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وهي جزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

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Chapter One

Basic Concepts and Preliminaries

1.1 Introduction

The purpose of this chapter is to introduce some fundamental inverse problem concepts. The inverse problem, in general, is a category of problems where the cause is inferred from the observable effects. This chapter is where we'll describe the terms "direct problem," "inverse problem," and "Illposed problem" and provide examples for each.

1.2 Partial Differential Equations

Starting by giving some basic definitions:

Definition 1.2.1 (Partial Differential Equation) [33]:

A partial differential equation (PDE) resembles an ordinary differential equation (ODE), with the exception that the dependent variable in a PDE depends on a number of independent variables rather than just one. A PDE's order is the ordering of the equation's highest order derivative. If a region's first order partial derivatives are all continuous, then in order to define the gradient of

$$gradu \equiv \nabla_u = \frac{\partial_u}{\partial_x}\vec{\imath} + \frac{\partial_u}{\partial_y}\vec{\jmath}$$

The Laplacian of u is

$$\Delta u = \nabla . \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

If $\vec{n}$ denotes a unit vector in R^2 the directional derivative of u in the direction n is

$$\frac{\partial u}{\partial n} = \nabla u . n$$

Definition 1.2.2 (Initial and Boundary Conditions)

Physical system modeling PDE typically have an infinite number of solutions. There are several requirements that must be met in order to choose one function to represent the solution. They may be beginning or boundary conditions, The boundary circumstance must be met at a location on the region's edge, denoted by the Determine this symbol, in which the PDE exists. The initial condition must be met throughout the entire area when the physical system starts. Typically, this entails combining and its time derivatives at t=0 **[30].**

Types of boundary conditions:-

1. Dirichlet-type Boundary Condition

This type of boundary conditions was named after the German mathematician Dirichlet, in case that. $u(x, y)$ is given at all point $(x, y) \in \partial\Omega$

For example: $\nabla^2 y + y = 0$

$$y(x) = f(x) \;\; \forall x \in \partial\Omega$$

$f(x)$ is a function defined on $\partial\Omega$ [27]

2. Neumann-type Boundary Condition:

When the gradient of the dependent variable is normal derivative on the boundary $\frac{\partial u}{\partial n}$, these conditions are named after the German mathematician Neumann, For example: $\nabla^2 y + y = 0$

$$\frac{\partial y}{\partial n}(x) = g(x)$$

where, $g(x)$ is known function defined on $\partial\Omega$

There are also other types of boundary condition, which are (Robin ,Mixed and Cauchy)[27].

1.3 Direct and In□erse □roblems

Definition 1.3.1 (Direct problem) [10]:

Finding the unknown solution within a domain using the beginning and boundary conditions that are known is the main focus in a direct problem. Throughout the last two centuries, direct issues have been thoroughly researched, leading to a large body of literature detailing methods for solving them.

In a direct problem formulation like this, the following details must be known:

1. The boundary of the solution domain
2. The governing equation in the domain
3. The boundary condition for the entire boundary and initial condition
4. The material properties
5. The forces acting in the domain

Definition 1.3.2 (□ ell-posed problem) [3]:

If a problem fits the following criteria, it is said to be well-posed in Hadamard's sense:

1.The existence of the solution at all the data.

2.The uniqueness of the solution.

3.The continuously dependence of the solution on the given (data)

(which means the stability of the solution). That means, in case that the solution depends continuously on the data so a small error in the given data produces just a small error in the obtained solution.

Definition 1.3.3 (Ill-posed problem) [3]:

Under Hadamard's definition, a problem is deemed ill-posed if any one of the well-posedness three conditions is not met. If the first condition is not met, then there is no solution to the problem in the intended location.

The task is to select the best answer if the second criterion is not met, which suggests that the issue has more than one solution in the area. If the third criterion is not met, it follows that the solution process is unstable (unstable), meaning that even minor changes in the Cauchy data might cause a significant departure from the exact solution in the numerical solution.

1.3.□□ome □□amples of Ill-posed □roblems [3□]:

1. The Cauchy problem for Laplacian is the first.

2. The modified Helmholtz equation or Cauchy problem for Helmholtz.

3. Resolving a badly constrained linear algebraic system.

4. The issue with the reversed heat equation.

5. Problems with minimization.

The definition of an inverse Cauchy problem is reviewed below.

Definition1.3.□(In□erse Cauchy problem) [20]:

In these types of problems, the boundary of a solution domainΩ of the researched problem is only known on a portion of the boundary, Γ_1, and some boundary conditions are over-specified while on the remaining portion of the boundary $\Gamma 2 = \partial\Omega / \Gamma 1$ is either not given or not specified.

Definition 1.3.□(In□erse problem) [10]:

The inverse problem is an illustration of an ill posed problem. In fact, the inputs and output are separated into an inverse issue. The reverse of a direct problem is known as an inverse problem. As soon as one or more of the circumstances for immediate issues possible unknowns or incomplete knowledge makes it possible to create an inverse problem. Using the above mentioned conditions will help you identify the unidentified condition. The noise of the provided Cauchy data in such cases may have an impact on the resolution. The problem can be solved in large part because to the overly specific data.

In□erse problems typically play one of the follo□in□roles:

1. The boundary's partial determination.

2. Drawing conclusions about the equation that controls the issue.

3. Determining the border condition and/or beginning condition.

4. Determining the material's characteristics.

5. The domain forces must be determined.

Definition 1.3.□(□ome □inds of In□erse □roblems)[2□]:

1. Inverse boundary value problems: For instance, the modified Helmholtz equation or the Cauchy problem for Helmholtz.

2. Inverse initial –value problems: For instance, the initial temperature displacement or velocity is unavailable in the backward heat conduction problem.

