A numerical Treatment for solving quadratic Riccati Differential Equations

Doctor Teacher Wafaa Abd Ibrahim
waffam@uodiyala.edu.iq
Teacher Ohood Fadil Alwan
Ohoodm@uodiyala.edu.iq
Department of Mathematics, Al Muqdad College of Education, University of Diyala, Baqubah, Iraq

Abstract

In this paper, a numerical technique is introduced to find a numerical solution(NS) of Quadratic Riccati Differential equations(QRDEs) by using Taylor serie's method(TSM).Different kinds of linear and non linear differential equations have been solved utilizing the proposed method with the initial conditions, at any point t_k this technique transformed the QRDEs in to a series. Then it solved by using soft ware-MATLAB.Two different examples have introduced to shown the efficiency and accuracy of the method. The QRDE$_s$ existence and uniqueness are studied.The stability and the upper bound of error are proven.The solutions of the proposed method also compared with those obtained by the Bezier curves method(BCM) and the classical Runge Kutta method of order four (RK4. The results are given by tables and figures of the proposed method outperformed the other methods in terms of accuracy and convergence.

Keywords: Tayler series, Quadratic Riccati Differential equation, Exact solution ,Numerical solution, absolute error.

المعالجة العددية لحل معادلات ريكاتي التفاضلية التربيعية

وفاء عبد ابراهيم سليمان
عهود فاضل علوان
قسم الرياضيات/ كلية التربية المقداد/ جامعة ديالى

الخلاصة
في هذا البحث تم تقديم تقنية عددية لايجاد الحل العددي لمعادلات ريكاتي التفاضلية التربيعية باستخدام طريقة متسلسلة تايلر. انواع مختلفة من المعادلات التفاضلية الخطية وغير الخطية تم حلها باستخدام هذه الطريقة المقترحة مع الشروط الابتدائية عند اي نقطة هذه الطريقة تحول معادلات ريكاتي التفاضلية التربيعية الى متسلسلة ثم تم حلها بأستخدام برنامج الماتلاب تم تقديم مثالين مختلفين لاظهار كفاءة ودقة الطريقة .تمت دراسة الوجود ووحدانية الحل لمعادلات ريكاتي التفاضلية التربيعية. تم اثبات الاستقرارية والحد الاعلى للخطأ. تمت مقارنة حلول الطريقة المقترحة مع تلك التي تم الحصول عليها بواسطة طريقة رنك- كتا من الدرجة الرابعة وطريقة منحنيات بيزير النتائج اعطيت من خلال الجداول والاشكال حيث تفوقت الطريقة المقترحة على الطرق الاخرى من حيث الدقة والتقارب .

1.Introduction

Riccati differential equations(RDEs) are a class of nonlinear differential equations of much importance, and play a significant role in many fields of applied science[1].The RDEs is named after the Italian nobleman Count Jacopo Francesco Riccati(1676-1754)[2].Usually these types of equations are needed to be solved numerically; of course there are many different ways to solve them. Such as; in 2020 the legendre collocation method(LCM)[5],the least-squares approximation and the operational matrices[6].Later in 2021 the variational Iteration method[7],the Fourier Transform and A domian methods[8],the Homotopy perturbation method[9].Recently in 2023 the Cubic B-Spline[10] and in 2024 the fourth-order-Runge-Kutta methods(RK4)[11].All the above methods motivated us to think about the study of the NS by using the proposed method TSM.

This paper begins with the description of the QRDEs, the NS is obtained by using TSM, at any point $t_k$this technique transformed the QRDEs in to a series. Then it solved by using soft ware-MATLAB. The existence and uniqueness of the QRDE are studied. The upper bound of error and the stability are studied. Finally, two examples are given to solve different problems using the TSM, the results were given by tables and figures which show the efficiency and accuracy of the proposed method.

