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Resolving an Inverse Cauchy Problem of Mixed Boundary Value of Bi-harmonic using a meshless collocation method

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Abstract

In this paper, a meshless collocation method using a polynomial expansion is proposed to solve a mixed fourth order differential boundary value problem. By employing the CGLS and PCG algorithms to solve some examples with different exact solutions, and by introducing noise for the input boundary data, the numerical stability of the suggested approaches is demonstrated.

Keywords: Inverse Cauchy problem, mesh less approach, polynomial expansion, CGLS,PCG, Bi-Laplacian differential equation.

حل مشكلة معكوس كوشي لقيمة الحدود المختلطة للتوافقي الثنائي باستخدام طريقة التجميع غير الشبكية

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الخلاصة

في هذا البحث ، تم اقتراح طريقة التجميع غير الشبكية باستخدام متعدد الحدود الموسعة لحل مشكلة القيمة الحدية التفاضلية من الدرجة الرابعة المختلطة. من خلال استخدام خوارزميات CGLS و PCG لحل بعض الأمثلة مع حلول دقيقة مختلفة ، ومن خلال إدخال ضوضاء لبيانات حدود الإدخال ، يتم توضيح الاستقرار العددي للطرق المقترحة

الكلمات المفتاحية:-مشكله كوشي العكسية, طريقه لا شبكيه ,مفكوك متعدد الحدود ,المعادلات التفاضلية ثنائية لا بلاس

CGLS,PCG

Introduction

Many applications of the fourth order differential equation exist in the domains of mathematics, engineering mathematics, fluid and solid mechanics, and computer sciences. Many iterative and non-iterative techniques that satisfy the Dirichlet and/or Dirichlet and Neumann conditions on the boundary have been developed in recent years. While some authors chose to divide the fourth order problem into two second order problems, i.e., two problems with the Laplace equation, this allowed them to take advantage of the second order equation's advantages and all the results obtained for them. Other authors chose to treat the fourth order problem directly and solve it in its original form. Boujaj et al. Finite difference method (FDM)-based numerical approaches were proposed by various authors (see [4] Finite difference method-based numerical approaches were proposed by certain authors (FDM).

By dividing the bi-harmonic issue into two decoupled Poisson equations, (see [1], [2]). Other works based on the finite element method (FEM) include those by [2] and the references cited therein). An iterative method based on the fixed point theory to solve bi-harmonic-type equation with mixed boundary conditions see [6]. Many study have done to prove that the bi-harmonic problem can be solved see ([3], [5], [7], [10], [11], [12], [13]).

In this work, we provide a mesh less collecting approach based on [15]'s method. With the help of the provided Cauchy data, Mostafa1 et. al. [14]. A novel iterative technique has just been developed, based on the transformation of the bi-harmonic due to the work of Mostafa et. Al [14], them work was limited to an annular domain, but in this work, we study another kind of domain with different types of non-accessible part of the boundary.

We construct a linear system, solve it using the CGLS and PCG algorithms by applying the suggested method to a few cases with exact solutions that are either polynomial or not, and then apply some noise to the provided data to check the stability of the numerical results.

The remainder of this article is organized as follows: in section 2, the bi-harmonic equation for the inverse Cauchy problem is given. Using a numerical approach that approximates the solution by solving the boundary value problem of Bi-Laplacian differential equation using the approximation of the solution as a polynomial expansion.

Inverse Cauchy problem bi-harmonic equation

We analysis the bi harmonic equation with an annular domain Ω with

$$\Omega = \{(r,\theta): 0 \le r < \rho_e(\theta)\,, \qquad 0 \le \ \theta \le \ 2\pi\}$$

For some given real function $\rho_e(\theta)$, with the boundary $\partial\Omega = \ \Gamma_1 \cup \Gamma_2$

$$\Gamma_1 = \{(r,\theta) \quad : r = \rho_e(\theta) \quad 0 \ \le \ \ \theta \le \beta\pi\}$$

$$\Gamma_2 = \{(r,\theta) \quad : r = \rho_e(\theta) \quad \beta\pi \ \le \ \ \theta \le 2\pi\}$$

The problem is given as follows:

$$\Delta^2 u = F(x,y) \qquad in\ \Omega \qquad (1)$$

$$u(\rho,\theta) = u0(\theta) \qquad on\ \Gamma_1 \qquad (2)$$

$$\partial_n u(\rho,\theta) = h0(\theta) \qquad on\ \Gamma_1 \qquad (3)$$

$$\Delta u(p,\theta) = w_0 \qquad on\ \Gamma_1 \qquad (4)$$

$$\partial_n \Delta u(\rho,\theta) = w_0' \qquad on\ \Gamma_1 \qquad (5)$$

The accessible section of the boundary is the one where the Cauchy data u,$\partial_n u, \Delta u, \partial_n \Delta u$ are given $on\ \Gamma_1$; however, because there are no data on the boundary conditions in this part, the accessible part of the boundary is over-determined (there are four boundary conditions in this part). In fact, this problem is an inverse problem since it is ill-posed in the sense of Hadamard [8], [9].

