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Constraints Optimal Control Problem for Quaternary Nonlinear Elliptic System

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Abstract

This paper is concerned with the study of the constraints quaternary continuous classical optimal control vector problem (CQCCOCVP) with equality and inequality constraint controlled by quaternary nonlinear elliptic partial differential equations (QNLEPDEqs).The existence theorem of a CQCCOCV of the constrained problem is stated and proved under suitable hypotheses. The mathematical formulation of the adjoint quaternary equations (AQEqs) associated with the QNLEPDEqs is derived, the Fréchet derivative for the objective function and the EINC are derived. Finally, the necessary condition theorem and the sufficient condition theorem for the optimality are stated and proved.

Keywords: Constraint Continuous Classical Optimal Control Vector, Quaternary Nonlinear Elliptic Boundary Value Problem, Fréchet derivative, Necessary and Sufficient Conditions for Optimality.

مشكلة التحكم الأمثل المقيدة للنظام الإهليلجي غير الخطي الرباعي

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الخلاصة

يهتم هذا البحث بدراسة مشكلة متجه التحكم الرباعي التقليدي المستمر الأمثل المقيدة بقيدي المساواة وعدم المساواة التي تتحكم فيها المعادلات التفاضلية الجزئية غير الخطية الرباعية. تم ذكر نص وبرهان وجود متجه سيطرة تقليدية امثلية مستمرة مقيدة بقيدي المساواة وعدم المساواة بوجود فرضيات مناسبة. تم اشتقاق الصيغة الرياضية للمعادلات الرباعية المصاحبة و المرتبطة بالمعادلات التفاضلية الجزئية غير الخطية الرباعية ، تم ايجاد مشتقة فريشيه لدالة الهدف أخيرًا ، تم ذكر وإثبات نظرية الشروط الضروري ونظرية الشروط الكافية لوجود متجه سيطرة تقليدية امثلية مستمرة مقيدة.

كلمات مفتاحية: متجه السيطرة الامثلية التقليدية المستمرة المقيدة , مسالة قيم حدودية اهليجية رباعية غير خطية, مشتقة فريشيه , الشروط الضرورية والكافية للامثلية.

Introduction

Optimal control problems (OCPs) have been involved in many applications in practical life, such as medicine [1], economics [2], robotics [3], Aircraft [4], and other applied fields. Usually, many researchers have given extensive attention to the study of the OCPs in general, and to the study of optimal classical continuous control problems (OCCCPs) in particular. Furthermore, in the past decade, great attention has been given to the study of OCCCPs governed by three kinds elliptic[5], hyperbolic[6], and parabolic[7].

Later the study of this subject was expanded to deal with OCCCPs for systems controlled by a couple of NLPDES of the three kinds above[8], and then for systems controlled by a triple of NLPDES of these three kinds[9].

All these surveys stimulate us to research the CQCCOCV with EINC controlled by QNLEPDEqs. In this work, the existence theorem of a CQCCOCV of the considered problem is stated and proved under suitable HYPOTHESES. The mathematical formulation of the AQEqs associated with QNLEPDEqs is derived, and the Fréchet derivative (FD) for the objective function and the equality and inequality constraint (EINC) are derived. Finally, the necessary condition theorem (NCTH) and the sufficient condition theorem (SCTH) for the optimality are stated and proved.

Description of the Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded and open connected subset with Lipschitz boundary $\partial\Omega$. Consider the CQCCOCVP which consists of the QNLEPDEqs

$$-\Delta y_1 + y_1 + y_2 + y_3 - y_4 + b_1(x, y_1, u_1) = d_1(x, u_1) \text{ in } \Omega \quad (1)$$

$$-\Delta y_2 - y_1 + y_2 + y_3 - y_4 + b_2(x, y_2, u_2) = d_2(x, u_2) \text{ in } \Omega \quad (2)$$

$$-\Delta y_3 - y_1 - y_2 + y_3 - y_4 + b_3(x, y_3, u_3) = d_3(x, u_3) \text{ in } \Omega \quad (3)$$

$$-\Delta y_4 + y_1 + y_2 + y_3 + y_4 + b_4(x, y_4, u_4) = d_4(x, u_4) \text{ in } \Omega \quad (4)$$

with the Dirchlet boundary condition

$$y_r = 0\,, \quad r = 1,2,3,4 \qquad on \quad \partial\Omega \quad (5)$$

where $\Delta y_r = \sum_{i,k}^{2} \frac{\partial}{\partial x_i}\left(\Delta y_{ik} \frac{\partial y_r}{\partial x_r}\right)$, $(r = 1,2,3,4)$, $\Delta y_{ik} = \Delta y_{ik}(x) \in L^2(\Omega)$, $\forall i, k = 1,2$, $x = (x_1, x_2), \vec{y} = (y_1, y_2, y_3, y_4) \in \left(H_0^2(\Omega)\right)^4$ is the quaternary state solution (QSS) of the (1)-(5), $\vec{u} = (u_1, u_2, u_3, u_4) \in \left(L^2(\Omega)\right)^4$ is the CQCCCV, the function $b_i(x, y_i, u_i)$ is defined on $\Omega \times \mathbb{R} \times U_i$ $(\forall i = 1,2,3,4)$ respectively (resp.) the function $d_i(x, u_i)$ is defined on $\Omega \times U_i$ $(\forall i = 1,2,3,4)$ resp. with $U_i \subset \mathbb{R}$ $(\forall i = 1,2,3,4)$.

The control constraints (CQ) are

$\vec{U} = U_1 \times U_2 \times U_3 \times U_4$ is a convex set.

$$\vec{V} = \left\{\vec{u} \in \left(L^2(\Omega)\right)^4 : \vec{u} = (u_1, u_2, u_3, u_4) \in \vec{U}\ a.e\ in\ \Omega\right\}$$

The objective functions and the equality and inequality constraints are defined as:

$$J_0(\vec{u}) = \sum_{i=1}^{4} \int_\Omega j_{0i}(x, y_i, u_i)dx$$

$$J_1(\vec{u}) = \sum_{i=1}^{4} \int_\Omega j_{1i}(x, y_i, u_i)dx = 0$$

$$J_2(\vec{u}) = \sum_{i=1}^{4} \int_\Omega j_{2i}(x, y_i, u_i)dx \le 0$$

The set of the admissible quaternary controls is

$$\vec{U}_A = \{\vec{u} \in \vec{V} : J_1(\vec{u}) = 0,\ J_2(\vec{u}) \le 0\}$$

The CQCCOCP is to minimize the objective function $J_0(\vec{u})$ subject to the EINC $J_1(\vec{u}) = 0$ and $J_2(\vec{u}) \leq 0$ s.t. : $J_0(\vec{u}) = \inf_{\vec{u}\in\vec{U}_A} J_0(\vec{u})$

