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Chapter One

Basic Concepts and Preliminaries

1.1 Introduction

This chapter is devoted to present some basic concepts for inverse problem. In general, the inverse problem is a kind of problems in which the cause reconstructed from the observed effects. In this chapter we are going to give the definitions and an example of direct problem, inverse problem and ill-posed problem.

1.2 Partial differential equations [30]

Starting by giving some basic definitions:

Definition 1.2.1 (Partial differential equation) :-

A partial differential equation (PDE) is similar to an ordinary differential equation (ODE), except that the dependent variable is a function of not just one ,but of several independent variables. The order of a PDE is the order of the highest order derivative that appears in the equation . if all first order partial derivatives of $u = u(x, y)$are continuous in a region Ω of R^2, the gradient of u is define by

$$grad\ u \equiv \nabla u = \frac{\partial u}{\partial x}\vec{\imath} + \frac{\partial u}{\partial y}\vec{\jmath}$$

The Laplacian of u is

$$\Delta u = \nabla . \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

If $\vec{n}$ denotes a unit vector in $R^2$the directional derivative of u in the direction n is

$$\frac{\partial u}{\partial n} = \nabla u.n$$

Definition 1.2.2 (Initial and Boundary Condition) [27]

PDE that models physical systems usually have infinity many solutions .To select one function that represented the solution, certain additional conditions are imposed. These can be boundary or initial condition .the boundary condition must be satisfied at a point on the boundary of the region Ω in which the PDE exists ,notated as $\partial\Omega$. Initial condition must be satisfied throughout the region Ω at the instant the physical system begins. This usually involves some combinations of u and its time derivatives where t=0.

Types of Boundary Conditions [26] :-

1. **Dirichlet-type Boundary Condition:**

This type of boundary conditions are named after the German mathematician Dirichlet, in case that. $u(x, y)$ is given at all point $(x, y) \in \partial\Omega$

For example : $\nabla^2 y + y = 0$,

$$y(x) = f(x) \quad \forall x \in \partial\Omega$$

$f(x)$ Is known function defined on $\partial\Omega$

2. **Neumann-type Boundary Condition:**

This type of conditions are named after the German mathematician Neumann, in case the value of the gradient of the dependent variable is the normal derivative on the boundary $\frac{\partial u}{\partial n}$ it is given at all point $(x, y) \in \partial\Omega$.

For example : $\nabla^2 y + y = 0$

$\frac{\partial y}{\partial n}(x) = g(x)$ *where*, $g(x)$ Is known function defined on $\partial\Omega$

There are also other type of boundary condition ,which are (**Robin ,Mixed and Cauchy),** but not within the limits of our study

1.3 Direct and Inverse Problems

Definition 1.3.1 (Direct problem) [8]:

The major concern in a direct problem is to determine the unknown solution within a domain from the known initial and boundary conditions. direct problems have been extensively studied over the last two centuries, resulting in a wealth of literature of procedures relating to their solution.

The following information are required to be known in a direct problem formulation such as:

1. The boundary of the solution domain
2. The governing equation in the domain
3. The boundary condition for the entire boundary and initial condition
4. The material properties
5. The forces acting in the domain

Definition 1.3.2 (Well-posed problem) [1]:

A problem is known as a well-posed in Hadamard's sense if it satisfies the following:

1.The existence of the solution at all the data.

2.The uniqueness of the solution.

3.The continuously dependence of the solution on the given (data) (which means the stability of the solution). That means, in case that the solution depends continuously on the data so a small error in the given data produces just a small error in the obtained solution.

An example of well-posed problems is the direct problem.

Definition 1.3.3 (Ill-posed problem) [1]:

A problem is considered as an ill-posed problem in Hadamard's sense of any of the well-posedness three condition is not satisfied. In case that the first condition is not satisfied, this implies that the problem does not have a solution in the desired space. In case that the second condition is not satisfied, this implies that the problem has more than one solution in the space and the challenge is to choose the appropriate one. In case that the third condition is not satisfied, this implies that the solution procedure is unstable(**unstable):**it means that the small perturbations of the Cauchy data can lead to high deviation of the numerical solution from the exact solution.

1.3.4 Some Examples of Ill-posed problems [33]:

1. The Cauchy problem for Laplacian.
2. The Cauchy problem for Helmholtz or modified Helmholtz equation.
3. Solving ill-conditioned linear algebraic system.
4. The backwards heat equation problem.
5. Minimization problems.

In the following we recall the definition of inverse Cauchy problem.

Definition1.3.5 (Inverse Cauchy problem) [18]:

In these kinds of problems, the boundary of a solution domain Ω of the studied problem is known just on part of the boundary Γ_1, some boundary conditions are over-specified while on the remaining part of the boundary $\Gamma_2 = \partial \Omega / \Gamma_1$ either are nongiven or they are under specified.

Definition 1.3.6 (Inverse problem) [8]:

An example of ill-posed problem is the inverse problem. In fact, an inverse problem is stated to identify the inputs from the output. An inverse problem can be considered as an opposite of the direct problem. When one or more of the conditions of direct problems may be unknown, or do not completely known this gives arise to formulate an inverse problem. It is Important to determine the unknown condition by benefiting from the over specified conditions. In case of this kind of problems, the noise of the given Cauchy data may have an effect in the solution. The over specified data has an important role to solve the problem.

In general, inverse problems has one of the following roles:

1. Determination of a part of the boundary.
2. Inference of the equation that governs the problem.
3. Identification of the condition on the boundary and \or initial condition.
4. Determination of the properties of the material.
5. Determination of the domain forces.