The boundary's under-determined or inaccessible portion is referred to as such. To find the unknown function on the interior under-determined boundary, an inverse Cauchy problem for the bi-Laplacian is formulated [9].

As we remembered, ∂_n is the outer normal derivative in polar coordinates is given by:

$$\partial_n u(\rho,\theta) = \eta(\theta)\left[\frac{\partial u(\rho,\theta)}{\partial\rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial\theta}\right] \tag{6}$$

With

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}} \tag{7}$$

where is the derivative of the radius function with respect to θ, in our case we take the radius a constant number.

Actually, the inner product of the gradient and the normal vector yields the normal derivatives, i.e.$\frac{\partial u}{\partial n}=\nabla u.\vec{n}$, so we can express the normal derivate in terms of the derivative with respect to x and y:

$$\partial_n u = \eta(\theta)\left[cos\,(\theta)\; - \frac{\rho'}{\rho^2} sin\,(\theta)\right]\partial_x u + \eta(\theta)\left[sin\,(\theta)\; - \frac{\rho'}{\rho^2} cos\,(\theta)\right]\partial_y u. \tag{8}$$

Expression of Solution as a polynomial expansion

We consider that the solution(x, y) is expressed as the following polynomial expansion:

$$u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}\; x^{i-j} y^{j-1} \tag{9}$$

Now, we express the problem in (1-5) in form of the expansion in (9). To do so, we find

$$\partial_x u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}\,(i-j)\,x^{i-j-1}y^{j-1} \tag{10}$$

$$\partial_y u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}\,(j-1)\,x^{i-j}y^{j-2} \tag{11}$$

From which the different degree of derivatives are calculated to find Δu,$\partial_n\Delta$,$\Delta^2 u$

$$\Delta u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}[\,i-j(i-j-1)x^{i-j-2j-1}y^{j-2} + (j-1)(j - 2)x^{i-j}y^{j-3} \tag{12}$$

Then

$$\partial x(\Delta u) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}[\,i-j(i-j-1)x^{i-j-2j-1}y^{j-2} + (j-1)(j-2)x^{i-j}y^{j-3} \tag{13}$$

$$\partial y(\Delta u) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}[(\,i-j)(i-j-1)x^{i-j-2j-2}y^{j-3} + (j-1)(j - 2)x^{i-j}y^{j-4}$$

So

$$\partial_n \Delta u = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}\,\eta(\theta)\left[\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right]\left[(i-1)(i-j-1)(i-j-2)x^{i-j-3}y^{j-1} + (j-1)(j-2)(i-j)x^{i-j}\ y^{j-3}\right] + \eta(\theta)\left[\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right]\left[(i-1)(i-j-1)(j-1)x^{i-j-2}y^{j-2} + (j-1)(j-2)(j-3)x^{i-j}\ y^{j-4}\right]$$

And

$$\Delta^2(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(\,i-j)(i-j-1)(i-j-2)(i-j-3)x^{i-j-4}y^{j-1} + 2(i-j)(i-j-1)(j-1)(j-2)x^{i-j-2}y^{j-3} + (j-1)(j-2)(j-3)(j-4)x^{i-j}y^{j-5} \quad (16)$$

It is necessary to determine the coefficients cij and their total number is n=$\frac{m(m+1)}{2}$.