Consider $\vec{W} = W_1 \times W_2 \times W_3 \times W_4 = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$

$$= \left\{\vec{w}, \vec{w} = (w_1, w_2, w_3, w_4) \in \left(H_0^1(\Omega)\right)^4 with\, w_i = 0 \ \forall i = 1,2,3,4 \ on \ \partial\Omega\right\}$$

The weak form (WF) of the QNLEPDEqs

The WF of problem ((1)-(5)) is

$$(\nabla y_1, \nabla w_1) + (y_1, w_1) + (y_2, w_1) + (y_3, w_1) - (y_4, w_1) + (b_1(y_1, u_1), w_1) = d_1\big((u_1), w_1\big) \quad \forall w_1 \in W_1 \tag{6}$$

$$(\nabla y_2, \nabla w_2) - (y_1, w_2) + (y_2, w_2) + (y_3, w_2) - (y_4, w_2) + (b_2(y_2, u_2), w_2) = d_2\big((u_2), w_2\big) \quad \forall w_2 \in W_2 \tag{7}$$

$$(\nabla y_3, \nabla w_3) - (y_1, w_3) - (y_2, w_3) + (y_3, w_3) - (y_4, w_3) + (b_3(y_3, u_3), w_3) = d_3\big((u_3), w_3\big) \quad \forall w_3 \in W_3 \tag{8}$$

$$(\nabla y_4, \nabla w_4) + (y_1, w_4) + (y_2, w_4) + (y_3, w_4) + (y_4, w_4) + (b_4(y_4, u_4), w_4) = d_4\big((u_4), w_4\big) \quad \forall w_4 \in W_4 \tag{9}$$

By blending to gather the equations ((6)-(9)), once get

$$B(\vec{y}, \vec{u}) + (b_1(y_1, u_1), w_1) + (b_2(y_2, u_2), w_2) + (b_3(y_3, u_3), w_3) + (b_4(y_4, u_4), w_4) = (d_1(u_1), w_1) + (d_2(u_2), w_2) + (d_3(u_3), w_3) + (d_4(u_4), w_4) \tag{10}$$

Hypotheses A:

a) $B(\vec{y}, \vec{w})$ is coercive, i.e. $\exists\, \epsilon > 0$ s.t. $\frac{B(\vec{y},\vec{y})}{\|\vec{y}\|_1} \geq \epsilon \|\vec{y}\|_1 > 0$, $\vec{y} \in \vec{W}$

b) $B(\vec{y}, \vec{w})$ is continuous i.e. $\exists\, \epsilon > 0$ s.t.$|B(\vec{y}, \vec{w})| \leq \epsilon \, \|\vec{y}\|_1 \|\vec{w}\|_1$, $\forall\, \vec{y}, \vec{w} \in \vec{W}$

c) The functions $b_i(x, y_i, u_i)$ are of Carathéodory type (C-T) on $\Omega \times R \times U_i$, $\forall i = 1,2,3,4$ and satisfy the following conditions: $|b_i(x, y_i, u_i)| \leq \vartheta_i(x) + c_i|y_i| + \bar{c}_i|u_i|, \forall (x, y_i, u_i) \in \Omega \times R \times U_i \ with \ \vartheta_i \in L^2(\Omega),\ c_i\bar{c}_i \geq 0, \forall i = 1,2,3,4$

d) $b_i(x, y_i, u_i)$ $\forall i = 1,2,3,4$ are monotone w.r.t y_i for each $x \in \Omega$, $u_i \in U_i$ $\forall i = 1,2,3,4$

e) $b_i(x, 0, u_i) = 0$, $x \in \Omega \ and \ u_i \in U_i$ $\quad \forall i = 1,2,3,4$

f) The function $d_i(x,u_i)$ is of C-T on $\Omega \times U_i$ $\forall i = 1,2,3,4$ and satisfy $\forall i = 1,2,3,4$ $:|d_i(x,u_i)| \leq \vartheta_{i+4}(x) + c_{i+4}|u_i|,\ \forall (x,u_i) \in \Omega \times U_i\ , with\ \vartheta_{i+4} \in L^2(\Omega)\ , c_{i+4} \geq 0$.

Theorem 1[10]: In addition to the hypotheses. (A), if b_1, is strictly monotone. Then for fixed QCCCP $\vec{u} \in \vec{U}$, the WF of (10) has a unique QSVS $\vec{y} \in \vec{W}$.

Lemma 1[10]: In addition to the hypotheses. (A), if the functions. b_i and d_i $(\forall i = 1,2,3,4)$ are Lipchitz w.r.t. u_i resp., then the operator $\vec{u} \to \vec{y}_{\vec{u}}$ from $\vec{V}$ to $\left(L^2(\Omega)\right)^4$ is Lipchitz continuous, i.e. $\left\|\overrightarrow{\delta y}\right\|_0 \leq L\left\|\overrightarrow{\delta u}\right\|_0$, for $L > 0$.

Hypotheses B: Suppose that $j_{\ell i}$ $(for\ \ell = 0,1,2\ and\ i = 1,2,3,4)$ are of C-T on $\Omega \times \mathbb{R} \times U_i$ satisfy the following condition w.r.t. (y_i, u_i) , i.e.

$|j_{\ell i}(x, y_i, u_i)| \leq \vartheta_{\ell i}(x) + c_{\ell i} y_i^2 + \check{c}_{\ell i} u_i^2$

where $(y_i, u_i) \in \mathbb{R} \times U_i$, with $\vartheta_{\ell i} \in L^1(\Omega)$ and $c_{\ell i}, \check{c}_{\ell i} \geq 0$.

Lemma 2[10]: With hypotheses (B), the functional $\vec{u} \to J_{\ell i}(\vec{u})$, $\forall\ \ell = 0,1,2$, defines on $\left(L^2(\Omega)\right)^4$ is continuous.

