Intervals
An interval is asset of no.^{s} x having one of the following forms?
1. Open interval: $a<x<b \equiv (a, b)$
2. Close interval: $a \le x \le b \equiv [a,b]$
3. Half open from the left or half close from the right: $a < x \le b \equiv (a,b]$
4. Half close from the left or half open from the right: $a \le x < b \equiv [a,b)$

Notes:
1. $a<x<\infty \equiv a<x \equiv (a,\infty)$
2. $a \le x < \infty \equiv a \le x \equiv [a,\infty]$
3. $-\infty<x<a \equiv x<a \equiv (-\infty,a)$
4. $-\infty < x \le a \equiv x \le a$
5. $-\infty<x<\infty \equiv (-\infty,\infty)$

Defn.:-(function) A function f from a set D to set R "written $f \cdot D \to R$ " is a rule which assigns a single element $y\in k$ to each element $x\in D$
Note:
The element $y\in R$ denoted f(x).
The set D is called the domain of f.
The set R is called the range of f.
Ex(3):-Given the function $f(x)=x^2-2x+3$. find f(-1), f(0), f(2), f(x+1), f(f(x)), f(f(1))
Solu. $f(x)=x^2-2x+3$
$f(-1)=(-1)^2-2*-1+3=1+2+3=6$
$f(0)=0-0+3=3$
$f(2)=4-4+3=3$
$f(x+1)=(x+1)^2-2(x+1)+3=x^2+2x+1-2x-2+3=x^2+2$
$f(f(x))=(f(x))^2-2(f(x))+3=(x^2-2x+3)^2-2(x^2-2x+3)+3$
$=x^4+4x^2+9-4x^3+6x^2-12x-2x^2+4x-6+3=x^4-4x^3+8x^2+8x+6$
$f(f(1))=1^4-4(1)^3+8(1)^2+8*1+8=1-4+8-8+6=3$

Note: The domain D is the set of all values of x for which y is define. The range R is the set of all values of y for which x is define.

Ex(4):-Find the domain and range of the following function.
1. $y=f(x)=x^2$, D :all x or D : $-\infty<x<\infty$
$x=\pm\sqrt{y}$, R: $y \ge 0$
2. $y=\dfrac{x-1}{x-6}$, D: $x \ne 6$
$yx-6y=x-1$
$yx-x=6y-1$

$x = \frac{6y-1}{y-1}$, R: $y \neq 1$

3. $y = \sqrt{2-\sqrt{x}}$

$2-\sqrt{x} \geq 0 \Rightarrow 2 \geq \sqrt{x} \Rightarrow 4 \geq x$

D: $0 \leq x \leq 4$

$y^2 = 2-\sqrt{x} \Rightarrow \sqrt{x} = 2-y^2 \Rightarrow x = (2-y^2)^2$

R: all y

1.3 Graph of function

Defn.:- The solution set or locus of an eq. In two unknown consists of all point in plane whose coordinates satisfy the eq.

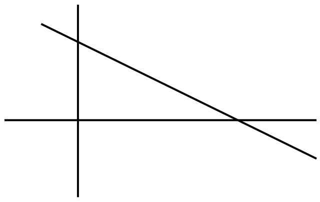
A geometrical representation of the locus is called the graph of the eq.

Ex(5):-Sketch the graph of the following eq.

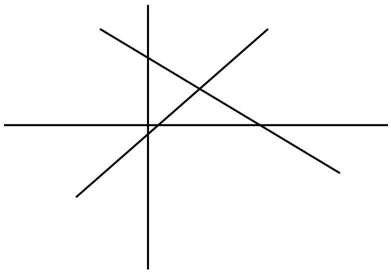
a. 2x+3y=6

at x=0 $\Rightarrow$ 3y=6 $\Rightarrow$ y=2 (0,2)

at y=0 $\Rightarrow$ 2x=6 $\Rightarrow$ x=3 (3,0)



b. $y = \begin{cases} x, 0 \leq x < 1 \\ 2-x, 1 < x \leq 2 \end{cases}$



Limits and Continuity

Defn.:- if the values of a function f of x approach the value L as x approaches a, we say F has limit L as x approaches a and we write $\lim_{x\to a} f(x) = L$.