Description of the problem

Consider the nonlinear RDE as the following form

$$f(t,w) = \frac{dw}{dt} = \alpha_1(t) + \alpha_2(t)w(t) + \alpha_3(t)w^2(t), t_0 \le t \le t_f \qquad (1)$$

With the initial condition

$$w(t_0) = \alpha_0 \qquad (2)$$

Where $\alpha_1(t), \alpha_2(t)$ and $\alpha_3(t)$ are continuous with $\alpha_3(t) \neq 0, t_0, t_f$ and α_0 are arbitrary constants, and $w(t)$ is unknown function.

The RDE$_s$ are numerically solved using Taylor series

Consider the first order differentia1 equation(1).Differentiating(1),to obtain

$$\frac{d^2w}{dt^2} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial w}\frac{dw}{dt},$$

Which means

$$w'' = f_t + f_w f \qquad (3)$$

$$w''' = \frac{\partial^2 f}{d^2 t} + \frac{2\partial^2 f}{\partial t \partial w}w' + (\frac{\partial f}{\partial w}w')^2 \qquad (4)$$

By differentiating this in steps, to get w^{iv} etc. let $t = t_0$ and $w = 0$, the values of w_0', w_0'', w_0''' can be obtained .Hence The TS

$$w_{TS}(t) = w_0 + (t - t_0)w_0' + \frac{(t-t_0)^2}{2}w_0'' + \frac{(t-t_0)^3}{6}w_0''' + \cdots + \qquad (5)$$

Gives the values of w for every value of t for which (5) converges.

4.The Algorithm for solving(QRDE)

Input :a, b, n ,initial condition, ES.

Output: The numerical solution of the QRDE.

Step1: Subsitute t_0 in $w_{TS}(t)$

Step2: Calculate $w_0, w_0^{'}, w_0^{''}, w_0^{'''}, \dots., w_0^{(n)}$.

Step3: Subsitute Step2 in $w_{TS}(t)$.

Step4: Calculate $t_k = a + k\frac{(b-a)}{n}$ $(k = 0, \dots, n)$.

Step5:Subsitute t_k in $w_{TS}(t)$.

Step6: Calculate the NS of The QRDE by using TSM.

Step7: Calculate absolute error is the comparison between the Exact and the numerical solutions.

Step8:End.

5.Theorem(existence and uniqueness of the QRDE)[14]

Suppose f(t ,w) be defined and continuous for all points (t ,w) in the region D given by $a \le t \le b, -\infty < w < \infty$, where a and b finite, and ∃ a constant σ such that(s.t) ,∀ t, w, w^* s.t (t, w) and (t, w^*) are both in D:

$$\left|f(t,w) - f(t,w^*)\right| \le \sigma\left|w - w^*\right| \quad (6)$$

Then, ∃ a unique solution w(t) of the problem (1), if ξ is any given number,where w(t) is continuous and differentiable for all (t,w) in D.The requirement(6) is called as a "lipschitz condition" and the constant σ as a "lipschitz constant".

6.Convergence of the QRDE

Consider Eq.(1) at the discretized point as :

$$f(t_k, w_k) = \alpha_1(t_k) + \alpha_2(t_k)w(t_k) + \alpha_3(t_k)w^2(t_k), \quad t_0 \le t \le t_f \quad (7)$$

Also, consider the linear first order experiment differential equation

$w^{'} = \beta w, w(t_0) = w_0$

Where β is a constant , and has its solution in the form of

$w(t) = w(t_0)e^{(\beta(t-t_0))}$ which at $t = t_0 + kh$

The solution becomes
$w(t_k) = w(t_0)e^{\beta kh} =$
$w_0(e^{\beta h})^k$ (8)
Let the non-linear QRDE of the form given in Eq.(1) and Eq.(7) written as
$w^{'} = f(t, w); w(t_0) = w_0 = \alpha_0$
(9)
The non- linear function Eq.(9) can linearized by expanding the function in TS about the point(t_0, w_0) and truncating it after the first term:

$$w^{'} = f(t_0, w_0) + (t - t_0)\frac{\partial f}{\partial t}(t_0, w_0) + (w - w_0)\frac{\partial f}{\partial w}(t_0, w_0) \quad (10)$$