Results

Existence of a CQCCOCV

Theorem 1: In addition to hypo (A & B), suppose that $\vec{U}$ is compact and $\vec{U}_A \neq \emptyset$, where b_i $(\forall i = 1,2,3,4)$ is independent of u_i $(\forall i = 1,2,3,4)$ resp. , and d_i $(\forall i = 1,2,3,4)$ is linear w.r.t. u_i $(\forall i = 1,2,3,4)$ resp. i.e. $b_i(x, y_i, u_i) = b_i(x, y_i)$, $d_i(x, u_i) = d_i(x) u_i$, s.t. $|b_i(x, y_i)| \leq \vartheta_i(x) + C_i |y_i|$ & $|d_i(x)| \leq n_i$ where $\vartheta_i \in L^1(\Omega)$, $n_i \in L^2(\Omega)$, and $\tilde{C}_i \geq 0$ j_{1i} is independent . of u_i and $j_{\ell i}$ for $(\ell = 0,2\ and\ i = 1,2,3,4)$ are con. w.r.t. u_i for fixed (x, y_i), then there exists a CQCCOCV.

Proof: The functions $J_\ell(\vec{u})$is continuous on$\left(L^2(\Omega)\right)^4$, for each$\ell = 0,1,2$ (by lemma 2.2).

Now, since j_{1i} is independent of u_i and since $y_{in} \to y_i$ ST in $L^2(\Omega), (\forall i = 1,2,3,4)$, (from the proof of theorem 2.1), then

$J_1(\vec{u}_n) = \int_\Omega j_{11}(x, y_{1n})dx + \int_\Omega j_{12}(x, y_{2n})dx + \int_\Omega j_{13}(x, y_{3n})dx + \int_\Omega j_{14}(x, y_{4n})dx \;\; \longrightarrow$

$\int_\Omega j_{11}(x, y_1)dx + \int_\Omega j_{12}(x, y_1)dx + \int_\Omega j_{13}(x, y_1)dx + \int_\Omega j_{14}(x, y_1)dx = J_1(\vec{u})$

i.e. $J_1(\vec{u}) = \lim_{x\to\infty} J_1(\vec{u}_n)$, but $J_1(\vec{u}_n) = 0 \;\; \forall n$ hence $J_1(\vec{u}) = 0$

Now, to prove $J_\ell(\vec{u}), (\forall \ell = 0,2)$ is weakly lower semicontinuous (W.L.Sc.) w.r.t. (y_i, u_i) $\forall i = 1,2,3,4$

Since $j_{\ell i}(x, y_i, u_i), (\ell = 0,2)$ *and* $(\forall i = 1,2,3,4)$ is continuous w.r.t. (y_i, u_i), in this case we have $u_{in} \in \vec{U}$, $\forall i = 1,2,3,4$ in Ω , and $\vec{U}$ is compact. , hence $J_\ell(\vec{u})$ is satisfied the hypotheses of lemma (3)[8], to get that

$$\int_\Omega j_{\ell i}(x, y_{in}, u_{in})dx \to \int_\Omega j_{\ell i}(x, y_i, u_{in})dx \qquad (11)$$

Since $j_{\ell i}(x, y_i, u_i)$, $(\forall \ell = 0,2)$ is continuous and convex w.r.t. u_i, then $\int_\Omega j_{\ell i}(x, y_i, u_i)dx$ is W.L.Sc. w.r.t. u_i, i.e.

$$\begin{aligned}\int_\Omega j_{\ell i}(x, y_i, u_i)dx &\le \lim_{n\to\infty} \int_\Omega j_{\ell i}(x, y_i, u_{in})dx \\ &= \lim_{n\to\infty} \int_\Omega (j_{\ell i}(x, y_i, u_{in})dx - j_{\ell i}(x, y_{in}, u_{in}))dx \\ &\quad + \lim_{n\to\infty} \int_\Omega j_{\ell i}(x, y_{in}, u_{in})dx \\ &= \lim_{n\to\infty} \int_\Omega j_{\ell i}(x, y_{in}, u_{in})dx\end{aligned}$$

Thus $J_\ell(\vec{u})$ is W.L.Sc w.r.t. $(\vec{y}, \vec{u})$, then $J_2(\vec{u}) \le \lim_{n\to\infty} J_2(\vec{u}_n) = 0$

On the other hand; we have that

$$J_0(\vec{u}) \le \lim_{n\to\infty} J_0(\vec{u}_n) = \lim_{n\to\infty} J_0(\vec{u}_n) = \inf_{\vec{u}\in\vec{U}_A} J_0(\vec{u})$$

$\Rightarrow \vec{u}$ is a CQCCOCV.

The NCTh and The SCTH for Optimality

The following hypotheses are useful to study the NCTH and the SCTH .

Hypotheses C:

a) The funs. b_{iy_i}, b_{iu_i} $\forall i = 1,2,3,4$ is of the C-T on $\Omega \times \mathbb{R} \times \mathbb{R}$ and satisfy:

$|b_{iy_i}(x, y_i, u_i)| \le k_i$, $|b_{iu_i}(x, y_i, u_i)| \le p_i$ where $x \in \Omega$ and $k_i, p_i \ge 0$, $\forall I = 1,2,3,4$

b) The function d_{iu_i} , is of the C-T on $\Omega \times \mathbb{R}$ and satisfy

$\left|d_{iu_i}(x,u_i)\right| \le q_i$ where $x \in \Omega$ and $q_i \ge 0$, $\forall i = 1,2,3,4$

d) The funs.$j_{\ell_{iy_i}}, j_{\ell_{iu_i}}$ for $\ell = 0,1,2$ and $i = 1,2,3,4$ is of the C-T on $\Omega \times \mathbb{R}^2$ and satisfy

$\left|j_{\ell_{iy_i}}\right| \le \eta_{\ell_i} + c_{\ell_i}|y_i| + d_{\ell_i}|u_i|$ and $\left|j_{\ell_{iu_i}}\right| \le \hat{\eta}_{\ell_i} + \hat{c}_{\ell_i}|y_i| + \hat{d}_{\ell_i}|u_i|$

Where $c_{\ell_i}, \hat{c}_{\ell_i}, d_{\ell_i}, \hat{d}_{\ell_i} \ge 0$, $\eta_{\ell_i}, \hat{\eta}_{\ell_i} \in L^2(\Omega)$

Theorem 1: With hypotheses (A, B &C), the Hamiltonian(Ham.) is defined by:

$H(x,\vec{y},\vec{z},\vec{u}) = \sum_{i=1}^{4}(z_i(d_i(x,u_i) - b_i(x,y_i,u_i)) + j_{0i}(x,y_i,u_i))$.

