**جمهورية العراق**

**وزارة التعليم العالي والبحث العلمي**

**جامعة ديالى/كلية العلوم/ قسم الرياضيات**


# حل معادلات ديوفانتين

# مع أو بدون عدد غير نسبي بأستخدام الكسور المستمرة على عدد كاوسيان الصحيح

رسالة مقدمة

الى/جامعة ديالى/كلية العلوم/ قسم الرياضيات كجزء من متطلبات نيل شهادة الماجستير في علوم الرياضيات


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# Chapter One

## General Introduction

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## 1.1 Overview

Continuous fraction are one method of representing a real number (both finite and infinite) by the sums of consecutive division of the number. different regions have made use of continuous fraction. They gave us a way to construct approximations of rational numbers to ir-rational numbers. Some computer algorithms have used continuous fraction to solve these approximations.

The word Diophantine refers to the Hellenic mathematician from the third century who studied these equations and was the first mathematician to introduce symbolism in algebra.

The linear Diophantine equation is an algebraic equation, sometimes called an equation of infinity with variables, whose solution is integers only. In jargon that is more specific, they are said to be drawing an algebraic curve, an algebraic surface, or something more general.

In general there are different types of Diophantine equations ranging from linear to quadratic to some polynomials of higher order than 2.

The number in Gaussian field can be written in the form, where it represents the real part and the imaginary part. It is also called complex numbers, and these numbers are relative to the scientist Gauss, who touched on a number of properties of these numbers such as addition, subtraction, division, norm, unique factors, and so on.

Our study in this thesis will be limited to solve linear Diophantine equations of a type $\alpha x + \beta y = \delta$ using continuous fractions on the field of Gaussian integers to obtain an indefinite linear equation $\alpha x + \beta y = \delta$, and if ($x_0$+$y_0$) the basic solution, we can also obtain general solutions ($x_n$+$y_n$) using different approximations as illustrated in to the thesis, continuous fractions will be used to solve the types of indefinite equations mentioned above.

In these equations, he needs to mention some results with proofs, as well as some definitions to facilitate understanding of these proofs. After this introduction to the problems that we have to solve, we introduce the technique of continuous fractions in the Gaussian field.

## 1.2 Related Works

mathematically, new results are always constructed from previous ones. By doing so, one may improve upon existing success rather than starting from scratch. Therefore, it is necessary to learn about and contribute to the field of continuous fractions if needed in order to learn about its past.

- Euclid's approach for finding the greatest common divisor may be seen as the genesis of continuous fractions. As a byproduct, this method produces a continuing fraction. [27]

- A linear indeterminate equation had been solved by the Indian mathematician Aryabhata using continued fractions. Additionally, continuing fractions are often shown and hinted to in both Greek and Arabic literature. [27]

- In the field of mathematics, important advancements were achieved primarily by two pupils : Pietro Cataldi and Rafael Bombelli, both of them were born and raised in Bologna, Italy. Bombelli was the first mathematician to apply the concept of continuous fractions. He did so in his book L'Algebra, which was produced in the year 1572. When he was trying to figure out what the square root of 13 would be, he made an approximation that ended up giving him a continuous fraction. Cataldi had previously completed work along these lines for the square root of 18. He represented 18 as 4, where each dot denotes an addition of the next fraction to the denominator. This led to an incorrect representation of the number 18. It seems that he was the first one to suggest a symbolism for continuous fractions in his book that he authored entitled Trattato del modo brevissimo. While researching the Radici quadra delli numeri in 1613, However, other from those specific examples, none of them conducted a comprehensive investigation of the properties of continuous fraction.[27]

- One of the earliest mathematicians to make a significant contribution to finding the convergents of the continuous fractions was Daniel Schwenter, in 1625. In particular, he was interested in simplifying expressions involving huge numbers. The current procedures for computing consecutive convergents were established by him. [28 ]

- Viscount The first independent study of continuous fractions was conducted by William Brouncker in 1655; his work was published in 1656 by his friend John Wallis's Arithmetica infinitorum. Wallis defined several characteristics of convergents and documented how to compute the nth convergent in his book Opera Mathematica; for example, $\frac{4}{\pi} = 1 + \frac{1}{2+\frac{9}{2+\frac{25}{2+\frac{49}{2+\cdots}}}}$ .(1695)However, Brouncker found a solution to the Diophantine Equation $x^2 - Ny^2 = 1$. [27,29]

- The Dutch mathematician Huygens in 1687 used continuous fractions for the first time. He used the convergents of a continuous fraction to determine the optimum rational approximations for gear ratios, allowing him to build a functional mechanical planetarium.[31]

- Mathematicians like Leonard Euler, Johan Heinrich Lambert, and Joseph Louis Lagrange helped to expand continued fraction theory in successive decades. For the most part, the modern theory can be traced back to Euler's work De Fractionlous Continious (1737). In order to express irrational and transcendental number, he constructed infinite series in which the terms were linked by continuing division.
- Such sequences were given the name "fractions continuae" by him, perhaps in homage to John Wallis's usage of the phrase "fractions continuae fractae" (continuously shattered fraction) in his Arithmetica Infinitorum. To prove the irrationality of $e$ and $e^2$, he also discovered an equation for e in continuing fraction form. He utilized continuing fractions to differentiate between rationals

and irrationals and proved that every rational can be written as a finite, simple continued fraction. After that, Euler provided the procedure that is still used today to turn a simple fraction into a continuing fraction.