Using Eq.(7) and by the chain rule differentiation, it yield
$w^{'} = \alpha_{10} + \alpha_{20}w_0 + \alpha_{20}{w_0}^2 + (t - t_0)\big[\alpha_{10}^{'} + \alpha_{20}^{'}w_0 + \alpha_{20}w_0^{'} + \alpha_{30}^{'}w_0^2 + 2\alpha_{30}w_0w_0^{'} + \alpha_{20}^{'}w_0 + \alpha_{30}w_0^{'} + \alpha_{30}^{'}w_0^2 + 2\alpha_{30}w_0w_0^{'}\big] + w\big[\alpha_{20} + 2\alpha_{30}w_0w_0^{'}\big] - \alpha_{20}w_0 - 2\alpha_{30}w_0w_0^{'}$
For simplicity let $\alpha_1(t_0) = \alpha_{10}, \alpha_2(t_0) = \alpha_{20}, \alpha_3(t_0) = \alpha_{30}, w(t_0) = w_0, w^{'}(t_0) = w_0^{'}$
And applying Eq.(2) $w(t_0) = w_0 = \alpha_0 =$ constant; So, $w_0^{'} = 0$
$\rightarrow w^{'} = \alpha_{10}w + \alpha_{20} + \alpha_{30}w_0^2 + (t - t_0)\big[\alpha_{10}^{'} + \alpha_{20}^{'}w_0 + \alpha_{30}^{'}w_0^2\big]$ (11)
This can be rewritten as:
$w^{'} = \beta w + \Theta$
Where $\beta = \alpha_{10}, \Theta = \alpha_{20} + \alpha_{30}w_0^2 + (t - t_0)\big[\alpha_{10}^{'} + \alpha_{20}^{'}w_0 + \alpha_{30}^{'}w_0^2\big]$ (12)
Dividing both sides of Eq.(12) by β, to obtain $\frac{w^{'}}{\beta} = w + \frac{\Theta}{\beta}$, if $y = w + \frac{\Theta}{\beta}$, then to get

$$w = y - \frac{\Theta}{\beta} \quad (13)$$

Sub. Eq.(13) in to Eq.(12) gives:

y'
$= \beta y$
Which is called the linear experiment equation for the non-linear Eq.(1)
The solution of this experiment equation, Eq.(15),is:
$y =$
$\gamma e^{\beta t}$ (
Now, by considering Eq.(5),to have

$$w_{k+1} = w_k + \beta h w_k + \frac{(\beta h)^2}{2} w_k + \frac{(\beta h)^3}{6} w_k + \frac{(\beta h)^4}{24} w_k$$

$$\Rightarrow w_{k+1} = (1 + \beta h + \frac{(\beta h)^2}{2} + \frac{(\beta h)^3}{6} + \frac{(\beta h)^4}{24}) w_k$$

$$\Rightarrow w_{k+1} = E(\beta h) w_k$$
(16)

Where $E(\beta h) = 1 + \beta h + \frac{(\beta h)^2}{2} + \frac{(\beta h)^3}{6} + \frac{(\beta h)^4}{24}$ is TS numerical to $e^{\beta h}$.

If $\beta < 0$, from Eq.(8),the ES $w(t_k)$ decreases at t_k increases. If $\beta > 0$, from Eq.(8),the ES $w(t_k)$ increases with t_k.from Eq.(16) we find the TSM absolutely stable if $\left|E(\beta h)\right| \leq 1$ and relatively stable if $\left|E(\beta h)\right| \leq e^{\beta h}$.

7.Upper bound of error

Here, we give error bound of NS $w_{TS}(t)$

Theorem(4.1): let $w(t)$ and $w_{TS}(t)$ represents NS and the ES of the problem(1),respectively. if $w(t) \in C^r[0,1], r = 0,1, \dots, r + 1$ then

$$\left|w(t) - w_{TS}(t)\right| \leq \left|R_n^T(t, t_0)\right| + \left|w_{TS}^T(t) - w(t)\right|$$

When $w_{TS}^T(t)$ exhibits the nth degree TS of w a round the point $t = t_0$ and $R_n^T(t, t_0)$ explain its reminder term.