The AQEqs in Ω $(z_i = z_{iu_i}), \forall i = 1,2,3,4$ of the QNLEPDEqs((1)- (5)) are:

$-\Delta z_1 + z_1 - z_2 - z_3 + z_4 + z_1 b_{1y_1}(x,y_1,u_1) = j_{01y_1}(x,y_1,u_1),$ (12)

$-\Delta z_2 + z_1 + z_2 - z_3 + z_4 + z_2 b_{2y_2}(x,y_2,u_2) = j_{02y_2}(x,y_2,u_2)$, (13)

$-\Delta z_3 + z_1 + z_2 + z_3 + z_4 + z_3 b_{3y_3}(x,y_3,u_3) = j_{03y_3}(x,y_3,u_3)$, (14)

$-\Delta z_4 - z_1 - z_2 - z_3 + z_4 + z_4 b_{4y_4}(x,y_4,u_4) = j_{04y_4}(x,y_4,u_4)$, (15)

$z_1 = z_2 = z_3 = z_4 = 0 \quad on\ \partial\Omega$ (16)

Then the FD of J_0 is given by $\acute{\vec{J}}_0(\vec{u})\overrightarrow{\delta u} = \int_\Omega H_{\vec{u}}^T.\overrightarrow{\delta u}\,dx$ Where

$$H_{\vec{u}} = \begin{pmatrix} H_{u_1}(x,\vec{y},\vec{z},\vec{u}) \\ H_{u_2}(x,\vec{y},\vec{z},\vec{u}) \\ H_{u_3}(x,\vec{y},\vec{z},\vec{u}) \\ H_{u_4}(x,\vec{y},\vec{z},\vec{u}) \end{pmatrix} = \begin{pmatrix} z_1(b_{1u_1} - d_{1u_1}) + j_{01u_1} \\ z_2(b_{2u_2} - d_{2u_2}) + j_{02u_2} \\ z_3(b_{3u_3} - d_{3u_3}) + j_{03u_3} \\ z_4(b_{4u_4} - d_{4u_4}) + j_{04u_4} \end{pmatrix}$$

Proof :- The WF of the QAEqs ((12)-(15)) is

$(\nabla z_1,\nabla w_1) + (z_1,w_1) - (z_2,w_1) - (z_3,w_1) + (z_4,w_1) + (z_1 b_{1y_1}(y_1,u_1),w_1) =$

$(j_{01y_1}(y_1,u_1),w_1) \quad \forall w_1 \in W_1$ (17)

$(\nabla z_2,\nabla w_2) + (z_1,w_2) + (z_2,w_2) - (z_3,w_2) + (z_4,w_2) + (z_2 b_{2y_2}(y_2,u_2),w_2) =$

$(j_{02y_2}(y_2,u_2),w_2) \quad \forall w_2 \in W_2$ (18)

$(\nabla z_3,\nabla w_3) + (z_1,w_3) + (z_2,w_3) + (z_3,w_3) + (z_4,w_3) + (z_3 b_{3y_3}(y_3,u_3),w_3) =$

$(j_{03y_3}(y_3,u_3),w_3) \quad \forall w_3 \in W_3$ (19)

$$(\nabla z_4, \nabla w_4) - (z_1, w_4) - (z_2, w_4) - (z_3, w_4) + (z_4, w_4) + \left(z_4 b_{4y_4}(y_4, u_4), w_4\right) = \left(j_{04y_4}(y_4, u_4), w_4\right) \quad \forall w_4 \in W_4 \tag{20}$$

blending together ((17)-(20)), then substituting $\vec{w} = \overrightarrow{\delta y}$, once has

$$(\nabla z_1, \nabla \delta y_1) + (z_1, \delta y_1) - (z_2, \delta y_1) - (z_3, \delta y_1) + (z_4, \delta y_1) + (\nabla z_2, \nabla \delta y_2) + (z_1, \delta y_2) + (z_2, \delta y_2) - (z_3, \delta y_2) + (z_4, \delta y_2) + (\nabla z_3, \nabla \delta y_3) + (z_1, \delta y_3) + (z_2, \delta y_3) + (z_3, \delta y_3) + (z_4, \delta y_3) + (\nabla z_4, \nabla \delta y_4) - (z_1, \delta y_4) - (z_2, \delta y_4) - (z_3, \delta y_4) + (z_4, \delta y_4) + \left(z_1 b_{1y_1}(y_1, u_1), \delta y_1\right) + \left(z_2 b_{2y_2}(y_2, u_2), \delta y_2\right) + \left(z_3 b_{3y_3}(y_3, u_3), \delta y_3\right) + \left(z_4 b_{4y_4}(y_4, u_4), \delta y_4\right) = \left(j_{01y_1}(y_1, u_1), \delta y_1\right) + \left(j_{02y_2}(y_2, u_2), \delta y_2\right) + \left(j_{03y_3}(y_3, u_3), \delta y_3\right) + \left(j_{04y_4}(y_4, u_4), \delta y_4\right) \tag{21}$$

Substituting $\vec{y}, \vec{y} + \overrightarrow{\delta y}$ in the WF ((6)-(9)), then subtracting each equation from its corresponding one with setting $w_i = z_i \ \forall i = 1,2,3,4$ to get , then blending the obtain equation:

$$(\nabla \delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) - (\delta y_3, z_1) + (\delta y_4, z_1) + (\nabla \delta y_2, \nabla z_2) + (\delta y_1, z_2) + (\delta y_2, z_2) - (\delta y_3, z_2) + (\delta y_4, z_2) + (\nabla \delta y_3, \nabla z_3) + (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_3, z_3) + (\delta y_4, z_3) + (\nabla \delta y_4, \nabla z_4) - (\delta y_1, z_4) - (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4) + (b_1(y_1 + \delta y_1, u_1 + \delta u_1) - b_1(y_1, u_1), z_1) + (b_2(y_2 + \delta y_2, u_2 + \delta u_2) - b_2(y_2, u_2), z_2) + (b_3(y_3 + \delta y_3, u_3 + \delta u_3) - b_3(y_3, u_3), z_3) + (b_4(y_4 + \delta y_4, u_4 + \delta u_4) - b_4(y_4, u_4), z_4)$$

$$= (d_1(u_1 + \delta u_1) - d_1(u_1), z_1) + (d_2(u_2 + \delta u_2) - d_2(u_2), z_2) + (d_3(u_3 + \delta u_3) - d_3(u_3), z_3) + (d_4(u_4 + \delta u_4) - d_4(u_4), z_4) \tag{22}$$