To be able to present our problem as a linear system, the matrix c_{ij} with C_k with $k=\frac{i(i+1)}{2}+j$, so the unknowns function can be expressed as an inner product of a row of variables, , say Ψ, with a column of coefficient vector

$$u = \Psi^T.C \quad (17)$$

where

$$\Psi = [1, x, y, x^2, xy,\ y^2,\ x^3, x^2y,\ xy^2,\ \ y^3\ \ldots],\ \ C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ . \\ . \\ c_n \end{bmatrix}$$

Similarly, the normal derivative $\partial_n u$ can be represented as a scalar product of a row of variables, say Φ , with a column of coefficient vector c ,i.e. $u = \Phi^T.C$ where the component of e are given by:

$$\Phi_l = \eta(\theta)\left[(i-j)x^{i-j-1}y^{j-1}\left(cos\,(\theta)\ -\frac{\rho'}{\rho^2}sin(\theta)\right) + (j - 1)x^{i-j}y^{j-2}\left(sin\,(\theta)\ -\frac{\rho'}{\rho^2}cos(\theta)\right)\right] \tag{18}$$

Another way to express the Laplacian is as the inner product of a row of variables, such as d. with a column of coefficient vector c, i.e. $\Delta u = \Theta^T$.c where the component of d are given by

$$\Theta_k = (i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} \tag{19}$$

and the normal derivative of the Laplacian can be expressed as an inner product of a row of variables, say ẹ , with a column of coefficient vector c ,i.e.$\partial_n\ \Delta u = \Xi^T.c$ where the component of ẹ are given by

$$\Xi_k = \eta(\theta)\left[\cos(\theta) - \frac{\bar{p}}{p^2}\sin(\theta)\right]\left[(i-j)(i-j-1)(i-j-2)x^{i-j-3}\,y^{j-1} + (j-1)(j-2)(i-j)x^{i-j-1}y^{j-3}\ \right] + \eta(\theta)\left[\cos(\theta) - \frac{\bar{p}}{p^2}\sin(\theta)\right](i-j)(i-j-1)(j-1)x^{i-j-2}\,y^{j-2} + (j-1)(j-2)(j-3)x^{i-j}y^{j-4}\Big]$$

Finally, a row of variables can be used to express the bi-Laplacian as an inner product. Say Υ,with a column of vector coefficients c, i.e. $\Delta^2 u = \Upsilon^T.c$ where the component of ẹ are given by

$$\Upsilon_k = (j-1)(i-j-1)(i-j-2)(i-j-3)\, x\hat{}(i-j-4)\, y\hat{}(i-1) + 2(i-j)(i-j-1)(j-1)(j-2)\, x\hat{}(i-j-2)\, y\hat{}(j-3) + (j-1)(j-2)(j-3)(j-4) x\hat{}(i-j)\, y\hat{}(j-5)\,) \tag{21}$$

We are now prepared to build the linear system.

$$AC = B \tag{22}$$

is built so that A and B are matrices with five blocks:

1. To create the first one, use the formula in (17) for a given function $u_0$and a few chosen spots on the surface to meet the first boundary condition stated in (2). utilizing the equation in (17) for a particular function and for a few chosen locations .
2. In order to meet the second boundary condition in (3) for any given functionh_0 and for some chosen locations onΓ_1, the second block must use the formula in (18).
3. By meeting the third boundary condition in (4) using the formula in (19) for some given functionw_0 and for some chosen locations onΓ_1, the third block is completed.
4. The formula in (20) is used to satisfy the fourth boundary condition in (5) for some given function$\acute{w}_0$ and for a few chosen places onΓ_1 the fourth block.
5. By solving the bi-Laplacian differential equation in (1) using the formula in (21) for a given function F and for a few chosen locations in the domain, the fifth block is reached Ω:

For this we select points on the boundaryΓ_1,say(xi, yi),i=2,….n_1 to satisfy the condition(2-5) and we select n_2 point in the domain Ω , say (xj,yj), j=1,….n_2 to satisfy the equation(1).So the vector b is of order $4n_1 + n_2$ and A is $(4n_1 + n_2)$ x n matrix.

$$A = \begin{bmatrix} \acute{\Psi}_1 \\ \vdots \\ \acute{\Psi}_{n_1} \\ \acute{\Phi}_1 \\ \vdots \\ \acute{\Phi}_{n_1} \\ \acute{\Theta}_1 \\ \vdots \\ \acute{\Theta}_{n_1} \\ \acute{\Xi}_1 \\ \vdots \\ \acute{\Xi}_{n_1} \\ \acute{\Upsilon}_1 \\ \vdots \\ \acute{\Upsilon}_{n_2} \end{bmatrix} \qquad b = \begin{bmatrix} u_0\,(\theta_1\,) \\ \vdots \\ u_0(\theta_1) \\ h_0(\theta_1) \\ \vdots \\ w_0(\theta_1) \\ \vdots \\ w_0(\theta_{n_1}) \\ \acute{w}_0(\theta_1) \\ \vdots \\ \acute{w}_0(\theta_{n_1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In order to solve the inverse problem related to the boundary differential bi-Laplacian differential equation, we need to solve the linear system given in (23).