3. Inverse source \force problems: locating dispersed sources of heat production, pollutants, or wave excitation.

4. Inverse coefficient problems: measurement of other physical quantities in a single, reliable way in order to determine unmeasurable physical qualities.

5. Inverse geometry problems: Identification of obstacles, such as tomography, inverse scattering, corrosion, and crack detection.

Definition 1.3.□(Condition Number) [3□]:

The condition number of a matrix A, denoted by$k(A)$ is defined by the following:

$$k(A) = \|A\|\|A^{-1}\|$$

$$K(A) \geq 1 \quad for\ all\ A$$

A is well- conditioned matrix if $k(A)$ is small this means (stable)

(small change in the data produce small changes in the solution)

A is well-conditioned matrix if $k(A)$ is large this means (unstable)

(small change in the data produce large changes in the solution)

1.□□ odified □elmholt□□□uation [22]

The Helmholtz and the modified Helmholtz equation problems arise in a variety of applications of science and engineering such as acoustic radiation, electromagnetic fields, wave propagation and heat conduction. That is why it appears frequently.

1.□.1 Definition (□elmholt□and modified □elmholt□e□uation):

The following problem:

$$\nabla u + k^2 u = F$$

Helmholtz equation is a term used under specific circumstances. Despite a problem identified by:

$$\nabla u - k^2 u = F$$

Modified Helmholtz equation refers to a situation under circumstances where $k \in R$.

1.□□ome Numerical □nalysis Concepts

Definition 1.□.1 (□ eshless □ ethod) [2□]:

Meshless techniques are first used to address astrophysics issues in the 1970s. The domain is discretized for these approaches by a collection of nodes. Unlike mesh-based techniques, where the domain is discretized by components (for instance, the finite volume, finite element, and finite difference methods).

There is always some error introduced into each numerical approach by the approximate solution that is obtained using one of the numerical methods.

Definition 1.□.2 (□ome Types of □rrors) [3□]:-

1. Absolute Error: The difference between the precise and approximate values. Finding the measurement with the highest likelihood is crucial.

2. Relative Error: Absolute error divided by exact value.The main purpose of using this type of error is to determine measurement is accurate or not is crucial.

Definition 1.□.3 (□toppin□Criterion) [3□]:

In order to identify certain answers using each numerical approach, we compute a series of approximations that converge to the desired solution; in reality, we apply the stopping criteria listed below, which are defined by utilizing the Relative and absolute residual:

The absolute error = $\| u_{ex} - u_{app} \| < Tol$

$$\text{The Relative error} = \frac{\|u_{ex} - u_{app}\|}{\|u_{ex}\|} < Tol$$

Where u_{ex} is the exact solution and $u_{app}$is the approximate solution.

Where Tol is an error tolerance that the user has specified.

1.□ □ome □inear □l□ebra Concepts

Definition 1.□.1 (□ymmetric □ atri□) [3□]:

a square matrix equal to the transposition of that matrix $A^T = A$ if$A$ is symmetric, thenA^T is also symmetric such as $a_{ij} = a_{ji}$.

Definition 1.□.2 (□ositi□e Definite □ atri□) [3□]:

An$n \times n$ matrix A is called positive definite if it is symmetric, ,$A^T = A$ and satisfies the positive condition $x^T Ax > 0 \; for \; all \; x \in R^n \; x \neq 0$ we will sometime write $A > 0$ to mean that A is positive definite matrix .

Definition 1.□.3 (Inner □roduct)[3□]:

Let x, y are $n \times 1$column vectors, the inner product is defined by

$$< x, y > = x^T . y = [x_1 \dots \dots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots x_n y_n$$

Every study of the geometry of n-dimensional space includes the combination. It is occasionally referred to as the scalar product or dot product of the two vectors, and is denoted by (x, y) or x, y, but we prefer to call it the inner product and make things simple the symbol $x^T y$.

المستخلص

احد انواع المسائل العكسية هي مسألة كوشي العكسية لمعادلة هيلمهولتز المعدلة, ينشأ هذا النوع من المشاكل في احد تطبيقات الحياة الواقعية , وهو توصيل الحرارة في الزعنفة. الهدف من هذه الرسالة هو تحديد درجة الحرارة على حدود غير محددة بالاستفادة من الجزء الذي يمكن الوصول اليه من الحدود مع بيانات كوشي و التي هي درجة الحرارة على الحدود التي يمكن الوصول اليها والتدفق الحراري في هذا الجزء.

يتم حل هذه المسألة بأستخدام بعض الطرق العددية التي هي الطريقة اللاشبكية من خلال التعبير عن الحل كتوسع متعدد الحدود والتحقق من مشكلتنا لهذا التوسع الذي ينتج نظام خطي يتم حله من خلال اثنين من الخوارزميات العددية المختلفة يتم مقارنة الحلول التقريبية التي تم الحصول عليها من قبل هاتين الخوارزميتين مع الحل الدقيق.التحقق من دقة هذه الطريقة المقترحة. تم دراسة العديد من الامثلة مع بعض المشاكل متعددة حدود وغير متعددة حدود على المجالات المنتظمة من خلال الاستفادة من كفاءة الطريقة اللاشبكية.

من المعروف ان مسألة كوشي المعكوسة هي مسألة معتلة وبالاضافة الى ذلك مشكلة غير مشروطة للغاية لذلك يتم تأكيد الاستقرار من خلال تطبيق الضوضاء لبيانات كوشي. لتقليل تأثير هذه الحالة السيئة للغاية يتم تطبيق تنظيم تيكانوف والتكييف النسبق.