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قسم علوم الرياضيات

قابلية الحل لمسالة السيطرة التقليدية الرباعية المثلى و التي تحكمها المعادلات التفاضلية الجزئية الإهليجية

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Chapter One
Basic Concepts

Introduction :

This chapter deals with several basic mathematical concepts including definitions, examples, lemmas, propositions, and theorems that important and useful in our work.

Definition(1.1)/[20]: A **Vector Space** $V \neq \emptyset$ is a set on a field $\mathbb{R}$, such that for all $x, y, z \in V$, and scalars $\alpha, \beta, \dots \in \mathbb{R}$, the following are held:

1. $\alpha x + \beta y \in V$
2. $x + y = y + x$,
3. $(x + y) + z = x + (y + z)$,
4. There is a zero vector, such that x $+ 0 = x$,
5. There is a vector $-x$ for any vector , such that $x + (-x) = 0$,
6. $\alpha(x + y) = \alpha x + \alpha y$,
7. $(\alpha + \beta)x = \alpha x + \beta x$,
8. $\alpha(\beta x) = (\alpha\beta)x$,
9. $1x = x$.

Definition (1.2)/[20]: The spanning set of a vector space V is a subset $\mathcal{A}$ of consisting of all linear combinations:

$$\sum_{i=1}^{k} a_i \alpha_i \,, \qquad \alpha_i \in \mathbb{R} \,\&\, a_i \in \mathcal{A} \,,\ i = 1,2, \dots k \,.$$

Definition(1.3)/[20]: Let V be a vector space, a finite collection $\mathcal{X}_i$, $i = \{1,2, \dots, n\}$ of elements in V is called **linearly independent** if : $\exists\, a_1, a_2, \dots, a_n \; in \; \mathbb{R}$, such that $a_1\mathcal{X}_1 + a_2\mathcal{X}_2 + \cdots + a_n\mathcal{X}_n = 0$ implies $a_1 = a_2 = \cdots = a_n = 0$.

Definition (1.4)/[20]: **A basis** of a vector space V is a finite subset in V, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that this set is linearly independent and span V, the number of elements of this set is called **the dimension** of V.

Definition (1.5)/[22]: **A Norm** on vector space V is function. $\|\cdot\| : V \longrightarrow [0,\infty)$, such that $\forall\, x, y \in V \;\&\; \alpha \in \mathbb{R}$, the following are held:

1) $\|x\| \geq 0 \;\&\; \|x\| = 0 \leftrightarrow x = 0$,

2) $\|\alpha x\| = |\alpha|\|x\|$,

3) $\|x + y\| \leq \|x\| + \|y\|$.

A Normed Space is a pair $(V, \|\cdot\|)$, where V is a vector space and $\|\cdot\|$ is a norm on V

Example(1.6): The vector space R^n with the norm :

$$\|u\|_2 = \left(\sum_{i=1}^{n} |u_i|^2\right)^{1/2}, \quad u = [u_1, \ldots u_n]^T \in R^n \quad \text{is a normed space.}$$

Definition (1.7)/[22]: An **Inner Product** in the vector space V is a function $(.,.) : V \times V \longrightarrow \mathbb{R}$ s.t

$\forall x, y, z, \in V$, and $\alpha, \beta \in \mathbb{R}$ the following are held:

1) $(x, x) \geq 0$; *moreover* $(x, x) = 0 \leftrightarrow x = 0$;

2) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$;

3) $(x, y) = (y, x)$.

An **Inner Product Space** is the vector space with inner product $(.,.)$.

Example(1.8)/[13]: Let $V = X_1 \times X_2 = \{(x, y) : x \in X_1,\; y \in X_2\}$ be the cartesian product of inner product spaces X_1 *and* X_2, the space V is an inner product space is defined by :

$$((x_1, y_1), (x_2, y_2)) = (x_1, x_2) + (y_1, y_2).$$

Theorem (1.9)/[11] :(C-S- I)

If x and y are two arbitrary vectors in an inner product space, then

$$|(x,y)| \leq \| x \| \| y \|.$$

Definition(1.10)/[17]: A Cauchy Sequence is a sequence of vectors $\{\alpha_n\}$ in a normed space if for every $\epsilon > 0$, there exists a positive integer K s.t. $\forall\, n,m > K$, $\| \alpha_n - \alpha_m \| < \epsilon$

Definition (1.11)/[17]: A space $\mathcal{D}$ is said to be **Compact** if every Cauchy sequence in $\mathcal{D}$ converges to an element in $\mathcal{D}$.