Solving the linear system by using CGLS and PCG algorithms

An important consideration when beginning and ending a numerical method is the initial guess which is assumed here to be the zero vector and the terminating criteria for these algorithms is given by :

$$absolute\ error < Tol \qquad (24)$$

$$relative\ error < Tol \qquad (25)$$

where Tol is the tolerance, which is given as small as possible.

Numerical results and discussion

We take a look at a few cases with exact answers that are either polynomial or non-polynomial to demonstrate the effectiveness of the suggested strategy. Calculations are made using the precise solution provided:

- the position it plays in the domain Ω
- The exact solution's trace is equal to $u_0$on Γ_1
- the standard Derivative The precise response is equal to h_0 onΓ_1
- The exact solution's Laplacian is equal to w_0 on Γ_1
- The Laplacian normal derivative is equal to $\acute{w}_0$ on Γ_1

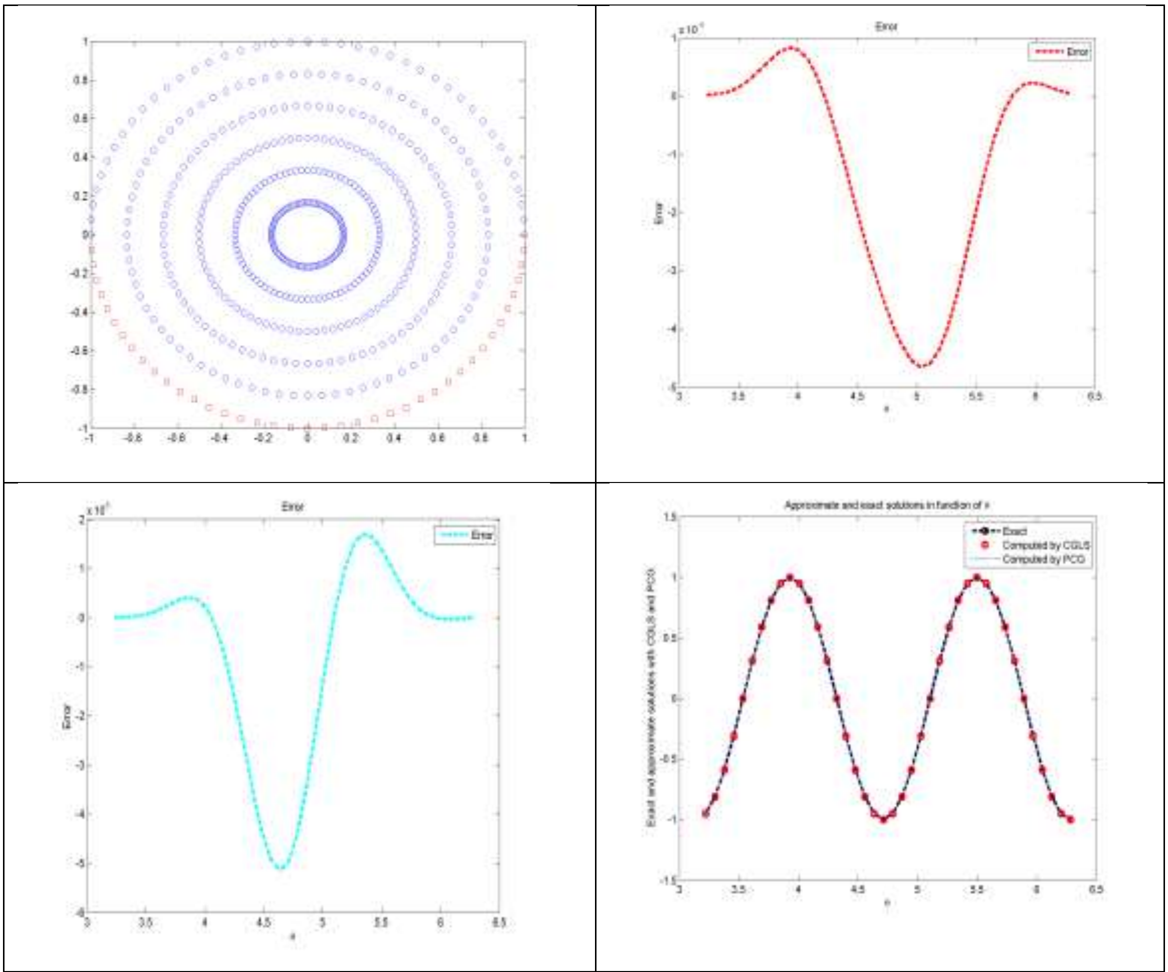
We use these data along with the zero initial guess and the CGLS and PCG: Similar techniques are employed together with certain appropriate tolerance and stopping standards.

