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Chapter One

Some Basic Concepts for

Bornological Structures

1.1 Introduction

In this chapter, we recall some basic concepts for bornological structures which it is used to solve the problem of boundedness for a set and a group in general way. It is a natural to study fundamental construction for this structure such as bornological subset, product bornological set, see [2].

1.2 Bornological Sets

In this section, we explain and clarify bornological structure with some detailed examples.

Definition [1.2.1](Bornological Set) [5]:

Let X be a nonempty set. A ***bornology*** β on X is a collection of subsets of X such that:

i. β forms a cover of X;

ii. β inclusion under hereditary, i.e. if $B \in \beta, \exists A \subseteq B$ then $A \in \beta$;

iii. β inclusion under finite union, i.e. if $\forall B_1, B_2 \in \beta$ then $B_1 \cup B_2 \in \beta$.

A pair (X, β) consisting of a set X and a bornology β on X is called ***a bornological set***, and the elements of β are called bounded subsets of X.

We can satisfy the first condition in different ways. First if the whole set X belong to the bornology or $\forall x \in X, \{x\} \in \beta$ or $X = \bigcup_{B \in \beta} B$.

Some Types of a bornology:

i. The **discrete bornology** (β_{dis}) is the collection of all subsets of X, i.e. discrete bornology P(X) =2^X.

ii. The **usual bornology** (β_u) is the collection of all usual bounded subsets of X, i.e. usual bornology = {B⊆X: B is usual bounded, with respect to X }.

iii. The **finite bornology** (β_{fin}) is the collection of all finite bounded subsets of X.

Definition[1.2.2](Base) [5]:

Let (X, β) be a bornological set. A ***base*** β_0 is a sub collection of bornology β, and each element of the bornology is contained in an element of the base.

Example [1.2.3]:

Let $X = \{1,3,5\}$, $\beta = \{\emptyset, X, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}\}$.

To satisfy that β is a bornology on X.

i. Since $X \in \beta$, then X is covering itself.

ii. If $B \in \beta, A \subseteq B$, then $A \in \beta$

Since β is the set of all subsets of X, i.e. $\beta = P(X) = 2^X$.

Then β inclusion under hereditary.

iii. β inclusion under finite union, i.e. $\bigcup_{i=1}^{n} B_i \in \beta, \forall\, B_1, B_2, \dots, B_n \in \beta$.

Since

$\{1\} \cup \{3\} = \{1,3\}$, $\{1,3\} \cup \{1,5\} = X$, $\emptyset \cup \{1,3\} = \{1,3\}$,

$\{1\} \cup \{5\} = \{1,5\}$, $\{1,3\} \cup \{3,5\} = X$, $\emptyset \cup \{3,5\} = \{3,5\}$,

$\{3\} \cup \{5\} = \{3,5\}$, $\{1,5\} \cup \{3,5\} = X$, $\emptyset \cup \{1\} = \{1\}$.

Then β is bornology on X. We can define only one bornology on finite set it is called discrete bornology.

Now, to find the base for bornology

$\beta_0 = \{\{1,3\}, \{1,5\}, \{3,5\}, X\}$, or $\beta_0 = \{X\}$.

Example [1.2.4]:

Let $\mathbb{R}$ be the set of real numbers (with Euclidean norm (absolute value)).

We want to define a usual bornology on $\mathbb{R}$ that mean β is the collection of all usual bounded sets on $\mathbb{R}$, in fact, (a subset B of $\mathbb{R}$ is bounded if and only if there exist bounded interval such that $B \subseteq (a,b)$).

So, $\beta = \{B: B \subseteq (a,b) : \forall a,b \in \mathbb{R}\}$.

i. Since $\forall\, x \in (a,b)$ and $\{x\} \subseteq (a,b)$ (the bounded intervals with respect to absolute value where the absolute value divided $\mathbb{R}$ into bounded interval), (every subset of bounded interval is bounded). Implies that, $\forall\, x \in \mathbb{R}, \{x\} \in \beta$. Then β covers $\mathbb{R}$.

ii. If $B \in \beta$ and $A \subseteq B$, there is bounded interval such that $A \subseteq B \subseteq (a,b)$.