From hypotheses (A-a&f) on b_i, d_i $(\forall i = 1,2,3,4)$ and by proposition (2) [11], we get that the FD of b_i, d_i exists , i.e.

$$\int_\Omega \left(b_i(x, y_i + \delta y_i, u_i + \delta u_i) - b_i(x, y_i, u_i)\right) z_i dx = \left(b_{iy_i}\delta y_i + b_{iu_i}\delta u_i, z_i\right) + \tilde{\varepsilon}_i(\overrightarrow{\delta u}) \left\|\overrightarrow{\delta u}\right\|_0, \quad \forall i = 1,2,3,4$$

And $\int_\Omega \left(d_i(x, u_i + \delta u_i) - d_i(x, u_i)\right) z_i dx = \left(d_{iu_i}\delta u_i, z_i\right) + \tilde{\varepsilon}_{i+4}(\overrightarrow{\delta u}) \left\|\overrightarrow{\delta u}\right\|_0$

By substituting in (22), to obtain

$(\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) - (\delta y_3, z_1) + (\delta y_4, z_1) + (\nabla\delta y_2, \nabla z_2) + (\delta y_1, z_2) + (\delta y_2, z_2) - (\delta y_3, z_2) + (\delta y_4, z_2) + (\nabla\delta y_3, \nabla z_3) + (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_3, z_3) + (\delta y_4, z_3) + (\nabla\delta y_4, \nabla z_4) - (\delta y_1, z_4) - (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4) + \left(b_{1y_1}\delta y_1 + b_{1u_1}\delta u_1, z_1\right) + \left(b_{2y_2}\delta y_2 + b_{2u_2}\delta u_2, z_2\right) + \left(b_{3y_3}\delta y_3 + b_{3u_3}\delta u_3, z_3\right) + +\left(b_{4y_4}\delta y_4 + b_{4u_4}\delta u_4, z_4\right) + \tilde{\varepsilon}_8\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0 =$

$$\left(d_{1u_1}\delta u_1, z_1\right) + \left(d_{2u_2}\delta u_2, z_2\right) + \left(d_{3u_3}\delta u_3, z_3\right) + \left(d_{4u_4}\delta u_4, z_4\right) + \tilde{\varepsilon}_9\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0 \quad (23)$$

Where $\tilde{\varepsilon}_8\left(\overrightarrow{\delta u}\right) = \sum_{i=1}^{4} \tilde{\varepsilon}_i\left(\overrightarrow{\delta u}\right)$, $\tilde{\varepsilon}_9\left(\overrightarrow{\delta u}\right) = \sum_{i=1}^{4} \tilde{\varepsilon}_{i+1}\left(\overrightarrow{\delta u}\right)$

Subtracting (21) from (23) to get

$$\sum_{i=1}^{4}\left(j_{0iy_i}(y_i, u_i), \delta y_i\right) + \sum_{i=1}^{4}\left(b_{iu_i}\delta u_i, z_i\right) = \sum_{i=1}^{4}\left(d_{iu_i}\delta u_i, z_i\right) + \tilde{\varepsilon}_{10}\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0 \quad \forall i = 1,2,3,4 \quad (24)$$

Where $\tilde{\varepsilon}_{10}\left(\overrightarrow{\delta u}\right) = \tilde{\varepsilon}_9\left(\overrightarrow{\delta u}\right) - \tilde{\varepsilon}_8\left(\overrightarrow{\delta u}\right)$

From hypotheses C, and lemma (2.1), we have

$$J_0\left(\vec{u} + \overrightarrow{\delta u}\right) - J_0(\vec{u}) = \sum_{i=1}^{4} \int_\Omega \left(j_{0iy_i}(y_i, u_i)\delta y_i + j_{0iu_i}(y_i, u_i)\delta u_i\right)dx + \tilde{\varepsilon}_{11}\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0 \quad (25)$$

Where$\tilde{\varepsilon}_{10}\left(\overrightarrow{\delta u}\right) = \sum_{i=1}^{4} \tilde{\varepsilon}_{i+8}\left(\overrightarrow{\delta u}\right) \to 0$ and $\left\|\overrightarrow{\delta u}\right\|_0 \to 0$ as $\overrightarrow{\delta u} \to 0$ $\forall i = 1,2,3,4$

From (24) & (25), once obtain

$$J_0\left(\vec{u} + \overrightarrow{\delta u}\right) - J_0(\vec{u}) = \sum_{i=1}^{4} \int_\Omega \left(z_i\left(b_{iy_i} - d_{iu_i}\right) + j_{0iu_i}\right)\delta u_i dx + \tilde{\varepsilon}\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0 \quad (26)$$

Where $\tilde{\varepsilon}\left(\overrightarrow{\delta u}\right) = \tilde{\varepsilon}_{10}\left(\overrightarrow{\delta u}\right) + \tilde{\varepsilon}_{11}\left(\overrightarrow{\delta u}\right) \longrightarrow 0$ $\quad as\ \delta u \longrightarrow 0$

But from the FD of J_0, we have that

$$J_0\left(\vec{u} + \overrightarrow{\delta u}\right) - J_0(\vec{u}) = \acute{J}_0(\vec{u})\overrightarrow{\delta u} + \tilde{\varepsilon}\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0 \quad (27)$$

Finally; from (26) and (27) once get

$\acute{J}_0(\vec{u})\overrightarrow{\delta u} = \int_\Omega H_{\vec{u}}^T \cdot \overrightarrow{\delta u} dx$, where

$$H_{\vec{u}} = \begin{pmatrix} H_{u_1}(x,\vec{y},\vec{z},\vec{u}) \\ H_{u_2}(x,\vec{y},\vec{z},\vec{u}) \\ H_{u_3}(x,\vec{y},\vec{z},\vec{u}) \\ H_{u_4}(x,\vec{y},\vec{z},\vec{u}) \end{pmatrix} = \begin{pmatrix} z_1(b_{1u_1} - d_{1u_1}) + j_{01u_1} \\ z_2(b_{2u_2} - d_{2u_2}) + j_{02u_2} \\ z_3(b_{3u_3} - d_{3u_3}) + j_{03u_3} \\ z_4(b_{4u_4} - d_{4u_4}) + j_{04u_4} \end{pmatrix}$$