Example (1): Suppose that the exact solution is $u_{exact}(x,y) = 6x^2y^2 - x^4 - y^4$ the domain is bounded by $\rho(\theta)$ =1 and is defined taking $\boldsymbol{\beta}$ = 1 the number of boundary collection used for discretizing the boundary is taken to be n1=80 and the number of internal collection is n2=400 .

Case 1: n1=80 and n2=400 with Tol= 10^{-15}

M	NO.OF ITERATION FOR CGLS	RELATIVE ERROR WITH CGLS	NO.OF ITERATION FOR PCG	RELATIVE ERROR WITH PCG
2	3	1.23045359E+00	3	1.23045359E+00
3	262	2.26243810E+15	7	1.58600200E+00
4	17	7.01342488E+00	15	7.01342488E+00
5	43	1.73958897E-14	46	3.30210077E-12
6	77	1.30191306E-13	112	3.74223534E-10
7	197	1.56788609E-12	313	1.59021906E-08
8	495	2.49907871E-11	838	9.57966992E-07
9	1322	2.43232513E-11	3031	3.69698855E-06
10	3066	3.09788058E-11	9621	2.86034216E-05

For m=5 the number of iteration is 43 and the relative error is equal 1.73958897E-14 for CGLS and 3.30210077E-10 for PCG that is a good approximation.



Approximate solution by CGLS and exact solution $u_{exact}(x,y) = 6x^2y^2 - x^4 - y^4$(in the right),error(in the left),for n1=80 and n2=400 with Tol= 10^{-15}

Case 2 : with n1=80 and n2=400, Tol=10^{-15}

M	No. of Iteration CGLS	Relative Error with CGLS	No.of Iteration for CGLS	Relative Error with CGLS
2	3	2.16871820E+00	3	2.16871820E+00
3	10	1.20872984E+00	8	1.20872984E+00
4	21	4.56326103E+00	19	4.56326103E+00
5	46	8.00066549E-14	63	7.81626640E-10
6	100	8.97886085E-14	162	4.23873567E-09
7	271	9.76607129E-13	407	2.04647232E-08
8	704	7.54719866E-12	1442	1.59651468E-06
9	1890	2.93418093E-11	5280	2.29215665E-05
10	4655	1.30082285E-10	22451	5.73572629E-05