Example (1.12): The sequence $\{v_n\} = \frac{n}{n+1}$ is Cauchy sequence in $\mathbb{R}$, and the space is compact.

Definition (1.13)/[11]: A Hilbert Space is a compact inner product space.

Example(1.14):Euclidean space$\mathbb{R}^n$:

The space $\mathbb{R}^n$ is a Hilbert space with inner product defined by:

$$(\alpha,\beta) = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$$

Where $\alpha = (\alpha_1,\alpha_2, \dots ,\alpha_n)$, $\beta = (\beta_1,\beta_2, \dots ,\beta_n)$.

Definition (1.15)/[17]: A sequence of vectors $\{\alpha_n\}$ in an inner product space is said to be convergent :

i. **Strongly** to $\alpha \in V$ if $\|\alpha_n - \alpha\| \to 0$ as $n \to \infty$,

ii. **Weakly** to $\beta \in V$ if $\forall\, \beta \in V, (\alpha_n,\beta) \to (\alpha,\beta)$ as $n \to \infty$.

Theorem (1.16)/[13]: A strongly convergent sequence is weakly convergent (the same limit).

Definition (1.17)/[39]: Let $\mathcal{V}$ be a normed space, the sequence $\{\alpha_n\}_{n=1}^{\infty} \subset \mathcal{V}$ is said to be **bounded** in $\mathcal{V}$ if there exists a $\epsilon > 0$ such that $\|\alpha_n\| \leq \epsilon$, for all n.

Theorem (1.18)/[13]: In a Hilbert space every weakly convergent sequence is bounded.

Theorem(1.19)/[11]: **(Alaoglu)** Let $\mathcal{H}$ be a Hilbert space, and $\{\alpha_n\}$ be bounded sequence of $\mathcal{H}$, then there exists a subsequence of $\mathcal{H}$ which convergent weakly to some $\alpha \in \mathcal{H}$.

Definition (1.20)/[23]: Let $\mathcal{V}$ be a real normed space and $\mathcal{A}$ is a nonempty set in $\mathcal{V}$. The fun. K: $\mathcal{V} \longrightarrow \mathbb{R}$ is **Weakly Lower Semicontinuous** if for every sequence $\{\alpha_n\}$ in $\mathcal{A}$ convergent weakly to some $\alpha \in \mathcal{A}$ we have:

$$\lim_{n\to\infty} \inf K(\alpha_n) \geq K(\alpha).$$

Example (1.21): The function $F(\alpha) = \begin{cases} \frac{6}{2+\alpha_n^2} , & \alpha \neq 0 \\ 2 , & \alpha = 0 \end{cases}$

is weakly lower semicontinuous because $2 = F(0) \leq \lim_{\alpha_n \to 0} inf(\alpha_n) = 3$, $as\ n \longrightarrow \infty$.

Definition(1.22)/[26]: **A Bilinear Form** is a mapping $\mathcal{B} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ ($\mathcal{H}$ is Hilbert Space) satisfying the following conditions:

- $\mathcal{B}(\alpha x_1 + \beta x_2, y) = \alpha\, \mathcal{B}(x_1, y) + \beta\, \mathcal{B}(x_2, y)$
- $\mathcal{B}(x, \alpha\, y_1 + \beta y_2) = a\, \mathcal{B}(x, y_1) + \beta\, \mathcal{B}(x, y_2)$

For any real number α , β and $x, x_1, x_2, y, y_1, y_2 \in \mathcal{H}$.

Example(1.23): Let $F = C^1[a, b]$. Then the BF $\mathcal{B}$ is defined a $\mathcal{B}: F \times F \longrightarrow \mathbb{R}$, $\mathcal{B}(x, y) = \int_a^b (xy + \hat{x}\hat{y})dt$.