Theorem 2: The NCTh for Optimality

(a) With hypotheses (A , B & C) and with U convex if $\vec{u} \in \vec{U}_A$ is CQCCOCV, then there exist multipliers $\lambda_\ell \in \mathbb{R}$, $\ell = 0,1,2$ with $\lambda_0, \lambda_2 \geq 0$, $\sum_{\ell=0}^{2}|\lambda_\ell| = 1$, such that the following Kuhn-Tucker-Lagrange conditions (K.T.L.C) are satisfied:

$$\int_\Omega H_{\vec{u}}^T \cdot \overrightarrow{\delta u}\, dx \geq 0 , \ \forall \vec{v} \in \vec{V} , \overrightarrow{\delta u} = \vec{v} - \vec{u} \qquad (28a)$$

Where $j_i = \sum_{\ell=0}^{2} \lambda_\ell j_{\ell i}$ and $z_i = \sum_{\ell=0}^{2} \lambda_\ell z_{\ell i}$, $\forall i = 1,2,3,4$ in the definition of H, and also

$$\lambda_2 J_2(\vec{u}) = 0 , \qquad (28b)$$

(b) (28a) is equivalent to the Minimum Principle form

$$H_{\vec{u}}^T \cdot \vec{u} = \min_{\vec{u} \in \vec{U}} H_{\vec{u}}^T \cdot \vec{v} \ e.a \ on \ \Omega \qquad (29)$$

Proof: (a) From the hypotheses (A , B & C), and from Lemma (2.2) get that the function $J_\ell(\vec{u})$ is continuous and is ρ- local continuous at each $\vec{u} \in \vec{V}$ $\forall \ell = 0,1,2$ and for each ρ . Also from the hypotheses (A , B & C) and Theorem (2.2), the funl. $J_\ell(\vec{u})$ ($\forall \ell = 0,1,2$) has a continuous FD at each $\vec{u} \in \vec{V}$, hence $J_\ell(\vec{u})$ is ρ - differentiable at each $\vec{u} \in \vec{V}$, for each ρ .

Since $\vec{u} \in \vec{U}_A$ is a CQCCOCV, then we can apply, the K.T.L.C with $\lambda_\ell \in \mathbb{R}$, $\forall \ell = 0,1,2$, $\lambda_0, \lambda_2 \geq 0$, $\sum_{\ell=0}^{2}|\lambda_\ell| = 1$ s.t.

$$(\lambda_0 \acute{J}_{0\vec{u}}(\vec{u}) + \lambda_1 \acute{J}_{1\vec{u}}(\vec{u}) + \lambda_2 \acute{J}_{2\vec{u}}(\vec{u})) \cdot (\vec{v} - \vec{u}) \geq 0 \ \forall \vec{v} \in \vec{V} \qquad (30\text{ a})$$

and

$$\lambda_2 J_2(\vec{u}) = 0 \qquad (30\text{ b})$$

By using Theorem (4.1), setting $\delta u_i = v_i - u_i$, $\forall i = 1,2,3,4$ and substituting the FD of J_ℓ , for ($\ell = 0,1,2$) in (34 a) to obtain

$$\sum_{i=1}^{4} \int_\Omega (\lambda_0 z_{i0} + \lambda_1 z_{i1} + \lambda_2 z_{i2})(d_{iu_i} - b_{iu_i}) + (\lambda_0 j_{iou_i} + \lambda_1 j_{i1u_i} + \lambda_2 j_{i2u_i})\delta u_i dx$$

$$\Rightarrow \sum_{i=1}^{4} \int_\Omega \left(z_i(d_{iu_i} - b_{iu_i}) + j_{iu_i}\right) \delta u_i dx \geq 0 \ , \forall \vec{v} \in \vec{V}$$

where $z_i = \sum_{\ell=0}^{2} \lambda_\ell z_{i\ell}$, $j_{iu_i} = \sum_{\ell=0}^{2} \lambda_\ell j_{i\ell u_i}$, $(\forall i = 1,2,3,4)$

$\Rightarrow \int_\Omega H_{\vec{u}}^T \cdot \overrightarrow{\delta u}\, dx \geq 0\,, \quad \forall \vec{v} \in \vec{V}\,, \quad \overrightarrow{\delta u} = \vec{v} - \vec{u}$

(b) Let μ is a Lebesgue measure on Ω , $\{\vec{u}_n\}$ be a dense sequence in $\vec{V}$ and let S⊂Ω be a measurable set s.t.

$$\vec{v}(x) = \begin{cases} \vec{u}_n(x) & if\ x \in S \\ \vec{u}(x) & if\ x \notin S \end{cases}$$

Hence (28a), become

$\int_S H_{\vec{u}}^T \cdot (\vec{u}_n - \vec{u}) \geq 0$ for each such set S

From Egorov's theorem [11] once has

$H_{\vec{u}}^T \cdot (\vec{u}_n - \vec{u}) \geq 0 \quad a.e\ on\ \Omega$

i.e. it holds in a set $\varphi_n = \Omega - \Omega_n$ with $\mu(\Omega_n) = 0$

$\Rightarrow H_{\vec{u}}(x, \vec{y}, \vec{z}, \vec{u}) \cdot (\vec{u}_n - \vec{u}) \geq 0\,, \ in\ \varphi = \bigcap_n \varphi_n$

And this hold for each n , since φ is independent of n , and we have

$\mu(\Omega/\varphi) = \mu\left(\bigcup_n \varphi_n\right) = 0$

But $\{\vec{u}_n\}$ is dense in $\vec{U}$, then

$H_{\vec{u}}^T \cdot (\vec{u}_n - \vec{u}) \geq 0\ in\ \varphi, \ \ (i.e.\ a.e\ on\ \Omega)$

$\Rightarrow H_{\vec{u}}^T \cdot \vec{u} = \min_{\vec{u} \in \vec{U}} H_{\vec{u}}^T \cdot \vec{v}\ \ a.e\ on\ \Omega$

The converse is easy.

Theorem 3: In addition to hypotheses (A, B & C), $\vec{U}$ is convex b_i is affine w.r.t. (y_i, u_i) resp. d_i are affine w.r.t. u_i $\forall i = 1,2,3,4$ resp. for each x. J_{1i} is affine w.r.t. (y_i, u_i) and $J_{\ell i}$ ($\ell =$ 0,1,2 , $i = 1,2,3,4$) is convex w.r.t. (y_i, u_i) for each x , then the NCTh in Theorem (4.2), with $\lambda_0 > 0$ are also sufficient.