- In addition, he proved a theorem that every such continuing fraction is the solution to a quadratic equation and calculated a continued fraction expansion of (2), which gives us a simple method for calculating the exact value of any periodic continued fraction. [27,29]

- Using a continuous portion of tan (x), Lambert proved the irrationality of $\pi$ in 1761. His work on e was an extension of Euler's, and he showed that if x is a nonzero rational, then $e^x$ and tan x are irrational. [27]

- A generic solution to Pell's Equation was found by Lagrange using continuous fractions. By proving that the expansion of an irrational number by regular continuous fractions is periodic, he proved the converse of Euler's Theorem, which states that if an irrational number x is a solution to a quadratic equation, then x is a solution to the equation. Lagrange used continued fractions to develop a general method for obtaining the continuing fraction expansion of the solution to a differential equation in a single variable during his work on integral calculus in 1776.[32]

- In the eighteenth century, the idea of convergents was established and the topic of continuing fractions became common knowledge among mathematicians. In 1813, Carl Friedrich Gauss used a clever identity utilizing the hypergeometric function to develop a highly broad complex-valued continuing fraction. The term "Pade approximant" was coined by Henri Pade. one may say that this century is the pinnacle of continuous fractions. Mathematics has benefited from the work of many great minds, including Jacobi, Perron, Hermite, Cauchy, Stieljes, and many more. [27]

- In the year 1997 the scientist Bosma, W., Cannon, J., Playoust, wrote a research titled The Magma algebra system i: The user language. J. Symb. Comput. [36]

- And In the year 2018 the scientist Bremner, A., Tho, N.X., wrote a research titled The equation $(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = n$. Int. J. Number Theory. [37]

- And In the year 2000 the scientist Bondarenko, A.Vwrote a research titled Investigation of one class of Diophantine equations. [38]

- And In the year 2022 the scientist Oleg N. Karpenkov a research titled Geometric Continued Fractions. [39]
- And In the year 2008 the scientist Luchko, Y.F.; Martinez, M.; Trujillo a research titled Fractional Fourier transform and some of its applications. [40]
- And In the year 2019 the scientist Upadhyay, S.K.; Khatterwani, K a research titled Upadhyay, S.K.; Khatterwani, K.[41]

## 1.3 Motivation

The searcher find themselves in important situations, including making decisions about how to deal with what they want, for example, a man who has some amount of money and needs to buy at least two different types of goods (for example) where he faces a quantity of different goods that can be bought with his money. This type of attitude is what is known Linear Diophantine Equation . Early mathematicians encountered many puzzles that led to equations Diophantine that made studying an interesting equation and often difficult.

## 1.4 Problem Statement

The Diophantine equation is an equation in several unknowns, and therefore the researcher tends to ask several questions, including:
1-Does the equation has a solution in the Gaussian integer field?
2-Are there finite or infinite solutions?
3- Can the researcher find solutions from a theoretical point of view?
4-Can the researcher practically complete the list of solutions?
For these questions and many other questions we will provide answers to them using continuous fraction techniques to solve equations DLP in a G.I. field.

## 1.5 Aim of Thesis

At the end of this thesis:
1.We determined to show that not all diophontain equations are solvable in a Gaussian integers with examples.
2. The convergence obtained from the continuous fractions of the numbers involved in the equations provides solutions to those equations.
3. Displaying different ways to make a decision.
4- Also to show that an equation has no solution.

## 1.6 Thesis Outline

This thesis contains the following chapters:

- **Chapter one:( General Introduction)**

This chapter contains an introduction, previous works, the problem, its solution, and the objective of the thesis

- **Chapter tow:( Definitions and Basic Concepts)**

This chapter explains some basic definitions and basic concepts that will help us in this letter

- **Chapter three: (Finite and infinite Simple Continued Fractions)**

This chapter explains the equation of continuous fractions with integers and how to deal with them, as well as the linear Diophantine equation and some examples and applications on it

- **Chapter four: (Solving The linear Diophantine Equation In Gaussian Integers)**

This chapter explains continuous fractions with Gaussian integers as well as the linear Diophantine equation with Gaussian integers

- **Chapter Five:(Conclusions and Suggestions for Future Works)**

In this chapter we give some concluding remarks which are derived from the outputs of the conducted tests are given in this chapter; also we give some suggestions for future works are presented.

# المستخلص

موضوع الكسور المستمرة هو فرع رئيسي من نظرية الأعداد بسبب تطبيقاتها في مجال نظرية الأعداد.

الكسور المستمرة هي طريقة لإيجاد أفضل التقريبات للأعداد المنطقية وغير المنطقية بمجموع القسمة المتتالية للأرقام.

في هذه الرسالة سوف ندرس الكسور المستمرة البسيطة ، والتقارب ، وخصائص التقارب وبعض الأمثلة عليها.

يتم استخدام التقارب وبعض النظريات لحل معادلة ديفوانتين الخطية في مجال الأعداد الغاوسية الصحيحة .

ندرس أيضًا الكسور المستمرة البسيطة اللانهائية وتقاربها وبعض خصائص التقارب للأرقام غير المنطقية (a + b) / c. ،e ، &) π٥) √ ،∜ ،∛

أخيرًا ، تمت دراسة التقارب وأفضل التقديرات التقريبية ، ثم يتم تطبيق بعض الخوارزميات الحاسوبية التي تستخدم الكسور المستمرة للتحقق من النتائج التي تم الحصول عليها في هذه الرسالة.