Definition(1.24)/[36]: A **Bilinear Form** $\mathcal{B}: \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ ($\mathcal{H}$ is Hilbert Space) is called:

i. **Continuous Bilinear Form** if:

$$\exists \epsilon \geq 0, \ s.t. \ |\mathcal{B}(x,y)| \leq \epsilon \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}}, \forall x, y \in \mathcal{H}.$$

ii. **Coercive** if:

$$\exists c > 0 \quad s.t. \ \mathcal{B}(x,x) \geq c\|x\|_{\mathcal{H}}^2, \forall x \in \mathcal{H}.$$

Definition(1.25)/[38]: An operator $\mathcal{B}: \mathcal{V} \rightarrow \mathcal{V}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_{\mathcal{V}}$, ($\mathcal{V}$ is vector space) is:

i. **Monotone** if:

$$(\mathcal{B}x - \mathcal{B}y, x - y) \geq 0, \forall x, y \in \mathcal{V},$$

ii. **Strictly monotone** if:

$$(\mathcal{B}x - \mathcal{B}y, x - y) > 0, \forall x, y \in \mathcal{V} \ with \ x \neq y.$$

Definition(1.26)/[38]: A operator $\mathcal{B}: \mathcal{K} \longrightarrow \mathcal{V}$ where $\mathcal{K}$ is a subset from vector space $\mathcal{V}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_{\mathcal{V}}$ is **Lipschitz Continuous** if :

$$\exists m > 0 \ s.t. \ \|\mathcal{B}x - \mathcal{B}y\| \leq m\|x - y\|, x, y \in \mathcal{K}.$$

Definition (1.27)/[36]: A subset $\mathcal{A}$ of a vector space $\mathcal{V}$ is **Convex set** if:

$$\forall x, y \in \mathcal{A} \ \& \ 0 < \vartheta < 1, , \vartheta x + (1 - \vartheta)y \in \mathcal{A}.$$

Definition (1.28)/[36]: A function $\mathcal{K}: \mathcal{V} \longrightarrow \mathbb{R}$ ($\mathcal{V}$ is vector space) is:

i. **Convex** if:

$$\mathcal{K}(\vartheta x + (1 - \vartheta)y) \leq \vartheta\mathcal{K}(x) + (1 - \vartheta)\mathcal{K}(y), \forall 0 \leq \vartheta \leq 1 \ \& \ \forall x, y \in \mathcal{V},$$

ii. **Strongly Convex** if

$$\mathcal{K}(\vartheta x + (1 - \vartheta)y) < \vartheta\mathcal{K}(x) + (1 - \vartheta)\mathcal{K}(y), \ \forall 0 \leq \vartheta \leq 1 \ \& \forall x, y \in \mathcal{V} \ with \ x \neq y$$

Example (1.29): The function $Y: C[0,1] \longrightarrow \mathbb{R}$ which defined as:

$$Y(f) = \int_0^1 (f(\alpha))^2 d\alpha\,,\ f \in C[0,1]$$

is Convex function.

Example (1.30): The function $F(x) = x^4$ is strongly convex on $\mathbb{R}$.

Definition(1.31)/[13]: A **Compact set** is a subset $\mathcal{C}$ of a normed space if every sequence $\{a_n\}$ in $\mathcal{C}$ contains a convergent subsequence whose limit belongs to $\mathcal{C}$.

Definition(1.32)/[9]: let $\mathcal{K} : \mathcal{S} \longrightarrow \mathbb{R}$ is a function then the closure of the set $f = \{\vartheta \in \mathcal{S} : \mathcal{K}(\vartheta) \neq 0\}$ is called the **Support** of $\mathcal{K}$ and is denoted by support $\mathcal{K}$ (support $\mathcal{K} = \bar{f}$). A function is said to have **compact support** if its support is a compact set.

Definition (1.33)/[13]: A **test function** is mean $C_c^\infty(\Omega)$ an infinitely differentiable function on Ω with compact Support $\mathcal{D}(\Omega)$ is denote to the space of all test function.

Example (1.34): The function

$$K(x) = \begin{cases} \frac{1}{e^{(x^2-1)}} & if\ |x| < 1 \\ 0 & \end{cases}$$

is a test function ,with support $|x| < 1$

Definition (1.35)/[22]: A **Banach Space** is a complete normed space.

Definition(1.36)/[10]: A Banach space $(\mathcal{B})$ with the function $F : \mathcal{B} \longrightarrow \mathcal{B}^{**}$ is **reflexive,** if $F(\mathcal{B}) = \mathcal{B}^{**}$.

Theorem (1.37)/[10]: Every nonempty convex bounded closed subset of a reflexive Banach space is weakly compact.