Proof: From the proof of Theorem (4.2), once obtain that

$\int_\Omega H_{\vec{u}}(x, \vec{y}, \vec{z}, \vec{u}) \cdot \overrightarrow{\delta u}\, dx \ \geq 0\,, \ \forall \vec{v} \in \vec{V}$, and $\lambda_2 J_2(\vec{u}) = 0$

Now, suppose $\vec{u} \in \vec{U}_A$, and let $J(\vec{u}) = \sum_{\ell=0}^{2} \lambda_\ell J_\ell(\vec{u})$, then for $\ell = 0,1,2$, and $i = 1,2,3,4$

$\vec{J}(\vec{u})\overrightarrow{\delta u} = \sum_{\ell=0}^{2} \lambda_\ell \vec{J}_\ell(\vec{u})\, \overrightarrow{\delta u}$

$$= \sum_{\ell=0}^{2} \sum_{i=1}^{4} \int_{\Omega} \lambda_{\ell} \left(z_{i\ell} \left(d_{u_i} - b_{u_i} \right) + j_{i\ell u_i} \right) \delta u_i dx$$

$$= \int_{\Omega} H_{\vec{u}}(x, \vec{y}, \vec{z}, \vec{u}) \cdot \overrightarrow{\delta u}\, dx \ \geq 0$$

From the hypotheses on b_i, d_i , $\forall i = 1,2,3,4$,

$b_i(x, y_i, u_i) = b_{i1}(x) y_i + b_{i2}(x) u_i + b_{i3}(x)$,

$d_i(x, u_i) = d_{i1}(x) u_i + d_{i2}(x)$

Let $\vec{u}$ & $\vec{\bar{u}}$ be two CQCCC, then $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{\bar{y}} = \vec{\bar{y}}_{\vec{u}}$, are their corresponding QSS by theorem (2.1) , i.e.

$-\Delta y_1 + y_1 - y_2 - y_3 + y_4 + b_{11}(x) y_1 + b_{12}(x) u_1 + b_{13}(x) = d_{11}(x) u_1 + d_{12}(x)$ (31a)

$-\Delta y_2 + y_1 + y_2 - y_3 + y_4 + b_{21}(x) y_2 + b_{22}(x) u_2 + b_{23}(x) = d_{21}(x) u_2 + d_{22}(x)$ (31b)

$-\Delta y_3 + y_1 + y_2 + y_3 + y_4 + b_{31}(x) y_3 + b_{32}(x) u_3 + b_{33}(x) = d_{31}(x) u_3 + d_{32}(x)$ (31c)

$-\Delta y_4 - y_1 - y_2 - y_3 + y_4 + b_{41}(x) y_4 + b_{42}(x) u_4 + b_{43}(x) = d_{41}(x) u_4 + d_{42}(x)$ (31d)

$y_i = 0 \quad in\ \partial\Omega \quad \forall i = 1,2,3,4$ (31e)

And

$-\Delta \bar{y}_1 + \bar{y}_1 - \bar{y}_2 - \bar{y}_3 + \bar{y}_4 + b_{11}(x) \bar{y}_1 + b_{12}(x) \bar{u}_1 + b_{13}(x) = d_{11}(x) \bar{u}_1 + d_{12}(x)$ (32a)

$-\Delta \bar{y}_2 + \bar{y}_1 + \bar{y}_2 - \bar{y}_3 + \bar{y}_4 + b_{21}(x) \bar{y}_2 + b_{22}(x) \bar{u}_2 + b_{23}(x) = d_{21}(x) \bar{u}_2 + d_{22}(x)$ (32b)

$-\Delta \bar{y}_3 + \bar{y}_1 + \bar{y}_2 + \bar{y}_3 + \bar{y}_4 + b_{31}(x) \bar{y}_3 + b_{32}(x) \bar{u}_3 + b_{33}(x) = d_{31}(x) \bar{u}_3 + d_{32}(x)$ (32c)

$-\Delta \bar{y}_4 - \bar{y}_1 - \bar{y}_2 - \bar{y}_3 + \bar{y}_4 + b_{41}(x) \bar{y}_4 + b_{42}(x) \bar{u}_4 + b_{43}(x) = d_{41}(x) \bar{u}_4 + d_{42}(x)$ (32d)

$\bar{y}_i = 0 , \quad in\ \partial\Omega \quad \forall i = 1,2,3,4$ (32e)

By multiplying (31) by , $\alpha \in [0,1]$ and (32) by $(1 - \alpha)$, then blending together the obtained equations from each sum of ((31 a)&(32 a)),((31 b)&(32 b)),((31 c)&(32 c)),((31 d)&(32 d)),((31 e)&(32 e)) once has

$-\nabla(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_1+(1-\alpha)\bar{y}_1)-(\alpha y_2+(1-\alpha)\bar{y}_2)-(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_4+(1-\alpha)\bar{y}_4)+b_{11}(x)\big((\alpha y_1+(1-\alpha)\bar{y}_1)\big)+b_{12}(x)(\alpha u_1+(1-\alpha)\bar{u}_1)+b_{13}(x)=d_{11}(x)(\alpha u_1+(1-\alpha)\bar{u}_1)+d_{12}(x)$ (33a

$\alpha y_1+(1-\alpha)\bar{y}_1=0$ (33b)

$-\nabla(\alpha y_2+(1-\alpha)\bar{y}_2)+(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_2+(1-\alpha)\bar{y}_2)-(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_4+(1-\alpha)\bar{y}_4)+b_{21}(x)\big((\alpha y_2+(1-\alpha)\bar{y}_2)\big)+b_{22}(x)(\alpha u_2+(1-\alpha)\bar{u}_2)+b_{23}(x)=d_{21}(x)(\alpha u_2+(1-\alpha)\bar{u}_2)+d_{22}(x)$ (34a)

$\alpha y_2+(1-\alpha)\bar{y}_2=0$ (34b)

$-\nabla(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_1+(1-\alpha)\bar{y}_1)+(\alpha y_2+(1-\alpha)\bar{y}_2)+(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_4+(1-\alpha)\bar{y}_4)+b_{31}(x)\big((\alpha y_3+(1-\alpha)\bar{y}_3)\big)+b_{32}(x)(\alpha u_3+(1-\alpha)\bar{u}_3)+b_{33}(x)=d_{31}(x)(\alpha u_3+(1-\alpha)\bar{u}_3)+d_{32}(x)$ (35a)