Ex(1):-Let f(x)=2x+5 evaluate f(x) at x=1.1,1.01,1.001,…

f(1.1)=2(1.1)+5=7.2

f(1.01)=2(1.01)+5=7.02

f(1.001)=2(1.001)+5=7.002

⋮

we see that f(x) tends to 7 as x approach to 1 so we say $f(x) \to 7$ as $x \to 1$

Theorems on Limits

1. Ungeness of limit

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ then $L_1 = L_2$

2. Limit of constant

If f(x)=c where c is constant then $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$

3. Obvious limit

If f(x)=x then $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$

4. limit of a sums and differences

If $f(x) = f_1(x) \mp f_2(x) \mp \ldots \mp f_n(x)$ and $\lim_{x\to a} f_i(x) = L_i$,i=(1,2,…,n)

$$\lim_{x\to a} f(x) = \lim_{x\to a}[f_1(x) \mp f_2(x) \mp \ldots \mp f_n(x)] = \lim_{x\to a} f_1(x) \mp \lim_{x\to a} f_2(x) \mp \ldots \mp \lim_{x\to a} f_n(x)$$

$$= L_1 \mp L_2 \mp \ldots \mp L_n = \sum_{i=1}^{n} L_i$$

5. Limit of a products

If $f(x) = f_1(x) \times f_2(x) \times \ldots \times f_n(x)$ and $\lim f_i(x) = L_i$, i=(1,2,…, n)

then

$$\lim_{x\to a} f(x) = \lim_{x\to a}[f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x)] = \lim_{x\to a} f_1(x) \cdot \lim_{x\to a} f_2(x) \cdot \ldots \cdot \lim_{x\to a} f_n(x)$$

$$= L_1 \cdot L_2 \cdot \ldots \cdot L_n = \prod_{i=1}^{n} L_i$$

6. Limit of quotients

If $f(x) = \dfrac{g(x)}{h(x)}$ and $\lim_{x\to a} g(x) = L_1$ and $\lim_{x\to a} h(x) = L_2$

then $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{g(x)}{h(x)} = \dfrac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \dfrac{L_1}{L_2}$, $h(x) \neq 0$

the limits are all taken as $\mathbf{x} \to a$ and L_1 and L_2 are real numbers.

Ex(2): Evaluate the following limits.

i. $\lim_{x\to -1} 3x^2 = 3 \lim_{x\to -1} x^2 = 3(-1)^2 = 3$ (constant multiples).

ii. $\lim_{x\to 1}(4x^2 + 5) = \lim_{x\to 1} 4x^2 + \lim_{x\to 1} 5 = 4 + 5 = 9$ (sum).

iii. $\lim_{x\to 2}\frac{x^3+4x^2-3}{x^2+5}=\frac{\lim_{x\to 2}x^3+4x^2-3}{\lim_{x\to 2}x^2+5}=\frac{(2)^3+4(2)^2-3}{(2)^2+5}=\frac{8+16-3}{9}=\frac{21}{9}=\frac{7}{3}$

Ex(3):- Evaluate the following limits if they exist

1. $\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$

$D=R\neq 2$, so the denominator is 0 when x=2

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}=\frac{4-3(2)+2}{2-2}=\frac{0}{0}$$

We have to rewrite the fraction $\frac{x^2-3x+2}{x-2}$ first then

$$\lim_{x\to 2}f(x)=\lim_{x\to 2}\frac{x^2-3x+2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)}=\lim_{x\to 2}(x-1)=(2-1)=1$$

2. $\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\frac{(2)^3-8}{(2)^2-4}=\frac{0}{0}$

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x^2+2x+4}{x+2}=\frac{(2)^2+2(2)+4}{(2)+2}=\frac{12}{4}=3$$