<u>Definition(1.38)/[22]:</u> The collection of subsets of the set Ω (let denotes by $\mathcal{M}$) is $\boldsymbol{\sigma}$ **− algebra** if the following properties hold:

a) $\emptyset \in \mathcal{M}$,

b) $\mathcal{T} \in \mathcal{M} \Longrightarrow \mathcal{T}^c \in \mathcal{M}$ ($\mathcal{M}$ is closed under complementation),

c) $\forall \mathcal{T}_n \in \mathcal{M}$ $(n = 1,2, \dots n)$ $\bigcup_{n=1}^{\infty} \mathcal{T}_n \in \mathcal{M}$ ($\mathcal{M}$ is closed under finite union).

A set Ω with $\sigma -$ algebra $\mathcal{M}$ is called **Measurable Space.**

<u>Definition(1.39)/[22]:</u> A **Measure** μ is a mapping $\mu : \mathcal{M} \longrightarrow [0, \infty)$ satisfies the following properties:

i. $\mu(\emptyset) = 0$

ii. $\mu(\mathcal{B}_n) \geq 0, \forall \mathcal{B}_n \subset \mathcal{M}$

iii. $\mu\left(\bigcup_{n=1}^{\infty} \mathcal{B}_n\right) = \sum_{n=1}^{\infty} \mu(\mathcal{B}_n)$, wherever $(\mathcal{B}_n)$ is disjoint sequence of $\mathcal{M}$.

<u>Definition(1.40)/[22]:</u> **A measure Space** is the triple $(\Omega, \mathcal{M}, \mu)$, where Ω is nonempty set, $\mathcal{M}$ is $\sigma -$ algebra of subsets of Ω, and μ is a measure defined on $\mathcal{M}$.

<u>Example(1.41)/[9]:</u> Consider the nonempty set Ω, and $\mathcal{M} = \mathcal{P}(\lambda)$, ($\mathcal{P}(\lambda)$ has the property that the elements λ is involving of Ω). Fix an element $c \in \Omega$, and define $\mu : \mathcal{M} \longrightarrow [0, \infty)$ by:

$$\mu(\mathcal{B}) = \begin{cases} 0 & if\ c \notin \mathcal{B} \\ 1 & if\ c \in \mathcal{B} \end{cases}$$

Then $(\Omega, \mathcal{M}, \mu)$ is measure space.

Definition (1.42)/[9]: A **measurable set (μ-mear)** is a subset $\mathcal{K}$ of Ω if

$$\mu(\mathcal{A}) = \mu(\mathcal{A} \cap \mathcal{K}) + \mu(\mathcal{A} \cap \mathcal{K}^c), \forall \mathcal{A} \subseteq \Omega.$$

A **measurable function** is $F : \Omega \longrightarrow \mathbb{R}$ such that $F^{-1}(\Omega)$ is a measurable set for every open subset Q of $\mathbb{R}$

Definition (1.43)/[13]: A two functions $\mathcal{F}, \mathcal{G}$ defined on $\mathbb{R}$ are **equals almost everywhere** (denoted $\mathcal{F} = \mathcal{G}$ a.e.), if the set of all $u \in \mathbb{R}$ for which $\mathcal{F}(u) \neq \mathcal{G}(u)$ is a set of measure zero.

Definition(1.44)/[33]: The **space** $\mathcal{L}^p(\Omega)$ is a space of the measurable Functions such that these functions defined as $\mathcal{F}(\alpha) : \Omega \to \mathbb{R}^n$ and integral $\int_\Omega |\mathcal{F}(\alpha)|^p dx$ exists, ($\mathcal{P} \geq 1$ any real number).

Example (1.45)/[33]: The function $F(\alpha) = \alpha^{-1/3}$ belongs to $\mathcal{L}^p(0,1)$ for any $p < 3$, since:

$$\int_0^1 |f(\alpha)|^p d\alpha = \int_0^1 \left|\alpha^{-1/3}\right|^p d\alpha = \frac{3}{3-p}\left[\alpha^{(3-p)/3}\right]_0^1$$

Definition(1.46)/[33]: If $p \to \infty$ the **space** $\mathcal{L}^\infty(\Omega)$ is a space of all measurable Functions and bounded a.e. on Ω:

$$\mathcal{L}^\infty(\Omega) = \{\mathcal{F} : |\mathcal{F}(\alpha)| \leq \mathcal{K}, a.e\ on\ \Omega\ for\ some\ \mathcal{K} \in \mathbb{R}\}$$

Note(1.47)/[34]: For a bounded domain Ω apparent, $\mathcal{L}^\infty(\Omega) \subset \mathcal{L}^p(\Omega)$,

($\forall 1 \leq p \leq \infty$), since any $\mathcal{F} \in \mathcal{L}^\infty(\Omega)$ satisfies:

$$\int_\Omega |\mathcal{F}(\alpha)|^p\, d\alpha \leq \int_\Omega \mathcal{K}^p\, d\alpha < \infty,$$

so that $\mathcal{F} \in \mathcal{L}^p(\Omega)$ also.