$\alpha y_3+(1-\alpha)\bar{y}_3=0$ (35b)

$-\nabla(\alpha y_4+(1-\alpha)\bar{y}_4)-(\alpha y_1+(1-\alpha)\bar{y}_1)-(\alpha y_2+(1-\alpha)\bar{y}_2)-(\alpha y_3+(1-\alpha)\bar{y}_3)+(\alpha y_4+(1-\alpha)\bar{y}_4)+b_{41}(x)\big((\alpha y_4+(1-\alpha)\bar{y}_4)\big)+b_{42}(x)(\alpha u_4+(1-\alpha)\bar{u}_4)+b_{43}(x)=d_{41}(x)(\alpha u_4+(1-\alpha)\bar{u}_4)+d_{42}(x)$ (36a)

$\alpha y_4+(1-\alpha)\bar{y}_4=0$ (36b)

Now, if we have the CQCCV $\vec{\bar{\bar{u}}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\bar{\bar{u}}_3,\bar{\bar{u}}_4)$ with $\bar{\bar{u}}_i=\alpha u_i+(1-\alpha)\bar{u}_i,\quad \forall i=1,2,3,4$

Then from (33) - (36), once get that

$\bar{\bar{y}}_i=y_{i\bar{\bar{u}}_i}=y_{i(\alpha u_i+(1-\alpha)\bar{u}_i)}=\alpha y_{iu_i}+(1-\alpha)\bar{y}_{i\bar{u}_i}\quad \forall i=1,2,3,4$

Are either corresponding QSS, i.e.

$-\Delta\bar{\bar{y}}_1+\bar{\bar{y}}_1-\bar{\bar{y}}_2-\bar{\bar{y}}_3+\bar{\bar{y}}_4+b_{11}(x)\bar{\bar{y}}_1+b_{12}(x)\bar{\bar{u}}_1+b_{13}(x)=d_{11}(x)\bar{\bar{u}}_1+d_{12}(x)$ (37a)

$-\Delta\bar{\bar{y}}_2+\bar{\bar{y}}_1+\bar{\bar{y}}_2-\bar{\bar{y}}_3+\bar{\bar{y}}_4+b_{21}(x)\bar{\bar{y}}_2+b_{22}(x)\bar{\bar{u}}_2+b_{23}(x)=d_{21}(x)\bar{\bar{u}}_2+d_{22}(x)$ (37b)

$-\Delta\bar{\bar{y}}_3+\bar{\bar{y}}_1+\bar{\bar{y}}_2+\bar{\bar{y}}_3+\bar{\bar{y}}_4+b_{31}(x)\bar{\bar{y}}_3+b_{32}(x)\bar{\bar{u}}_3+b_{33}(x)=d_{31}(x)\bar{\bar{u}}_3+d_{32}(x)$ (37c)

$-\Delta\bar{\bar{y}}_4-\bar{\bar{y}}_1-\bar{\bar{y}}_2-\bar{\bar{y}}_3+\bar{\bar{y}}_4+b_{41}(x)\bar{\bar{y}}_4+b_{42}(x)\bar{\bar{u}}_4+b_{43}(x)=d_{41}(x)\bar{\bar{u}}_4+d_{42}(x)$ (37d)

$\bar{\bar{y}}_i=0\quad in\ \partial\Omega\quad \forall i=1,2,3,4$ (37e).

i.e. the operator $\vec{u} \to \vec{y}_{\vec{u}}$ is convex- linear w.r.t. $(\vec{y},\vec{u})$, for each $x \in \Omega$.

Now, since $j_{1i}(x,y_i,u_i)$ is affine w.r.t. (y_i,u_i) , for each $\in \Omega$, then $J_1(\vec{u})$ is convex linear w.r.t. $(\vec{y},\vec{u})$. Also, since $j_{\ell i}$, $\forall x \in \Omega$, $(\ell = 0,1,2 \quad , i = 1,2,3,4)$, is convex w.r.t. (y_i,u_i) , i.e. . $J_1(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u})$.

Since the FD of $J_\ell(\vec{u})$, $(\ell = 0,1,2)$, exists for each $\vec{u} \in \vec{V}$, and it is continuous, and since $\vec{V}$ is convex, thus $J(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u})$, in the convex $\vec{V}$, and it has a continuous FD, and satisfy $\acute{\vec{J}}(\vec{u})\overrightarrow{\delta u} \geq 0$.

Then, $J(\vec{u})$ has a minimum at $\vec{u}$, i.e.

$J(\vec{u}) \leq J(\vec{v})$, $\forall \vec{v} \in \vec{V} \Rightarrow$

$$\lambda_0 J_0(\vec{u}) + \lambda_1 J_1(\vec{u}) + \lambda_2 J_2(\vec{u}) \leq \lambda_0 J_0(\vec{v}) + \lambda_1 J_1(\vec{v}) + \lambda_2 J_2(\vec{v}) \qquad (38)$$

Now, let $\vec{v}$ be an admissible control, since $\vec{u}$ is also admissible, by substituting these facts in (38), once get that

$\lambda_0 J_0(\vec{u}) + \lambda_2 J_2(\vec{u}) \leq \lambda_0 J_0(\vec{v}) + \lambda_2 J_2(\vec{v}) \leq \lambda_0 J_0(\vec{v})$, $\forall \vec{v} \in \vec{U}$

since $J_2(\vec{v}) \leq 0$, with $\lambda_2 \geq 0$,

$\lambda_2 J_2(\vec{u}) = 0$, then $\lambda_0 J_0(\vec{u}) \leq \lambda_0 J_0(\vec{v})$, $\forall \vec{v} \in \vec{U} \Rightarrow J_0(\vec{u}) \leq J_0(\vec{v})$, $\forall \vec{v} \in \vec{V}$

i.e. $\vec{u}$ is CQCCOCV.

Conclusions

The CQCCOCVP with EINC controlled by QNLEPDEqs is considered in this paper. The existence theorem of a CQCCOCV with EINC of the problem is stated and proved under suitable hypotheses. The mathematical formulation of the AQEqs associated with the QNLEPDEqs is formulated, and the FD for the objective function and the EINC are derived. Finally; The NCTH and the SCTH for the optimality are stated and proved.

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