Ex(4):- Evaluate the following limits if they exist

a. $\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1}, x\neq -1, x\geq -2$

$$\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1}\cdot\frac{\sqrt{2+x}+1}{\sqrt{2+x}+1}=\lim_{x\to -1}\frac{(2+x-1)}{(x+1)(\sqrt{2+x}+1)}=\lim_{x\to -1}\frac{1}{(\sqrt{2+x}+1)}=\frac{1}{\sqrt{2-1}+1}$$

$$=\frac{1}{2}$$

b. $\lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}}, x\neq 2, x\geq 0$

$$\lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}}\cdot\frac{2+\sqrt{2x}}{2+\sqrt{2x}}=\lim_{x\to 2}\frac{(2-x)(2+\sqrt{2x})}{4-2x}=\lim_{x\to 2}\frac{2+\sqrt{2x}}{2}=\frac{2+\sqrt{4}}{2}=\frac{4}{2}=2$$

Ex(5):- Evaluate the following limits

a. $\lim_{x\to 1}\frac{x^3-1}{x-1}, x\neq 1$

$$=\lim_{x\to 1}\frac{(x-1)(x^2+x-1)}{(x-1)}=\lim_{x\to 1}(x^2+x-1)=3$$

b. $\lim_{h\to 0}\frac{1}{h}(\frac{1}{x+h}-\frac{1}{x}), h\neq 0$

$$=\lim_{h\to 0}\frac{1}{h}(\frac{x-x-h}{(x+h)x})=-\lim_{x\to 0}\frac{1}{x(x+h)}=-\frac{1}{x(x+0)}=-\frac{1}{x^2}$$

One-sided and two-sided limits(Right and Left limits)
Some times the values of a function f(x) tends to different limits as x approaches a from different sides, when this happens, we call the limit of f(x) as x approach a from the right (the right hand limit of f at a) and denoted by $\lim_{x \to a^+} f(x)$ and (+) means that x approaches a through values above a on the number line.
And the limit of f(x) as x approaches a from the left (left hand limit of f at a) and denoted by $\lim_{x \to a^-} f(x)$ and (-) means that x approaches a through values below a on the number line.
Note: A function f has a limit as x approaches a if and only if the right-hand and left-hand limits at a exist and are equal in symbols. $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Ex(6):-let $f(x) = \sqrt{x}$ find $\lim_{x \to 0} f(x) = 2$

Solu. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt{x} = 0$

we shall explain this limit

$f(x) = \sqrt{x} \quad D: x \geq 0$

since $\sqrt{x}$ is not define for –ive values

so we restrict to +ive values of x

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \sqrt{x} = 0 = \lim_{x \to 0} f(x)$

Note: this example of one-sided limits

Ex(7):-If $f(x) = \sqrt{1-x}$ find $\lim_{x \to 1} f(x)$

$\lim_{x \to 1} f(x) = \lim_{x \to 0} \sqrt{1-x} = \sqrt{1-1} = 0$

we shall explain this example

$f(x) = \sqrt{1-x}$, $D: 1-x \geq 0 \Rightarrow x \leq 1$

since $\sqrt{1-x}$ is not define for x>1

so we restrict on values of $x \leq 1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1-x} = \sqrt{1-1} = 0 = \lim_{x \to 1} f(x)$

this example of one-sided limits

Limits at Infinity
We note that , when the limit of a function f(x) exist as x approach infinity, we write $\lim_{x \to \infty} f(x) = l$,also two write

$\lim_{x \to +\infty} f(x) = l$ for +ive values of x.

$\lim_{x \to -\infty} f(x) = l$ for –ive values of x.

for one-sided limit and two sided limits, we have $\lim_{x\to\infty} f(x) = L$ iff $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$.