Therefore $A \in \beta$, and β stable under hereditary.

iii. If $B_1, B_2, \dots, B_n \in \beta$, then there is $L_1, L_2, \dots, L_n$ least upper bound and $g_1, g_2, \dots, g_n$ greater lower bound, such that

$\bigcup_{i=1}^{n} B_i$ has finite union of upper and lower bounds.

Assume that $L = \max\{L_i\}$, and $g = \min\{g_i\}$

$\bigcup_{i=1}^{n} B_i$ has least upper bound L and greater lower bound g.

That mean $\bigcup_{i=1}^{n} B_i$ is bounded subset of $\mathbb{R}$.

That mean $\bigcup_{i=1}^{n} B_i \in \beta$. Then β is bornology on $\mathbb{R}$.

And the base of this bornology is:

$$\beta_0 = \{B_r(x): r \in \mathbb{R}, x \in \mathbb{R}\} = \{(x-r, x+r): r \in \mathbb{R}, x \in \mathbb{R}\}.$$

It is clear that every element of that bornology β is contained in an element of the base β_0.

Example[1.2.5]:

Let $X = \mathbb{R}^2$ with Euclidean norm

$$\|p\| = \left(\sum_{i=1}^{2} |x_i|^2\right)^{1/2}.$$

$$D_r(x_\circ) = \{x \in \mathbb{R}^2 : \|p - x\| \leq r, for\, r \geq 0\}$$

be a disk of the radius r with center at $x_\circ = (x_1, x_2)$.

A subset B of $\mathbb{R}^2$ is bounded if there exists a disk with center x_0 $D_r(x_0)$ such that $B \subseteq D_r(x_0)$.

Let β_u be a family of all bounded subset of $\mathbb{R}^2$.

$\beta_u = \{B \subseteq \mathbb{R}^2 : B \text{ is usual bounded subset } of\ \mathbb{R}^2\}$.

Then $(\mathbb{R}^2, \beta_u)$ is a bornological set and β_u is called usual bornology on $\mathbb{R}^2$.

i. It is clear that every disk in $\mathbb{R}^2$ is bounded set that means, $B \in \beta_u$ since $\mathbb{R}^2$ is covered by the family of all disks. Then β_u covers $\mathbb{R}^2$.

ii. If $B \in \beta_u$ and $A \subseteq B$, then there is disk such that $A \subseteq B \subseteq B_r$. Therefore $A \in \beta_u$.

iii. If $B_1, B_2 \in \beta_u$ then B_{r_1}, B_{r_2} are disks such that $B_1 \in B_{r_1}, B_2 \in B_{r_2}$.

And $r = Max_{i=1,2}\{r_i\}$. Then $B_1 \cup B_2 \subseteq B_{r_2}$.

Therefore, the determinate union subsets of $\mathbb{R}^2$. Then β_u is a bornology on $\mathbb{R}^2$.

A base of bornology on $\mathbb{R}^2$ is any sub family such that $\beta_0 = \{D_r(x_0) : r \geq 0\}$.

Definition [1.2.6](Sub Base)[8]:

Let (X,β) be a bornological set, a family $\mathbb{S}$ of members of a bornology β is said to be ***subbase*** for β if the family of all finite unions of members of $\mathbb{S}$ is a base for β.

Example[1.2.7]:

Let $=\{1,2,3\}$, $\beta=\{X,\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$.

Let $S=\{\{1\},\{2\},\{3\}\}$.

Then S is subbase for β where $\beta_0=\{\{1,2\},\{1,3\},\{2,3\},X\}$, or $\beta_0=\{X\}$ are bases for β.

Remarks[1.2.8]:

1. Not every family of subsets of a set X will form a base for a bornology on X.

   For example $=\{1,3,5\}$, $\beta_0=\{\{1\},\{3\},\{5\}\}$, then β_0 cannot be a base for any bornology because the base must be contain a large subsets of X see [12].