Proof

Since w is n+1 times derivative, using the TS as a representation, to obtain

$$w(t) = \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} w^k(t_0) + R_n^T(t, t_0),$$

Where

$$R_n^T(t, t_0) = \frac{(t-t_0)^{n+1}}{(n+1)!} w^{(n+1)}(\delta_t), 0 \leq \delta_t \leq t \leq 1$$

Is the reminder term of the TS of w. let us denotes the nth degree TS of w on the point $t = t_0$ by $w_{TS}^T(t)$. Thus , $w(t) - w_{TS}^T(t) = R_n^T(t, t_0)$. By using it and "triangle inequality" we get

$$\left|w(t) - w_{TS}(t)\right| = \left|w(t) - w_{TS}(t) + w_{TS}^T(t) - w_{TS}^T(t)\right|$$
$$\leq \left|w(t) - w_{TS}^T(t)\right| + \left|w_{TS}^T(t) - w_{TS}(t)\right|$$
$$= \left|R_n^T(t, t_0)\right| + \left|w_{TS}^T(t) - w_{TS}(t)\right|$$

Then, an "UBOE''of this method is found for the NS.

8. Illustrative Examples

Example(1):solve the QRDE[12]

$w'(t) = 8tw(t) + w^2(t) + 16t^2 - 5, (0 \leq t \leq 1), the\ initial\ condition\ is\ w(0) = 1,$ where the ES is w(t)=1-4t.

From TSM

$w'(t) = 8tw(t) + w^2(t) + 16t^2 - 5 \Rightarrow w'(0) = -4$

$w''(t) = 8tw'(t) + 8w(t) + 2w(t)w'(t) + 32t \Rightarrow w''(0) = 0$

$w'''(t) = 8tw''(t) + 8w'(t) + 8w'(t) + 2w(t)w''(t) + 2w'(t)w'(t) + 32 \Rightarrow w'''(0) = 0$

$w''''(t) = 8tw'''(t) + 8w''(t) + 8w''(t) + 8w''(t) + 2w(t)w'''(t) + 2w'(t)w''(t) + 2w'(t)w''(t) + 2w'(t)w''(t) + 2w'(t)w''(t) \Rightarrow w''''(0) = 0$

TS is

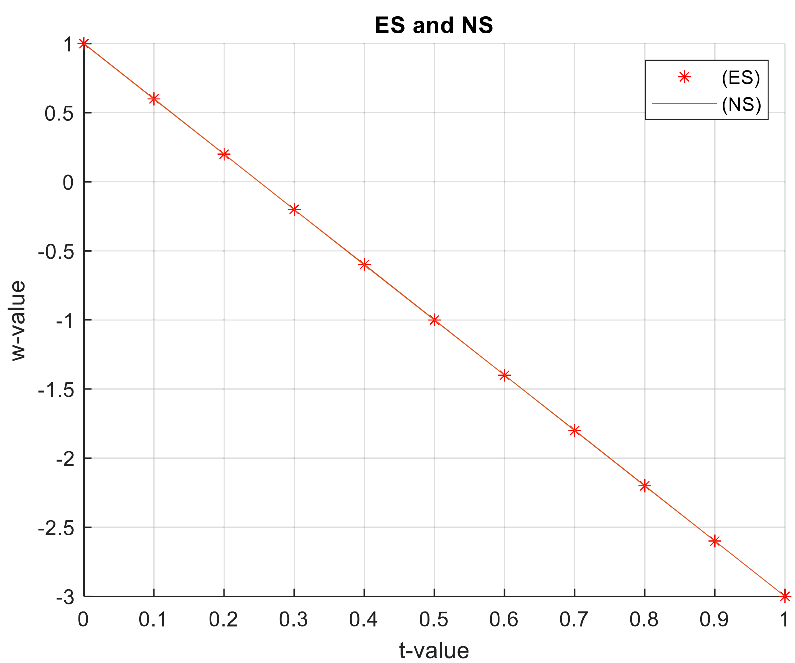
$w_{TS}(t) = w(0) + tw'(0) + \frac{t^2}{2}w''(0) + \frac{t^3}{6}w'''(0) + \frac{t^4}{24}w''''(0) + \dots$