Some obvious theorems

1. If k is constant, then $\lim_{x\to\infty}(k) = k$, $\lim_{x\to-\infty}(k) = k$

2. $\lim_{x\to\infty}\frac{1}{x} = \frac{1}{\infty} = 0$, $\lim_{x\to+\infty}\frac{1}{x} = 0$, $\lim_{x\to-\infty}\frac{1}{x} = 0$

3. $\lim_{x\to 0}\frac{1}{x} = \infty$, $\lim_{x\to 0^+}\frac{1}{x} = +\infty$, $\lim_{x\to 0^-}\frac{1}{x} = -\infty$

Limits of rational functions as $x \to \mp\infty$

To find the limit of a rational function as $x \to \mp\infty$ (when the limit exists) we divide the numerator and denominator by the highest power of x.

Ex(8):-find the following limits

1. $\lim_{x\to\infty}\frac{x}{2x+3} = \lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{2x+3}{x}} = \lim_{x\to\infty}\frac{1}{2+\frac{3}{x}} = \frac{1}{2+\frac{3}{\infty}} = \frac{1}{2+0} = \frac{1}{2}$

2. $\lim_{x\to\infty}\frac{2x^2+1}{3x^3-2x^2+5x-2} = \lim_{x\to\infty}\frac{\frac{2}{x}-\frac{1}{x^2}}{3-\frac{2}{x}+\frac{5}{x^2}-\frac{2}{x^3}} = \frac{0+0}{3-0+0-0} = \frac{0}{3} = 0$

3. $\lim_{x\to\infty}\frac{2x^3+3x^2+1}{x^2-5x+2} = \lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}+\frac{2}{x^3}} = \frac{2+0+0}{0-0+0} = \frac{2}{0} = \infty$

4. $\lim_{x\to-\infty}\frac{2x^2-3}{7x+4} = \lim_{x\to-\infty}\frac{2x-\frac{3}{x}}{7+\frac{4}{x}} = -\infty$

5. $\lim_{x\to\infty}\sqrt{x} = \lim_{x\to+\infty}\sqrt{x} = +\infty$

6. $\lim_{x\to\infty}(2+\frac{\sin x}{x}) = \lim_{x\to\infty}(2) + \lim_{x\to\infty}\frac{\sin x}{x}$

but $\lim_{x\to\infty}(2) = 2$ and $\lim_{x\to\infty}\frac{\sin x}{x} = 0$ because $-1 \le \sin x \le 1$

$\lim_{x\to\infty}(2+\frac{\sin x}{x}) = 2+0 = 2$

7. $\lim_{x\to-\infty}(2x+\frac{3}{x}) = \lim_{x\to-\infty} 2x + \lim_{x\to-\infty}\frac{3}{x} = -\infty + 0 = -\infty$

8. $\lim_{x\to 2^-}\frac{1}{x^2-4} = \frac{1}{4-4} = \frac{1}{0} = -\infty$

Continuity
Defn. (Continuous function)
A function f(x) is continuous at x=a if and only if the following statements are true.
1. f(a) exists (a lies in the domain of f).
2. $\lim_{x\to a} f(x)$ exists (f has a limit as $x \to a$).
3. $\lim_{x\to a} f(x) = f(a)$ (the limit equals the function value).

Ex(9):-
1. Every polynomial of the from
$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is cont. for all x

2. let $f(x) = \begin{cases} \frac{x^2+x-6}{x^2-4}, x \neq 2 \\ \frac{5}{4}, x = 2 \end{cases}$ is f(x) cont. at x=2?

1- $f(2) = \frac{5}{4}$

2- $\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x+3}{x+2} = \frac{2+3}{2+2} = \frac{5}{4} = f(2)$

$\therefore$ f(x) is cont. of x=2

3. $f(x) = \frac{x+3}{(x-1)(x+2)}$ at is continuous at every value of x except x=1 and x=-2.

4. The function $y = \frac{1}{x}$ is continuous at every value of x except x=0 because $f(0) = \frac{1}{0} = \infty$ so the function is not defined at x=0.