2. Let $(X,\beta),(Y,\beta')$ be bornological sets and $\beta_0,{\beta_0}'$ are bases for β and β', respectively. If $\beta_0\subseteq{\beta_0}'$, then $\beta\subseteq\beta'$.

For example: $X=\{1,2,3\}, Y=\{1,2,3,4\}$

$\beta=\{\emptyset,X,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$,

$\beta'=\{\emptyset,Y,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},$

$\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

$\beta_0=\{\{1,2\},\{1,3\},\{2,3\},X\}$,

${\beta_0}'=\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},Y,\{1,2\},\{1,3\},\{2,3\},X\}$.

It is clear that $\beta_0\subseteq{\beta_0}'$ then, $\beta\subseteq\beta'$.

Definition [1.2.9](Bounded Map) [5]:

Let (X,β) and (Y,β') be bornological sets. A map $\psi:(X,\beta)\to(Y,\beta')$ is called ***bounded map*** if the image for every bounded set in (X,β) is bounded set in (Y,β'). That means, $\forall B\in\beta\Rightarrow\psi(B)\in\beta'$.

Notice that, the composition of two bounded maps is bounded map.

Definition[1.2.10](Bornological Isomorphism)[9]:

A map ψ between two bornological sets (X,β) and (Y,β') is called a ***bornological isomorphism*** if ψ is bijective also, ψ , ψ^{-1} are bounded maps.

Consequently, for every $B\in\beta$ there is $B'\in\beta'$, such that

$B'=\psi(B), B=\psi^{-1}(B')$.

Example [1.2.11]:

Consider $X=\{10,11\}, Y=\{5,6\}$ with finite bornology on X,Y.

$$\beta=\{\emptyset,X,\{10\},\{11\}\},\beta'=\{\emptyset,Y,\{5\},\{6\}\}.$$

Define a function $\psi: X\longrightarrow Y$ such as,$\psi(X)=Y,\psi(\emptyset)=\emptyset$,$\psi(10)=5$,$\psi(11)=6$. It is clear that ψ is bijective and ψ and ψ^{-1} bounded map.

$\psi(\{10\})=\{5\}\ \Rightarrow\psi^{-1}(\{5\})=\{10\}$

$\psi(\{11\})=\{6\}\Rightarrow\psi^{-1}(\{6\})=\{11\}$

$\psi^{-1}(Y)=X, \psi^{-1}(\emptyset)=\emptyset$.

Proposition[1.2.12]:

If $\psi:(X,\beta)\longrightarrow(Y,\beta')$ and $\phi:(Y,\beta')\longrightarrow(Z,\beta'')$ are both bornological isomorphism, then the composition $\phi\circ\psi:(X,\beta)\longrightarrow(Z,\beta'')$ is bornological isomorphism.

Proof:

Since ψ and ϕ are one to one and onto,

then $\phi \circ \psi$ is one to one and onto (composition of any two one to one, onto maps is one to one, onto map respectively).

Since ψ and ϕ are bounded maps, then $\phi \circ \psi$ is bounded map.

Since ψ and ϕ are bornological isomorphism,

then ψ^{-1} and ϕ^{-1} are bounded also, $\psi^{-1} \circ \phi^{-1}$ is bounded,

since $\psi^{-1} \circ \phi^{-1} = (\phi \circ \psi)^{-1}$ is bounded.

Therefore, $\phi \circ \psi$ is bornological isomorphism.

Definition [1.2.13](Bornological Subset) [5]:

Let (X, β) be a bornological set and let $Y \subseteq X$. Then the collection $\beta_Y = \{B \cap Y : B \in \beta\}$ is a bornology on Y. The bornological set (Y, β_Y) is called bornological subset of a bornological set (X, β) also called ***relative bornology*** on Y.

Example [1.2.14]:

$X = \{2,4,6\}$, with discrete bornology on X.

$\beta = \{\emptyset, X, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}\}$.