Theorem (1.48)/[13]: (Minkowski Inequality)

For every two function $\varphi, \psi \in \mathcal{L}^p(\Omega)$ the following inequality satisfies:

$$\| \varphi + \psi \|_p \leq \| \varphi \|_p + \| \psi \|_p , \text{ Where } 1 \leq p < \infty.$$

Definition(1.49)/[27]: A **Linear Operator** $\mathcal{T}$ is a mapping $\mathcal{T} : V \longrightarrow \mathcal{W}$ (where $V, \mathcal{W}$ are a vector space) such that:

$$\forall \alpha_1, \alpha_2 \in V, \quad a, b \in \mathbb{R}, \mathcal{T}(a\alpha_1 + b\alpha_2) = a\mathcal{T}(\alpha_1) + b\mathcal{T}(\alpha_2).$$

Definition(1.50)/[33]: A linear operator $\mathcal{T} : V \longrightarrow \mathcal{W}$ ($V,\mathcal{W}$ are normed space) is **bounded** if there is a number $\mathcal{S} > 0$ s.t $\| \mathcal{T}(\alpha) \| \leq \mathcal{S}\|\alpha\|, \forall \alpha \in V$.

For $\mathcal{S} \neq 0$ we see that $k \geq \frac{\|\mathcal{T}(\alpha)\|}{\|\alpha\|}$. So the set $\left\{k: k \geq \frac{\|\mathcal{T}(\alpha)\|}{\|\alpha\|}, \alpha \neq 0\right\}$ is bounded below, and the least upper bound, taken over all members $\mathcal{S}$ of V , is called the norm of $\mathcal{T}$, which is written as $\| \mathcal{T} \|$, that is,

$$\|\mathcal{T}\| = \sup\left\{\frac{\|\mathcal{T}(\alpha)\|}{\|\alpha\|}, \alpha \neq 0\right\} \quad (1.1)$$

Definition (1.51)/[22]: $\mathcal{L}(V, \mathcal{W})$ is a set of all bounded linear operator from a normed space V into a normed space $\mathcal{W}$, this set itself is a normed space with norm defined by (1.1). The **Dual Space** of V is a space of bounded linear Functionals. from the normed space V into $\mathbb{R}$ and symbolized by V^* ($i.e. \mathcal{L}(V , \mathcal{W}) = V^*$).

Example(1.52)/[33]: Let $\mathcal{P} : \mathcal{L}^2(a, b) \longrightarrow \mathbb{R}$ be a function defined by

$$\langle p, w \rangle = \int_a^b w(x)dx$$

Then $\mathcal{P}$ is a linear functional $:\langle p, \xi w + \zeta u \rangle = \xi \langle p, w \rangle + \zeta \langle p, u \rangle$. Furthermore, using the C- S- I on $\mathcal{L}^2$,

$$|\langle p, w\rangle| = \left|\int_a^b w(x)dx\right| \leq \|1\|_0 \|w\|_0$$

and so $\mathcal{P}$ is bounded, and is thus a member of the dual space $[\mathcal{L}^2(a,b)]^*$

Definition(1.53)/[11]: The adjoint operator of a bounded linear operator $\mathcal{L} : V \to W$ is the operator $\mathcal{L}^* : W \longrightarrow V$ which defined by:

$$(\mathcal{L}v, w) = (v, \mathcal{L}^* w)\ for\ all\ v \in V\ and\ w \in W.$$

Example(1.54): Let the left and right shift operators on $\mathcal{L}^2$:

$$R(\alpha_1, \alpha_2, \dots) = (0, \alpha_1, \alpha_2, \dots),\ L(\alpha_1, \alpha_2, \dots) = (\alpha_2, \alpha_3, \alpha_4, \dots).$$