The Derivatives

Defn. let y=f(x) be a function then the derivative of f w.r.t x denoted by $f'(x)=\frac{dy}{dx}=y'$, is defined by the rule $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, h\neq 0$ where $h=\Delta x$

(Differentiation Rules)

Rule 1: If f(x)=k where k is content then $f'(x)=0$

Ex(1):- $y=\sqrt{12}\Rightarrow \frac{dy}{dx}=0$

Rule 2: If f(x)=x^n where n is integer then $f'(x)=nx^{n-1}$

Ex(2):-

1. $f(x)=x^5\Rightarrow f'(x)=5x^4$
2. $y=2x^{-6}\Rightarrow y'=-12x^{-7}=\frac{-12}{x^7}$
3. $y=\sqrt{x}=x^{\frac{1}{2}}\Rightarrow \frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

Rule 3: If f(x)=k u(x) where k is constant and u is differentiable function of x then $f'(x)=ku'(x)$ or $\frac{dy}{dx}=k\cdot\frac{du}{dx}$

Ex(3):- $f(x)=-4x^3\Rightarrow f'(x)=-4*3x^2=-12x^2$

Rule 4: If $f(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$ where $f_1(x)f_2(x)\ldots f_n(x)$ are differentiable function of x then $f'(x)=f_1'(x)+f_2'(x)+\ldots+f_n'(x)$

Ex(4):- $f(x)=4x^3-2x^2+x-5\Rightarrow f'(x)=-12x^2-4x+1$

Rule 5: If f(x)=u(x)v(x) where u(x) and v(x) are differentiable function of x then $f'(x)=u(x)\cdot v'(x)+v(x)\cdot u'(x)$

Ex(5):-

$$y=(x^2+3x-2)(x^3-8x^2+1)$$

$$\frac{dy}{dx}=(x^2+3x-2)(3x^2-16x)+(x^3-8x^2+1)(2x+3)$$

Rule 6: If $f(x)=\frac{u(x)}{v(x)}, v(x)\neq 0$ where u(x) and v(x) are differentiable functions of x then $f'(x)=\frac{v(x)u'(x)-u(x)v'(x)}{[v(x)]^2}$

Ex(6):- $y=\frac{2x^3-4x^2+8}{x^2+1}\Rightarrow\frac{dy}{dx}=\frac{(x^2+1)(6x^2-8x)-(2x^3-4x^2+8)(2x)}{(x^2+1)^2}$

Rule 7: If $f(x)=[u(x)]^n$ where n is integer and u(x) is a differentiable function of x then $f'(x)=n[u(x)]^{n-1}\cdot u'(x)$

Ex(7):-

1. $y=(2x^2+5x-3)^{-4}\Rightarrow\frac{dy}{dx}=-4(2x^2+5x-3)^{-5}(4x+5)$

2. $y=(3x^{\frac{1}{2}}+5)^{-\frac{1}{4}}\Rightarrow\frac{dy}{dx}=-\frac{1}{4}(3x^{\frac{1}{2}}+5)^{-\frac{5}{4}}(\frac{3}{2}x^{-\frac{1}{2}})$

Implicit Differentiation

Consider the function defined by the eq. f(x,y)=0 which may or may not be solved for y in terms of x. for example $y+x^3+2x-5=0$ can be written as $y=-(x^3+2x-5)$ and $\frac{dy}{dx}=3x^2+2$ while $y^5+4x^2y^2+x^3-2=0$ can not be solved for y in terms of x. Implicit differentiation enables us to find the derivative of such function whenever they exist.

Ex(8):- $y^2=x\ \Rightarrow 2y\frac{dy}{dx}=1\ \Rightarrow\frac{dy}{dx}=\frac{1}{2y}$

Ex(9):- $x^2-y^2=1\ \Rightarrow 2x-2yy'=0\Rightarrow y'=\frac{2x}{2y}=\frac{x}{y}$

The Second and Higher Derivatives

Given the function y=f(x) the derivative $y'=f'(x)=\frac{dy}{dx}$ is the 1st derivative of y.w.r. to x. And $y''=f''(x)=\frac{d^2y}{dx^2}$ is called the 2nd derivative of y.w.r. to x.