And the base of bornology is $\beta_0 = \{\{2,4\}, \{2,6\}, \{4,6\}, X\}$.

And let $Y = \{2,4\}$.

Then we have

$$\beta_Y = \{\emptyset \cap Y, X \cap Y, \{2\} \cap Y, \{4\} \cap Y, \{6\} \cap Y, \{2,4\} \cap Y, \{2,6\} \cap Y, \{4,6\} \cap Y\}$$

$$\beta_Y = \{\emptyset, Y, \{2\}, \{4\}\}.$$

Then the subset of bornology is β_Y.

[1.2.15] Definition (Product Bornology) [9]:

Let (X, β), and (Y, β') be two bornological sets, $\beta_0 = \{$The family of all $B \times B'$ where $B \in \beta, B' \in \beta'\}$.We say that β_0 is a base, and is called ***product bornology***.

If we defines a bornological structure on $X \times Y$, then the product set $X \times Y$ with this bornological structure is called a bornological product sets of (X, β), and (Y, β').

Example [1.2.16]:

Suppose $X = \{a, b\}, Y = \{1, 2\}$.

We defined a discrete bornology on X, Y.

$$\beta = \{\emptyset, X, \{a\}, \{b\}\}$$

$$\beta' = \{\emptyset, Y, \{1\}, \{2\}\}.$$

$$\beta \times \beta' = \big\{\emptyset, X \times \emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times \emptyset, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}, \{b\} \times \emptyset, \{b\} \times Y, \{b\} \times \{1\}, \{b\} \times \{2\}\big\}$$

$$\beta \times \beta' = \big\{\emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}, \{b\} \times Y, \{b\} \times \{1\}, \{b\} \times \{2\}\big\}$$

$\beta \times \beta' = \{\emptyset, X \times Y, \{(a, 1), (b, 1)\}, \{(a, 2), (b, 2)\}, \{(a, 1), (a, 2)\}, \{(a, 1)\},$

$\{(a, 2)\}, \{(b, 1), (b, 2)\}, \{(b, 1)\}, \{(b, 2)\}\}$

$\beta_0 = \beta \times \beta'$ is a base, and is called product bornology. And if

$X \times Y = \{(x, y): x \in X, y \in Y\}$

$= \{(a, 1), (a, 2), (b, 1), (b, 2)\}$, define a discrete bornology on $X \times Y$.

$\beta_{X \times Y} = \{\emptyset, X \times Y, \{(a, 1)\}, \{(a, 2)\}, \{(b, 1)\}, \{(b, 2)\}, \{(a, 1), (b, 1)\},$

$\{(a, 1), (b, 2)\}, \{(a, 2), (b, 1)\}, \{(a, 2), (b, 2)\}, \{(a, 1), (a, 2)\}, \{(b, 1), (b, 2)\},$

$\{(a,1),(a,2),(b,1)\},\{(a,1),(a,2),(b,2)\},\{(a,1),(b,1),(b,2)\},$

$\{(a,2),(b,1),(b,2)\}\}$. $(X \times Y, \beta_{X\times Y})$ is called a bornological product sets.

1.3 Bornological Group

In order to address the issues of boundedness for groups, we will review the idea of bornological group in this section.

Definition [1.3.1](Bornological Group) [4]:

A bornological group (G,β) is a set with two structures:

i. $(G,*)$ is a group;

ii. β is a bornology on G.

Such that the product map $\psi: (G,\beta) \times (G,\beta) \longrightarrow (G,\beta)$ is bounded and $\psi^{-1}: (G,\beta) \longrightarrow (G,\beta)$ is bounded.

In the other words, a bornological group is a group G together with a bornology on G such that the group binary operation and the group inverse maps are bounded with respect to the bornology.

Let (G,β) be a bornological group and B_1, B_2 be two bounded subsets of G.

We denote the image of $B_1 \times B_2$ under product map

$$(G,\beta) \times (G,\beta) \longrightarrow (G,\beta)$$

By $B_1 * B_2 = \{b_1 * b_2 : b_1 \in B_1, b_2 \in B_2\}$, $*$ is the binary operation defined on the group which we want to bornologies it, and $B^{-1} = \{b^{-1} \in B\}$.