We will show that $R^* = L. \forall \alpha, \beta \in \mathcal{L}^2$,we have

$$(R\alpha, \beta) = ((0, \alpha_1, \alpha_2, \dots), (\beta_1, \beta_2, \beta_3 \dots))$$

$$= \sum_{n=1}^{\infty} \alpha_n \beta_{n+1}^*$$

$$= \left((\alpha_1, \alpha_2, \alpha_3 \dots), (\beta_1, \beta_2, \beta_3 \dots)\right)$$

$$= (\alpha, L\beta).$$

Definition(1.55)/[25]: Let X,Y be two normed space, and $A : \mathcal{S} \subset X \longrightarrow Y$ be a mapping, moreover suppose an element $x \in \mathcal{S}$. If there is a continuous linear mapping $\acute{A}(x) : X \longrightarrow Y$, satisfies

$$\lim_{\|h\|_X \to 0} \frac{\left\|A(x+h) - A(x) - \acute{A}(x)(h)\right\|_Y}{\|h\|_X} = 0$$

Then $\acute{A}(\mathcal{X})$ is called the **Fréchet Divertive (FD)** of A at x and A is called the Fréchet differentiable at x.

Example(1.56): The mapping $\mathcal{K}(\alpha) = \|\alpha\|^2$ is Fréchet differentiable on every Hilbert space.

<u>Definition(1.57)/[16]:</u> The **Sobolev Space** $W^{1,p}(Q)$ which defined as :

$$W^{1,p}(Q) \quad = \{\, \mathcal{X} \in \mathcal{L}^p(Q) | \exists\, \mathcal{Y}_1, \mathcal{Y}_2, \dots , \mathcal{Y}_n \in \mathcal{L}^p(Q) \ni \int_Q \mathcal{X} \frac{\partial \psi}{\partial x_i} dx = -\int_Q \mathcal{Y}_i \psi dx$$

$\psi \in C_c^\infty(Q)$, $\forall i = 1,2, \dots n$

Where Q is an open subset from $\mathbb{R}^n$, and $p \in \mathbb{R}$ with $1 \le p \le \infty$.

Set $H^1(Q) = W^{1,2}(Q)$. For $\mathcal{X} \in W^{1,2}(Q)$, define $\frac{\partial \mathcal{X}}{\partial x_i} = \mathcal{Y}_i$ and write

$$\nabla \mathcal{X} = \left(\frac{\partial \mathcal{X}}{\partial x_1}, \frac{\partial \mathcal{X}}{\partial x_2}, \dots \frac{\partial \mathcal{X}}{\partial x_n} \right)$$

The space is equipped with the norm $\|\mathcal{X}\|_{W^{1,2}(\Omega)} = \|\mathcal{X}\|_p + \sum_{i=1}^n \left\| \frac{\partial \mathcal{X}}{\partial x_i} \right\|_p$

The space $H^1(\Omega)$ is equipped with Inner Product $(\mathcal{X}, w)_1 = (\mathcal{X}, w) + (\nabla \mathcal{X}, \nabla w) = \int_Q \mathcal{X} w \, dx + \int_Q \frac{\partial \mathcal{X}}{\partial x_i} \frac{\partial w}{\partial x_i} dx$

The associated norm

$$\|\mathcal{X}\|_1 = \|\mathcal{X}\|_{H^1(Q)} = \left(\|\mathcal{X}\|_{\mathcal{L}^2(Q)}^2 + \sum_{i=1}^n \left\| \frac{\partial \mathcal{X}}{\partial x_i} \right\|_{\mathcal{L}^2(Q)}^2 \right)^{1/2}$$

is equivalent to the $W^{1,2}$ norm

<u>Definition (1.58)/[26]:</u> The space $\vec{W}$ is equipped with the product norm

$$\vec{w} = (w_1, w_2, \dots w_n) \to \|\vec{w}\|_1 = \left(\sum_{i=1}^n \|w_i\|_1^2\right)^{1/2}$$

For any $\vec{w} = (w_1, w_2, \dots , w_n) \in \left(H^1(Q)\right)^n$, where Q bounded and open connected subset in R^n

Theorem(1.59)(Generalized Greens Formula)/[34]: Let $Q \subset \mathbb{R}^n$, with boundary ∂Q, the following generalization of green's formula with meaning of $\mathcal{L}^2$-weak derivative is satisfied:

$$\int_Q \Delta w \cdot u \, dx = -\int_Q \nabla w \cdot \nabla u \, dx + \int_{\partial Q} \left(\frac{\partial w}{\partial n}\right) u \, dq$$

Where $\Delta = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}$, for $w \in H^2(\Omega), u \in H^1(\Omega)$

Theorem (1.60),(Egorov's theorem)/[25]: Let Q be a measurable subset of $\mathbb{R}^d$, $\psi : Q \longrightarrow \mathbb{R}$ and $\psi \in L^1(Q, \mathbb{R})$, if the following inequality is satisfied $\int_S \psi(x)dx \geq 0$ $(or \leq 0$ or $= 0)$, for each measurable subset $S \subset$ Q, then $\psi(x) \geq 0$ $(or \leq 0$ or $= 0)$, a.e. in Q.