Thus the 2nd derivative is derivative of the 1st derivative.

That $\frac{d^2y}{dx^2}=\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)$

In general if y=f(x) is differentiable function of x then the nth derivative of y.w.r to x is denoted by $y^n=f^n(x)=\frac{d^ny}{dx^n}$

Ex(10):- If y=3x^4-5x^3+6x-7 find $\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3},\ldots,\frac{d^ny}{dx^n}$

$$\frac{dy}{dx}=12x^3-15x^2+6$$

$$\frac{d^2y}{dx^2}=36x^2-30x$$

$$\frac{d^3y}{dx^3}=72x-30$$

$$\frac{d^4y}{dx^4}=72\Rightarrow\frac{d^5y}{dx^5}=0=\frac{d^6y}{dx^6}=\cdots=\frac{d^ny}{dx^n}=0$$

Ex(11):- If $y=(x^3+2x-1)^{\frac{1}{2}}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx}=\frac{1}{2}(x^3+2x-1)^{-\frac{1}{2}}(3x^2+2)$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left\{(x^3+2x-1)^{-\frac{1}{2}}(6x)+(3x^2+2)\left[-\frac{1}{2}(x^3+2x-1)^{-\frac{3}{2}}(3x^2+2)\right]\right\}$$

Chain Rule and Parametric Equations

If y is a function of x, say y=f(x)

And x is a function of t, say x=g(t)

Then y is a function of t and $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ -------- (1)

Ex(12):- let y=x^3-2x^2+3 and x=t^2+2 find $\frac{dy}{dt}$ at t=2

Solu. $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$=(3x^2-4x)(2t)

When t=2 $\Rightarrow$ x=4+2=6

$$\therefore \left.\frac{dy}{dt}\right|_{t=2}=\left[3(6)^2-4*6\right](2*2)=336$$

or

y=x^3-2x^2+3, x=t^2+2

y=(t^2+2)^3-2(t^2+2)^2+3

$$\frac{dy}{dt}=3(t^2+2)^2(2t)-4(t^2+2)(2t)$$

$$\left.\frac{dy}{dt}\right|_{t=2}=3(4+2)^2(2-2)-4(4+2)(2\cdot 2)=336$$

Indeterminate forms

$\frac{0}{0}, \frac{\infty}{\infty}, 1^{\infty}, 0^{0}, \infty^{0}, 0-\infty, \infty-\infty$ are called indeterminate forms.

Same times the $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} or \frac{\infty}{\infty}$ when we substitute x=a

L'Hopital Rule (1st form)

Suppose that f(a)=g(a)=o or ∞ and f', and $g'(a)$ exist with $g'(a) \neq 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$

Ex(13):- find the following limits

1. $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{2x}{1} = 2*2 = 4$

2. $\lim_{x \to 0} \frac{x-2x^2}{3x^2+5x} = \lim_{x \to 0} \frac{1-4x}{6x+5} = \frac{1-0}{0+5} = \frac{1}{5}$

3. $\lim_{x \to \infty} \frac{6x+5}{3x-8} = \lim_{x \to \infty} \frac{6}{3} = \frac{6}{3} = 2$

4. $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}-0}{1} = \lim_{x \to 0} \frac{1}{2\sqrt{1+x}} = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$

L'Hopital Rule (2nd form)

Suppose f(a)=g(a)=0 or ∞ and the function f(x) and g(x) are differentiable functions in some interval 1.

To find $\lim_{x \to a} \frac{f(x)}{g(x)}$, we proceed to differentiable f(x) and g(x)as we still get $\frac{0}{0} or \frac{\infty}{\infty}$.

But we step differentiation as soon as one or the other derivatives is different from.