Example [1.3.2]:

Let $G = (\begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}_{2\times 2}, \cdot)$ be a group $e, f \in \mathbb{R}, e \neq 0$, which is the set of matrix elements and multiplication operation.

We can define a finite bornology β on this group, which it is a collection of all finite subsets of G.

To prove G with finite bornology β is bornological group (G, β).

We must prove that, the product map and the inverse map are bounded.

i. $\psi: (G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$.

Let $D_1, D_2 \in (G, \beta)$ be two bounded subsets, we must prove that

$\psi(D_1 \times D_2)$ is bounded.

$\psi(D_1 \times D_2)= D_1 \cdot D_2 = \{d_1 \cdot d_2 : d_1, d_2 \in G\}$

$$=\{d_1 \cdot d_2 : d_1 = \begin{bmatrix} e_1 & f_1 \\ 0 & 1 \end{bmatrix}, d_2 = \begin{bmatrix} e_2 & f_2 \\ 0 & 1 \end{bmatrix}, e_1, e_2, f_1, f_2 \in \mathbb{R}\}$$

$$=\{\begin{bmatrix} e_1 & f_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_2 & f_2 \\ 0 & 1 \end{bmatrix}\} = \{\begin{bmatrix} e_1 \cdot e_2 & e_1 f_2 + f_1 \\ 0 & 1 \end{bmatrix}\} \subset (G, \beta)$$

the product map is bounded.

ii. $\psi^{-1}: (G, \beta) \longrightarrow (G, \beta)$.

Let $D \in (G, \beta), D = \{D: D = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}\}$ which is finite set.

Then $D^{-1} = \{d^{-1}: d^{-1} = \begin{bmatrix} \frac{1}{e} & \frac{-f}{e} \\ 0 & 1 \end{bmatrix}\} \subset (G, \beta)$.

So, ψ^{-1} is bounded. Then (G, β) is bornological group.

Example[1.3.3]:

Let $(\mathbb{Z}, +)$ be an additive group and β be the finite bornology define on $\mathbb{Z}$.

To prove $\mathbb{Z}$ with finite bornology β is a bornological group $(\mathbb{Z}, \beta)$.

We must prove that, the product map and the inverse map are bounded.

i. $\psi: (\mathbb{Z}, \beta) \times (\mathbb{Z}, \beta) \longrightarrow (\mathbb{Z}, \beta)$

Let $A, B \in \beta$ be two bounded subsets, to prove that $\psi(A \times B)$ is bounded.

Thus, $\psi(A \times B) = A + B$ is bounded subset belong to $\mathbb{Z}$, since $\mathbb{Z}$ group.

Thus, the image for every two bounded sets A, B under ψ is bounded set.

ii. $\psi^{-1}: (G, \beta) \to (G, \beta)$.
Let $A \in \beta$ be a bounded set, thus $\psi^{-1}(B) = B^{-1} \subset \mathbb{Z}$, since $\mathbb{Z}$ is a group.

Which it is finite bounded set . Then $B^{-1} \in \beta$.

Then ψ^{-1} is bounded map .

As we know, every group can be turned into bornological groups by providing it with the discreet bornology. But the problem it is with indiscreet bornology, there are such kind of group cannot be bornological group because the inverse map is not bounded that means, cannot solve the problem of boundedness. In [9] this problem was solved by introducing a related structure to bornological group which is bornological semi groups. The idea of this solution is come from since every group is semigroup.So, for these kind of groups.

Definition [1.3.4](Bornological semigroup)[9][10][11]:

Abornological semigroup is a set S with two structure:

i. S is semigroup with binary operation $f: S \to S$;

ii. S is bornological set.

Such that the function f is bounded map.

Example[1.3.5]:

If $(S,*)$ an infinite semigroup , then the bornology consisting of all finite subsets of S is a bornological semigroup.

Every bornological group is a bornological semigroup but the converse is not true.

