G=$(Z, +)$ be an additive group and let β be the finite bornology define on G.

To show G with finite bornology β is a bornological group (G, β).

We have to show that, the product map and the inverse map are bounded.

1. $f : (G, \beta) \times (G, \beta) \rightarrow (G, \beta)$.

المستخلص

الفضاء البرنولوجي هو بنية لحل مشاكل التقييد بالنسبة للمجموعات والزمر والدوال بصورة عامة. الهدف الرئيسي من هذه العمل هو دراسة اصناف جديدة من الزمر البرنولوجية فيما يتعلق بالدوال المقيدة من النوع S والدوال المقيدة من النوع *S والدوال المقيدة من النوع **S. تشكل هذه الهياكل الجديدة هياكل مرتبطة بالزمر البرنولوجية تسمى هنا الزمرة البرنولوجية من النوع S والزمرة البرنولوجية من النوع *S والزمرة البرنولوجية من النوع **S .ايضا في هذا العمل تمت دراسة اصناف جديدة من شبه الزمرة البرنولوجية تسمى شبه الزمرة البرنولوجية من النوع S وشبه الزمرة البرنولوجية من النوع **S. ان الدافع لدراسة هذه الهياكل الجديدة هو طلب شرطا أقل تقييدًا على عمليات الزمر، ولا يلزم تقييد أي من العملية. سنفترض ان عمليات الزمر هي دوال مقيدة من النوع S ودوال مقيدة من النوع *S ودوال مقيدة من النوع **S.

النتائج المهمة الرئيسية هي تم اثبات كل زمرة برنولوجية هي زمرة برنولوجية من النوع S، أعطينا شرطا معينًا لأي مجال مقابل لزمرة برنولوجية من النوع *S ليكون زمرة برنولوجية من النوع *S. بالإضافة الى ذلك ، اثبتنا كل تحويل أيسر (أيمن) يكون تماثلاً برنولوجياً من النوع S وتماثلاً برنولوجياً من النوع *S وتماثلاً برنولوجياً من النوع **S. واثبتنا كل شبه زمرة برنولوجية هي شبه زمرة برنولوجية من النوع S وشبه زمرة برنولوجية من النوع **S .

أخيراً ، منذ أن وضعت المعرفة الجديدة كل هيكل جديدة في نظرية الفئة لذلك ، أصبح مصدر اهتمام العديد من الباحثين ، لوضع كل هيكل جديد في نظرية الفئة و مناقشة مفهوم الضرب المباشر والضرب الاساسي والمعادلات. هذا يحفزنا على وضع هيكل المجموعة شبه البرنولوجية في نظرية الفئة.