Ex(14):-

1. $\lim_{x \to 1} \frac{x^3-3x+2}{x^3-x^2-x+1} = \lim_{x \to 1} \frac{3x^2-3}{3x^2-2x-1} = \lim_{x \to 1} \frac{6x}{6x-2} = \frac{6}{6-2} = \frac{3}{2}$

2. $\lim_{x \to 0} \frac{\sqrt{1+x}-\frac{x}{2}-1}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x} = \lim_{x \to 0} \frac{-\frac{1}{4}\frac{1}{(1+x)^{\frac{3}{2}}}}{2} = \frac{-\frac{1}{4}*1}{2} = -\frac{1}{8}$

3. $\lim_{x \to \infty} \frac{2x^3-x^2+3x+1}{x^3+2x^2-x-1} = \lim_{x \to \infty} \frac{6x^2-2x+3}{3x^2+4x-1} = \lim_{x \to \infty} \frac{12x-2}{6x-4} = \lim_{x \to \infty} \frac{12}{6} = \frac{12}{6} = 2$

Derivatives of the transcendental functions
a. Derivatives of the Trigonometric functions
Theorems:-

1. If y=f(x)=sinx then $\frac{dy}{dx}=f'(x)=\cos x$
2. If y=f(x)=cosx then $\frac{dy}{dx}=f'(x)=-\sin x$
3. If y=f(x)=tanx then $\frac{dy}{dx}=f'(x)=\sec^2 x$
4. If y=f(x)=cotx then $\frac{dy}{dx}=f'(x)=-\csc^2 x$
5. If y=f(x)=secx then $\frac{dy}{dx}=f'(x)=\sec x\tan x$
6. If y=f(x)=cscx then $\frac{dy}{dx}=f'(x)=-\csc x\cot x$

Now: If u=u(x) is a differentiable function of x and

1. y=sinu then $\frac{dy}{dx}=\cos u\cdot\frac{du}{dx}$

2. y=cosu then $\frac{dy}{dx}=-\sin u\cdot\frac{du}{dx}$

3. y=tanu then $\frac{dy}{dx}=\sec^2 u\cdot\frac{du}{dx}$

4. y=cotu then $\frac{dy}{dx}=-\csc^2 u\cdot\frac{du}{dx}$

5. y=secu then $\frac{dy}{dx}=\sec u\cdot\tan u\cdot\frac{du}{dx}$

6. y=cscu then $\frac{dy}{dx}=-\csc u\cdot\cot u\cdot\frac{du}{dx}$

Ex(1):- find $\frac{dy}{dx}$ of the following

1. $y=\sin(x^3+3x^2-2) \Rightarrow \frac{dy}{dx}=\cos(x^3+3x^2-2)(3x^2+6x)=(3x^2+6x)\cos(x^3+3x^2-2)$

2. $y=\cos^2(x^2+8x-9) \Rightarrow \frac{dy}{dx}=2\cos(x^2+8x-9)[-\sin(x^2+8x-9)](2x+8)$

3. $y=\tan 2x\cos(x^2+1) \Rightarrow \frac{dy}{dx}=\tan 2x[-\sin(x^2+1)(2x)]+\cos(x^2+1)\sec^2 2x(2)$

Ex(2):- find $\frac{dy}{dx}$ if $x^2+5x-\tan^2(xy)=0$

$$2x+5-2\tan(xy)\cdot\sec^2(xy)\cdot\left(x\frac{dy}{dx}+y(1)\right)=0$$

$$2x+5-2y\tan(xy)\cdot\sec^2(xy)=2x\tan(xy)\cdot\sec^2(xy)\cdot\frac{dy}{dx}$$

$$\therefore\frac{dy}{dx}=\frac{2x+5-2y\tan(xy)\cdot\sec^2(xy)}{2x\tan(xy)\cdot\sec^2(xy)}$$

Ex(3):- find $\frac{dy}{dx}$ if $2y=x^2+\sin y$

$$2y'=2x+\cos yy'$$

$$2y'-\cos yy'=2x\Rightarrow y'(2-\cos y)=2x\quad\Rightarrow y'=\frac{2x}{2-\cos y}$$