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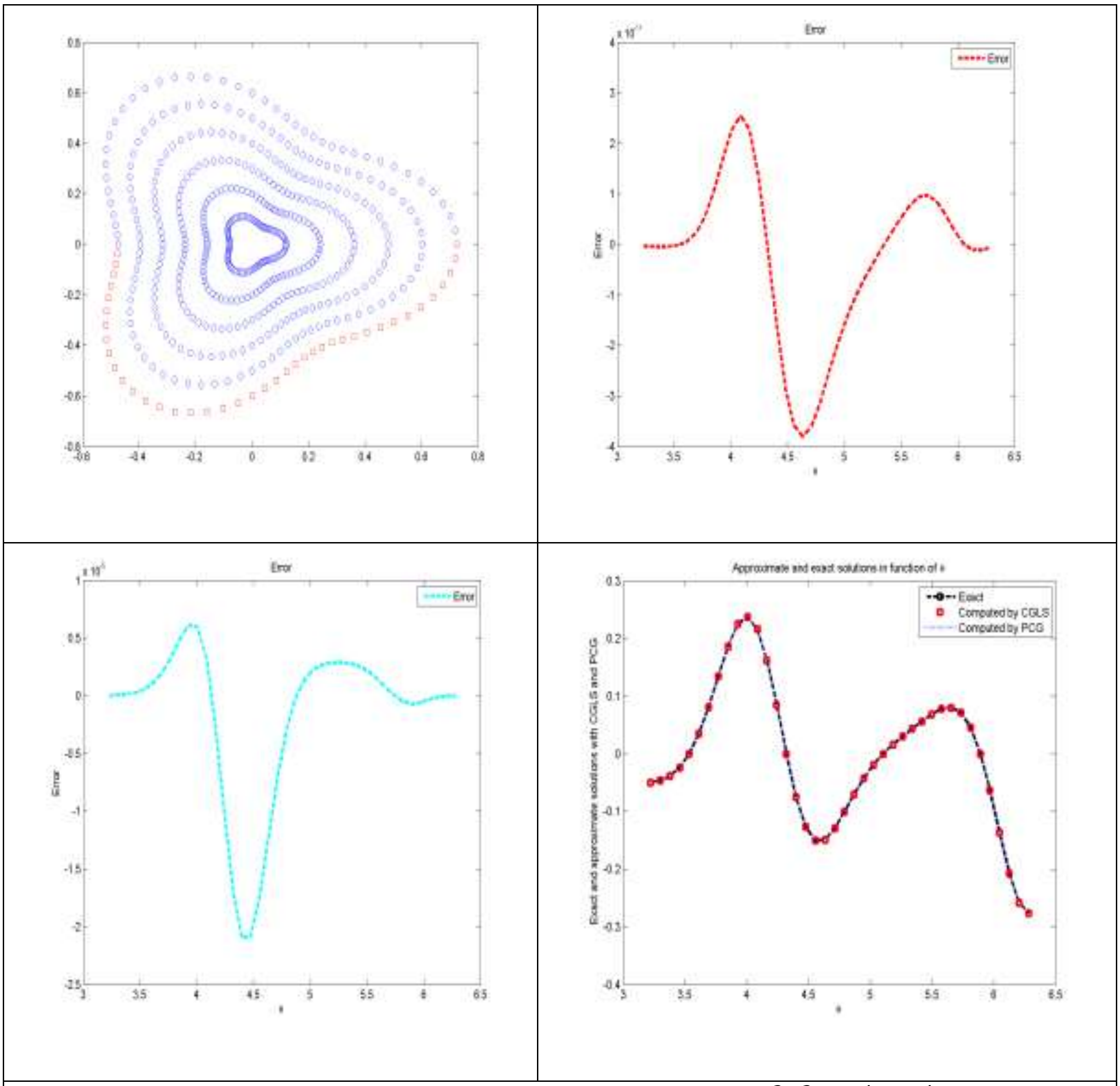
For m=5 the number of iteration =46 and the relative error is equal 8.00066549e-14 for CGLC and 7.81626640e-10 for CPC that is a good approximation.



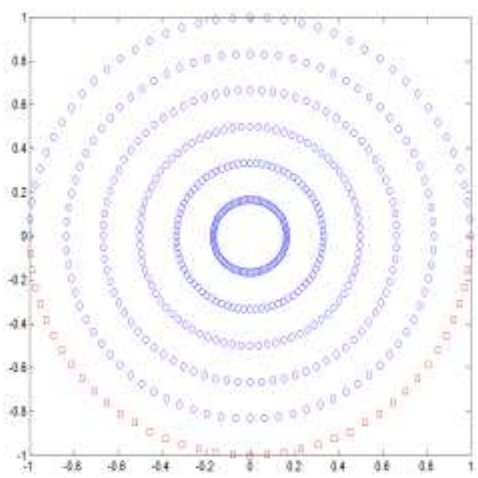
Approximate solution by CGLS and exact solution $u_{exact}(x,y) = 6x^2y^2 - x^4 - y^4$(in the right),error(in the left),for n1=80 and n2=400 with Tol= 10^{-15}

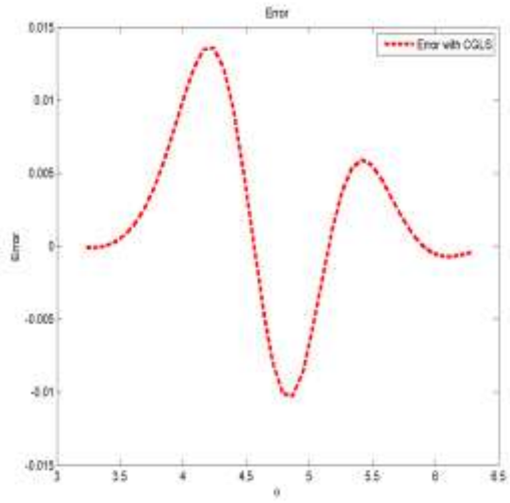
Example (2): Suppose that the exact solution is $u(x,y) = \exp(-x^\wedge 2)$the domain is bounded by $\rho(\theta)$ =0.5 and is defined taking $\boldsymbol{\beta}$ = 0.5 the number of boundary collection used for discretizing the boundary is taken to be n1=80 and nr=5 and the number of internal collection n2=400.