$w_{TS}(t) = 1 - 4t$

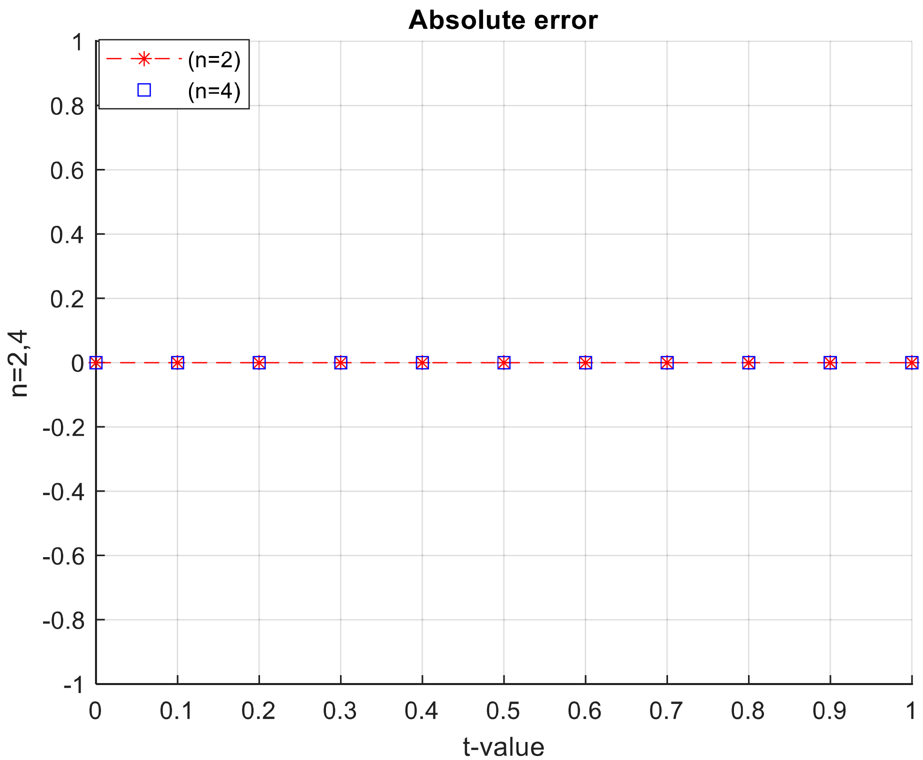
This problem is solved using the TSM for n=4,then the ES(w)and the NS(w_{TS}) at $t_k, k = 0,1,2, \dots, n$. The maximum absolute error in Table(1) is(0.0000)and are shown in figure(1).

Table(1): The ES, NS and absolute error

T	W(ES)	w_{TS}(NS)	$\lvert w - w_{TS} \rvert$
0	1.0000	1.0000	0.0000
0.1	0.6000	0.6000	0.0000
0.2	0.2000	0.2000	0.0000
0.3	-0.2000	-0.2000	0.0000
0.4	-0.6000	-0.6000	0.0000
0.5	-1.0000	-1.0000	0.0000
0.6	-1.4000	-1.4000	0.0000
0.7	-1.8000	-1.8000	0.0000
0.8	-2.2000	-2.2000	0.0000
0.9	-2.6000	-2.6000	0.0000
1	-3.000	-3.000	0.0000



Figure(1): The ES and NS when(n=4)



Figure(2) The absolute error of QRDE for n=2,4

Example(2): solve the QRDE[13]