Example (1.3.6)[9]:

consider the group (Z ,+) and the bornology β= $\{B \subset Z : B \subset (-\infty, b)$, for $b \in Z\}$, on Z. Then (Z,+ ,β) is bornological semigroup. However (Z,β) is not a bornological group since, the image $-B = [-b, +\infty)$ of $B = (-\infty, b] \in \beta$ under the inverse map is not bounded in β.

After that, the problem is if they define bornology, for example usual bornology or finite bornology on infinite total order set the whole set cannot belong to the collection of bornology. So, to solve this problem they introduce the concept of semi bounded set.

1.4 Semi Bounded Set [10]:

In this section, we will speak about the definition of semi bounded set and some details it.

Definition[1.4.1]Semi Bounded Set with respect to Bornological Set[10]:

A subset S of a bornological set X is said to be a **semi bounded set** if there is a bounded set B of X such that, $B \subseteq S \subseteq \bar{B}$, where $\bar{B}$={ all upper and lower bounds of B} $\cup B$.

$SB(X)$ is that collection of all semi bounded subsets of X.

Every bounded set is semi bounded set, but the converse is not true see [9].

In discrete bornology on infinite set, any semi bounded set is bounded set .

Definition[1.4.2][10]:

A map $f: X \to Y$ from a bornological set X into a bornological set Y is called:

1. ***S-bounded map*** if the image of every bounded subset of X is semi bounded set in Y;

المستخلص

تهدف الرسالة الى:

أولا: دراسة فئات جديدة من شبه المجموعة البرنولجية بحسب كل من الدوال المقيدة S والدوال المقيدة S * والدوال المقيدة S **. مما سيكون مفيداً لمنح الحل النظري للمشكلات في الحدود والتقييد بواسطة الحد من شرط التقييد. لذلك، فإن الدافع من هذه الدراسة هو وضع شرط أقل تقييدًا على عمليات شبه الزمرة ولا يلزم تحديد أي من العمليات. لقد أثبتنا من خلال النتائج الرئيسية المهمة ، أن كل شبه الزمرة البرنولوجية هي عبارة عن شبه زمرة البرنولوجية –S ، و شبه زمرة البرنولوجيه *S وشبه الزمرة البرنولوجية** S . علاوة على ذلك، الشرط المحدد لأي مجال زمري مقابل لشبه المجموعة البرولوجية – * S لتكون لشبه المجموعة البرولوجية– * S . بالإضافة إلى ذلك، كل تحويل يسار (يمين) هو تماثل البورنولوجي –S وتماثل البورنولوجي * S.

ثانياً: هو بناء هيكل جديد يسمى بالبرنولوجي المثالي , الدافع من بناء هذا الهيكل هو لتقليل شروط التقييد بصورة مثالية. واوضحت النتيجة الرئيسية انه بالإمكان بناء هيكل برنولوجي مثالي وحيد من الأساس المثالي، وأيضا تم اثبات الاستنتاج ان الأساس المثالي يمتلك خاصية الاتحاد المنتهي. إضافة الى ذلك برهانا ان اتحاد أي مجموعتين برنولوجية مثالية هو مثالي لكن اتحادهم ليس برنولوجي مثالي وأيضا، تم دراسة بعض أنواع البرنولوجي المثالي. لقد تم إعطاء شرط كافي ان أي صورة لبرنولوجي مثالي هو برنولوجي مثالي.

أخيرًا، ومنذ تحديد المعرفة الجديدة لكل هيكل حديث في إطار نظرية الفئة. مما جعل جل اهتمام للعديد من الباحثين من ادراج كل هيكل حديث ضمن نظرية الفئة ومناقشة مفهوم المنتج المباشر والمنتج المشترك. هذا يحفزنا على رسم حدود الهيكل البرنولوجي المثالي ضمن اطارنظرية الفئة.واجمل ما توج به هذا العمل هو بعض التطبيقات الحياتية للبرنولوجي المثالي كبصمة الاصبع والعين ..الخ