Case 1: n1=80 and n2=400 with Tol= 10^{-15}

M	No.of Iteration for CGLS	Relative Error with CGLS	No.of Iteration for CGLS	Relative Error with CGLS
2	3	5.93142786E-01	3	5.93142786E-01
3	8	9.00005112E-01	7	9.00005112E-01
4	16	3.37126055E-01	13	3.37126055E-01
5	35	1.30369360E+00	30	1.30369360E+00
6	98	9.78882346E-01	58	9.78882346E-01
7	242	2.80968384E-01	155	2.80968384E-01
8	792	1.90740910E-01	331	1.90740910E-01
9	1138	3.97901471E-02	874	3.97901480E-02
10	3474	6.04848209E-02	2097	6.04848249E-02

Similarly to the previous case, for m=9 the number of iteration =1138 and the relative error is equal 3.97901471e-2 for CGLC and 3,97901480e-2 for CPC that is a good approximation.





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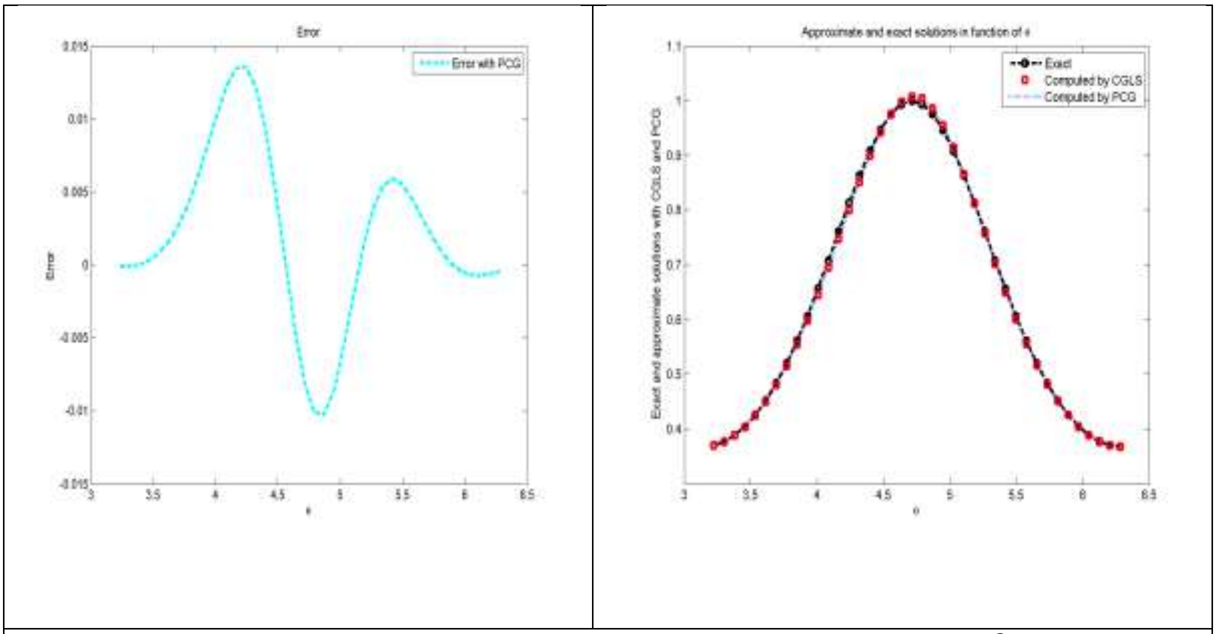
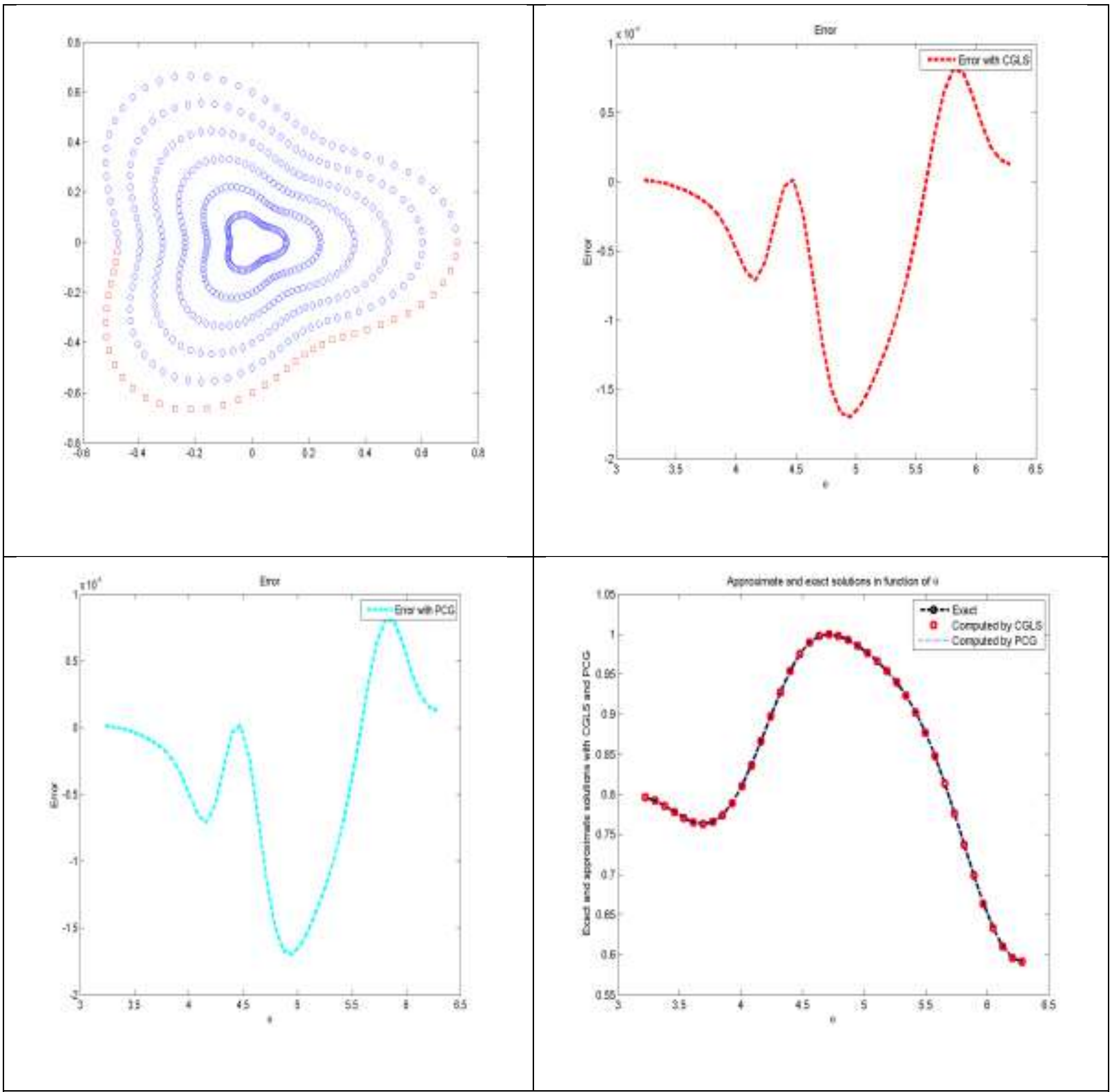