$w'(t) =$
$-e^{2t} + w + w^2, (0 \leq t \leq 1), \textit{the initial condition is } w(0) =$
1, with the ES
$w(t) = e^t.$

From TSM

$w'(t) = -e^{2t} + w(t) + w^2(t) \Longrightarrow w'(0) = 1$

$w''(t) = -2e^{2t} + w'(t) + 2w(t)w'(t) \Longrightarrow w''(0) = 1$

$w'''(t) = -4e^{2t} + w''(t) + 2w(t)w''(t) + 2w'(t)w'(t) \Longrightarrow$
$w'''(0) = 1$

$w''''(t) = -8e^{2t} + w'''(t) + 2w(t)w'''(t) + 2w''(t)w'(t) +$
$2w'(t)w''(t) + 2w'(t)w''(t) \Longrightarrow w''''(0) = 1$

$w'''''(t) = -16e^{2t} + w''''(t) + 2w(t)w''''(t) + 2w'(t)w'''(t) +$
$2w''(t)w''(t) + 2w'(t)w''(t) + 2w'(t)w'''(t) +$
$2w''(t)w''(t) + 2w'(t)w'''(t) + 2w''(t)w''(t) \Longrightarrow w'''''(0) = 1$

$w''''''(0) = 1$

$w'''''''(0) = 1$

$w''''''''(0) = 1$

TS is

$$w_{TS}(t) = w(0) + tw^{'}(0) + \frac{t^2}{2}w^{''}(0) + \frac{t^3}{6}w^{'''}(0) + \frac{t^4}{24}w^{''''}(0) + \frac{t^5}{120}w^{'''''}(0) + \cdots$$

$$w_{TS}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \frac{t^8}{40320} + \cdots$$

This problem is solved using the TSM for n=2,then the ES and the NS at t_k,k=0,1,2,…,n. The maximum absolute error in table(2) is(0.2183)and are shown in figure(2).

Table(2):The ES, NS and absolute error

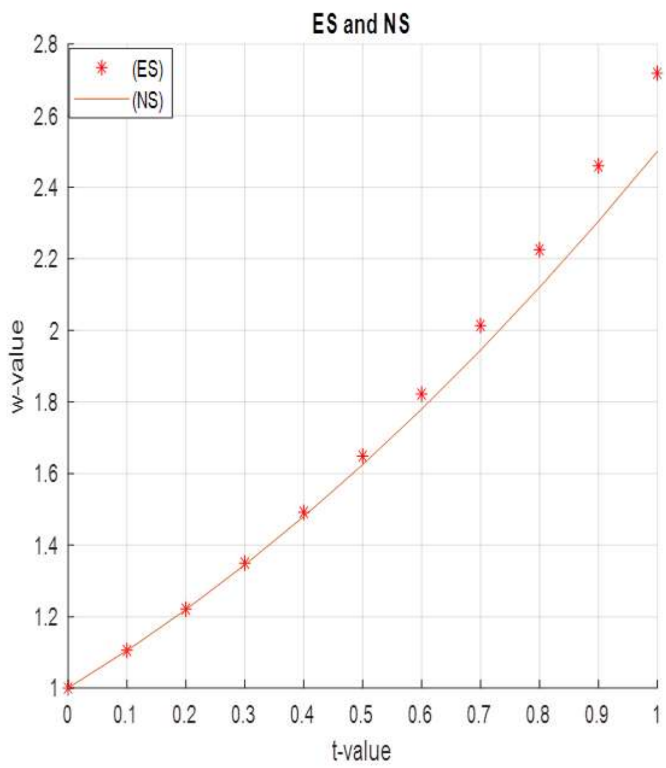
T	w	w_{TS}	$\lvert w - w_{TS} \rvert$ (n=2)
0	1.0000	1.0000	0.0000
0.1	1.1052	1.1050	0.0002
0.2	1.2214	1.2200	0.0014
0.3	1.3499	1.3450	0.0049
0.4	1.4918	1.4800	0.0118
0.5	1.6487	1.6250	0.0237
0.6	1.8221	1.7800	0.0421
0.7	2.0138	1.9450	0.0688
0.8	2.2255	2.1200	0.1055
0.9	2.4596	2.3050	0.1546
1	2.7183	2.5000	0.2183