Theorem(1.61),(Rellich-Kondrachov theorem)/[21]: Let Q be a bounded set of class C^1.Then the following compact injections are satisfied: $W^{1,\alpha}(Q) \subset \mathcal{L}^\beta(Q)$, $\forall \beta \in [\alpha, \infty)$, if $\alpha = n$. In particular, $W^{1,\alpha}(Q) \subset \mathcal{L}^\beta(Q)$ with compact injection for all α (and all n)

Definition(1.62)/[2]: Let $Q \subset \mathbb{R}^d$, a function $\mathcal{F}(x, u, v) : Q \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called of the **Carathéodory type** if it is continuous with respect to u and v for fixed $x \in$ Q, and measurable with respect to x for fixed

(u, v).

Proposition(1.63)/[2]: Let $\mathcal{F} : Q \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Carathéodory type, let $\mathcal{F}$ be a functional, such that $\mathcal{F}(u) = \int_Q f(x, u(x)) \, dx$, where Q is a measurable subset of $\mathbb{R}^n$, and suppose that

$$\|f(x, u)\| \leq \zeta(x) + \eta(x)\|u\|^\gamma, \forall (x, u) \in Q \times \mathbb{R}^n, u \in \mathcal{L}^\alpha(Q \times \mathbb{R}^n)$$

where $\zeta \in \mathcal{L}^1(Q \times \mathbb{R})$, $\eta\, \mathcal{L}^{\frac{\alpha}{\alpha-\gamma}} \in (Q \times \mathbb{R})$, and $\lambda \in [0, \alpha]$, if $\alpha \in [1, \infty)$, and $\eta \equiv 0$, if $\alpha = \infty$.

Then $\mathcal{F}$ is continuous on $\mathcal{L}^\alpha(Q \times \mathbb{R}^n)$.

المستخلص:

ينقسم الهدف الرئيسي الذي تم القيام به والإبلاغ عنه في هذه الرسالة إلى جزأين ، الجزء الأول هو دراسة مشكلة متجه السيطرة الرباعي التقليدي المستمر الامثل للمعادلات التفاضلية الجزئية الخطية الرباعية . تم ذكر نص و برهان وجود متجه حل الحالة الرباعية للصيغة الضعيفة للمعادلات التفاضلية الجزئية الخطية الرباعية وإثباتها لمتجه السيطرة الرباعي التقليدي المستمر الامثل باستخدام طريقة كليركن. علاوة على ذلك ذكر نص و برهان مبرهنة الوجود لمتجه سيطرة رباعي تقليدي مستمر امثل مسيطر بالمعادلات التفاضلية الجزئية الخطية الرباعية . تمت دراسة المعادلات الرباعية المرافقة المرتبطة بالمعادلات التفاضلية الجزئية الخطية الرباعية. و كذلك ايجاد مشتقة فريشيه . أخيرًا ، تم ذكر و إثبات نظرية الشرط اللازمة لتحقيق الأمثل .

الجزء الثاني هو دراسة متجه السيطرة الرباعي التقليدي المستمر الامثل للمعادلات التفاضلية الجزئية غير الخطية . ثم اثبات و وجود حل للصيغة الضعيفة للمعادلات التفاضلية الجزئية الرباعية غير الخطية باستخدام مبرهنة منتي براودر. تم إثبات مبرهنة وجود متجه السيطرة الرباعي التقليدي المستمر الامثل المرتبط بالمعادلات التفاضلية الجزئية غير الخطية الرباعية. تمت دراسة الوجود و وحدانية الحل للمعادلات الرباعية المرافقة المرتبطة بالمعادلات التفاضلية الجزئية غير الخطية الرباعية. وتم ايجاد مشتقة فريشيه . أخيرًا ، تم ذكر و اثبات نظرية الشروط الضرورية و كذلك نظرية الشروط الكافي لتحقيق أمثليه المتجه المقيد.