Approximate solution by CGLS and exact solution $u_{exact}(x,y) = \exp(-x^2)$ (in the right),error(in the left),for n1=80 and n2=400 with Tol= 10^{-15}

Case 2 : with n1=80 and n2=400, Tol=10^{-15}

M	No.of Iteration for CGLS	Relative Error with CGLS	No.of Iteration for CGLS	Relative Error with CGLS
2	3	2.94835693E-01	3	2.94835693E-01
3	8	1.36619542E-01	7	1.36619542E-01
4	22	1.59289805E-01	16	1.59289805E-01
5	41	1.32677152E-01	33	1.32677152E-01
6	89	2.72626648E-01	68	2.72626648E-01
7	197	2.19980949E-02	160	2.19980949E-02
8	597	1.15933050E-02	377	1.15933051E-02
9	991	1.13179460E-03	990	1.13179533E-03
10	2699	1.44363063E-03	2726	1.44362805E-03

For m=9 the number of iteration is 991 and the relative error is equal 1,13179460E-03 for CGLS and 1.13179533E-03 for PCG that is a good approximation.



Approximate solution by CGLS and exact solution $u_{exact}(x,y) = \exp(-x^2)$ (in the right),error(in the left),for n1=80 and n2=400 with Tol= 10^{-15}

Conclusion

On an annular domain, we resolve the inverse Cauchy problem of the bi-Laplacian differential equation. Unknown data are recovered for a portion of the boundary benefiting from the extra data on th other part of the boundary. A direct problem is obtained by replacing a polynomial expansion of the solution in the inverse Cauchy problem. To demonstrate that the inverse Cauchy problem is extremely ill-posed, many types of numerical examples with exact polynomial and non-polynomial solutions are provided. Applying various noise values to the procedure allows for the stability of the method to be evaluated Cauchy data.

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