Table(3):Comparison between the ES ,NS and the maximum absolute error is(0.0099)

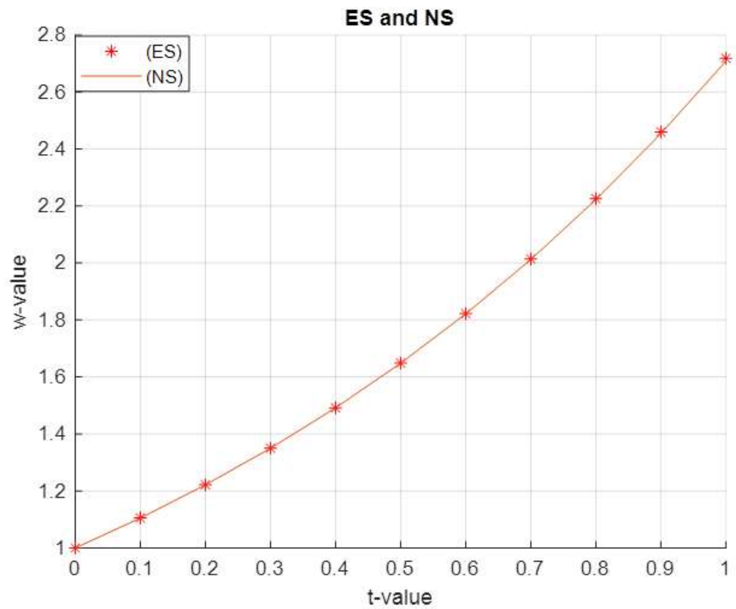
t	w	w_{TS}	$\lvert w - w_{TS} \rvert$ (n=4)
0	1.0000	1.0000	0.0000
0.1	1.1052	1.1052	0.0000
0.2	1.2214	1.2214	0.0000
0.3	1.3499	1.3498	0.0000
0.4	1.4918	1.4917	0.0001
0.5	1.6487	1.6484	0.0003
0.6	1.8221	1.8214	0.0007
0.7	2.0138	2.0122	0.0016
0.8	2.2255	2.2224	0.0031
0.9	2.4596	2.4538	0.0058
1	2.7183	2.7083	0.0099

able(4) Comparison between the ES ,NS and the maximum absolute error is(0.0000)

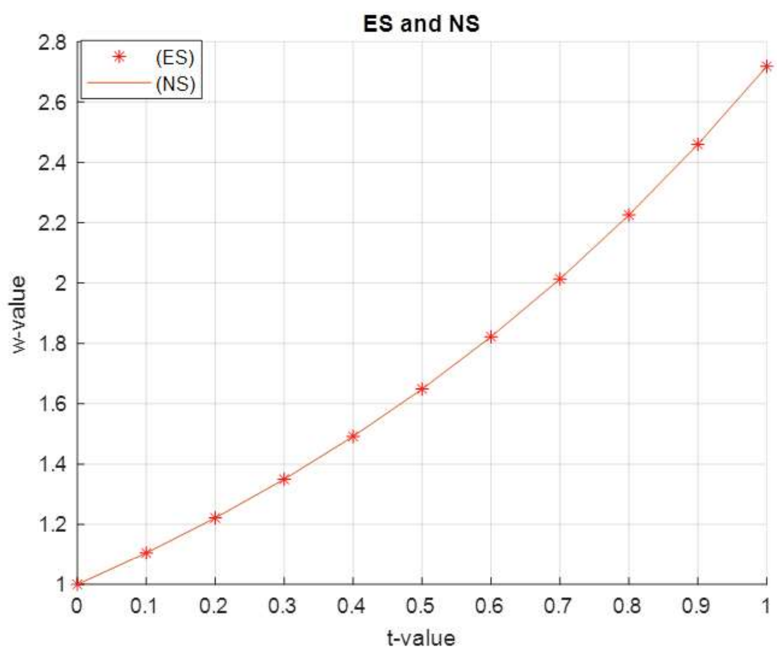
t	w	w_{TS}	$\left| w - w_{TS} \right|$ (n=8)
0	1.0000	1.0000	0.0000
0.1	1.1052	1.1052	0.0000
0.2	1.2214	1.2214	0.0000
0.3	1.3499	1.3499	0.0000
0.4	1.4918	1.4918	0.0000
0.5	1.6487	1.6487	0.0000
0.6	1.8221	1.8221	0.0000
0.7	2.0138	2.0138	0.0000
0.8	2.2255	2.2255	0.0000
0.9	2.4596	2.4596	0.0000
1	2.7183	2.7183	0.0000