1.3.7 Some Kinds of Inverse Problems [22]:

1. Inverse boundary value problems: For example, the Cauchy problem for Helmholtz or modified Helmholtz equation.

2. Inverse initial –value problems: For example, the backward heat conduction problem, in this case the initial temperature displacement or velocity are not available.

3. Inverse source \force problems: Identifying distributed source of pollution, heat generation or wave excitations.

4. Inverse coefficient problems: Determination of un measurable physical properties that can be determined by measuring other physical quantities in a single and stable manner .

5. Inverse geometry problems: Obstacle identification, for example crack detection, corrosion, inverse scattering and tomography.

Definition 1.3.8 (Condition Number) [35]:

The condition number of a matrix A, denoted by $\kappa(A)$, is defined by the following:

$$\kappa(A) = \|A\|\|A^{-1}\|$$

$\kappa(A) \geq 1$ for all A

A is **well-conditioned** matrix if $\kappa(A)$ is small this means (stable)

(small change in the data produce small changes in the solution)

A is **ill-conditioned** matrix if $\kappa(A)$ is large this means (unstable)

(small change in the data produce large changes in the solution)

1.4 Modified Helmholtz Equation [1]

1.4.1 Definition (Helmholtz and modified Helmholtz equation):

A problem stated by:

$$\Delta u + k^2 u = F$$

with some conditions is called Helmholtz equation. While a problem stated by:

$$\Delta u - k^2 u = F$$

with some conditions is called modified Helmholtz equation. Where $k \in R$

1.5 Some numerical analysis concepts

Definition 1.5.1 (Meshless Method) [26]:

In the 1970s the Meshless methods are introduced to solve problems in astrophysics. For this type of methods the domain is discretized by a set of nodes. un similar to the mesh-based methods in which the domain is discretized by elements (for example: finite difference method, finite element method, finite volume method).

In each numerical method there is always some error produced from the approximation of the exact solution by some calculated one using one of the numerical methods

1.5.2 Some Types of Errors [35]:-

1. Absolute Error: The difference between the exact value and the Approximate value $\Delta x = x_0 - x$. important to find the most probable of the measurement

2. Relative Error: The Absolute error divided by the exact value. important determine how good or bad our measurement .

Definition 1.6 (Stopping Criterion) [34]:

In each numerical method we aim to find some solutions $\boldsymbol{c}$ by computing a sequence of approximate solution that converges to the desired solution, in practice we use the following stopping criteria which are defined by using the absolute and relative residual:

$$\|r_i\| < Tol$$

$$\|r_i\|/\|b\| < Tol$$

Where Tol is an error tolerance chosen by the user.

Definition 1.7 (Symmetric Matrix) [31]:

A square matrix A that equal to its transpose $A^T = A$ if A is symmetric, then A^T is also symmetric such as $a_{ij} = a_{ji}$.

Definition 1.8 (Positive Definite Matrix) [31]:

An $n \times n$ matrix A is called positive definite if it is symmetric, $A^T = A$,and satisfies the positive condition $x^T Ax > 0 \ \ for\ all\ \ 0 \neq x \in R^n$ we will some time write $A > 0$ to mean that A is positive definite matrix .

Definition 1.9 (Inner product) [31]:

The product matrix of size $1 \times n$ (the row vector x^T) with an $n \times 1$ matrix (the column vector y). Let $x = [x_1, x_2, \dots, x_n], y = [y_1, y_2, \dots, y_n]$

The inner product is stated by $x^T . y = [x_1 \dots x_n] \begin{bmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$

The combination appears in every discussion of the geometry of n-dimensional space its sometimes called the scalar product or dot product of the two vectors ,and denoted by $(x, y)\ or\ x.y$ but we prefer to call it the inner product and keep the notation $x^T y$.

المستخلص

أحد أنواع المسائل العكسية هي مسألة كوشي العكسية لمعادلة هيلمهولتزالمعدلة ,هذا النوع من المشكلة ينشا في أحد التطبيقات الحياة , وهو التوصيل الحراري في الزعنفة .الهدف من هذه الدراسة هو تحديد درجة الحرارة على حدود غير محددة ,بالاستفادة من الجزء الذي يمكن الوصول اليه من الحدود مع بيانات كوشي والتي هي درجة الحرارة على الحدود التي يمكن الوصول اليها والتدفق الحراري في هذا الجزء .

يتم حل هذه المسالة باستخدام بعض الطرق العددية التي هي الطريقة اللاشبكية عن طريق التعبير عن الحل كتوسع متعدد الحدود والتحقق من مشكلتنا لهذا التوسع متعدد الحدود الذي ينتج نظام خطي يتم حله من قبل أثنين من الخوارزميات العددية المختلفة (CGM),(CGLS) يتم مقارنة الحلول التقريبية التي تم الحصول عليها من قبل هتين الخوارزميتين مع الحل الدقيق.التحقق من دقة هذه الطريقة المقترحة .تم دراسة العديد من الامثلة مع بعض المشاكل متعددة حدود وغير متعددة حدود على المجالات المنتظمة من خلال الاستفادة من كفأه الطريقة اللاشبكيه .

من المعروف أن مسألة كوشي المعكوسة هي مسألة مطروحة بشكل سي ء وبالإضافة الى ذلك مشكله غير مشروطه للغاية لذلك يتم التأكيد الاستقرار من خلال تطبيق الضوضاء لبيانات كوشي . لتقليل تأثير هذه الحالة السيئة للغاية يتم تطبيق تنظيم تيكانوف والتكييف المسبق .