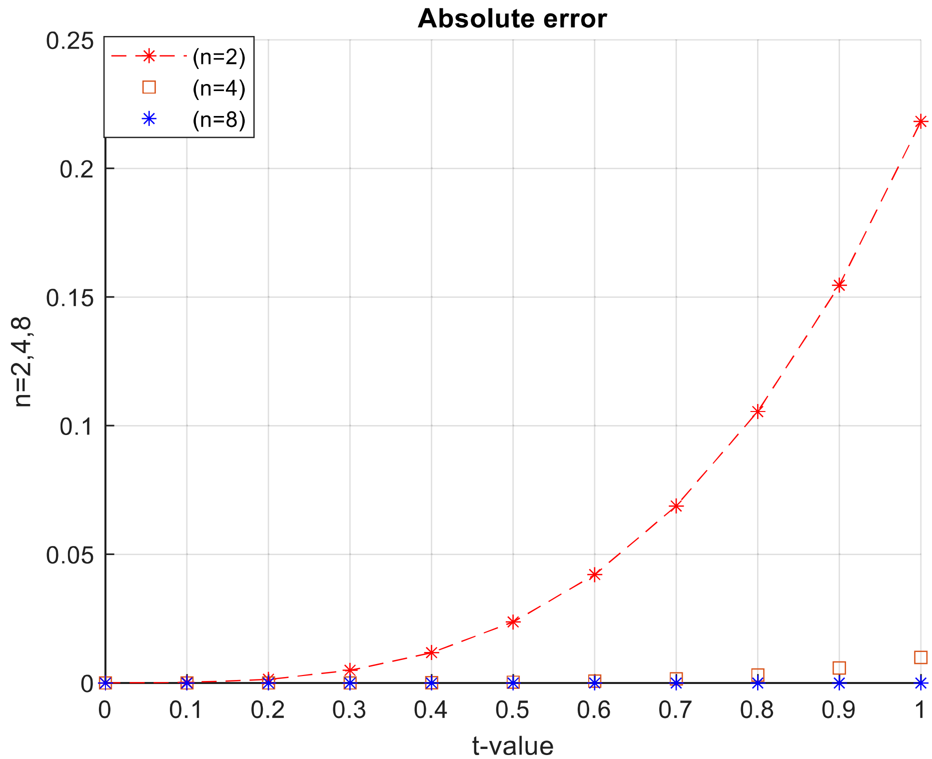
Figure(3):The ES and NS when(n=2)



Figure(4):The ES and NS when(n=4)



Figure(5):The ES and NS when(n=8)



Figure(6) The absolute error of QRDE for n=2,4,8

9.Conclusion

In this search,the TSM is utilized for solving the QRDEs numerically.The stability and the upper bound of the error for this method are studied.Two numerical examples have been applied To prove the efficiency and accuracy of the proposed method .The absolute error comparisons for those of the BCM in [3] and the RK4 in [4] seem to show that the proposed method is better than some of the current methods in terms of error.This method can be extended and applied to solve the system of QRDEs.

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