Ex(4):- Evaluate the following limit

$$\lim_{x\to0}\frac{\sin x}{x}=\lim_{x\to0}\frac{\cos x}{1}=\cos 0=1$$

3. The Derivative of lnx

If $y=\ln x$ then $\frac{dy}{dx}=\frac{1}{x}$

Now, if u=u(x) is a differentiable function of x and y=ln u then $\frac{dy}{dx}=\frac{1}{u}\cdot\frac{du}{dx}$

Ex(1):- find $\frac{dy}{dx}$ of the following

1. $y=\ln(x^2+3x-8)\Rightarrow\frac{dy}{dx}=\frac{1}{x^2+3x-8}\cdot(2x+3)=\frac{2x+3}{x^2 3x-8}$

2. $y=\ln(x^{-3}+\sin^2 2x)\Rightarrow\frac{dy}{dx}=\frac{(-3x^{-4}+2\sin 2x\cdot\cos 2x\cdot 2)}{(x^{-3}+\sin^2 2x)}$

3. $y=\ln(3x^2+4)\Rightarrow\frac{dy}{dx}=\frac{1}{3x^2+4}\cdot 6x=\frac{6x}{3x^2+4}$

Ex(2):- Evaluate the following limit

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}=0$$

4. The Derivative of exp(x)

Theorem: If y=e^x then $\frac{dy}{dx}=e^x$

Now, if u=u(x) is a differentiable function of x and $y=e^u$ then $\frac{dy}{dx}=e^u\cdot\frac{du}{dx}$

Ex(1):- find $\frac{dy}{dx}$ of the following

1. $y=e^{x^2+\sin 2x} \Rightarrow \frac{dy}{dx}=e^{x^2+\sin 2x}\cdot(2x+2\cos 2x)$

2. $y=e^{\tan^{-1}2x+\ln x}=e^{\tan^{-1}2x}\cdot e^{\ln x}=xe^{\tan^{-1}2x} \Rightarrow y'=xe^{\tan^{-1}2x}\cdot\frac{2}{1+4x^2}+e^{\tan^{-1}2x}\cdot 1$

3. $y=\cot^2(\tan^{-1}e^x) \Rightarrow y'=2\cot(\tan^{-1}e^x)\cdot\left[-\csc^2(\tan^{-1}e^x)\right]\cdot\frac{e^x}{1+e^{3x}}$

Ex(2):- Evaluate the following limit

1. $\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=\frac{6}{e^\infty}=\frac{6}{\infty}=0$

2. $\lim_{x\to 0}x^{\frac{1}{x}}=\lim_{x\to 0}e^{\ln x^{\frac{1}{x}}}=\lim_{x\to 0}e^{\frac{1}{x}\ln x}=e^{\lim\limits_{x\to 0}\frac{\ln x}{x}}=e^{\lim\limits_{x\to 0}\frac{\frac{1}{x}}{1}}=e^{\lim\limits_{x\to 0}\frac{1}{x}}=e^{\infty}=\infty$

Intercepts, Symmetry and more about graphing

1. To find x-intercepts, set y=0 and solve for y.
To find y-intercepts, set x=0 and solve for x.

2. The locus is symm. w.r.t the
- **a.** x-axis $\Leftrightarrow$ (x,y)≡(x,-y)
- **b.** y-axis $\Leftrightarrow$ (x,y)≡(-x,y)
- **c.** origin $\Leftrightarrow$ (x,y)≡(-x,-y)

Ex(7):-Find the domain ,range, intercepts, symmetry, for the following functions. (or we may write discuss the following function).

1. y=f(x)=x^3-x
D : all x
R : all y

Intercepts

Set y=0 $\Rightarrow$ x^3-x=0 $\Rightarrow$ x(x^2-1)=0 $\Rightarrow$ x(x-1)(x+1)=0 $\Rightarrow$ x=0,1,-1
x-int. are: (0,0),(1,0),(-1,0)
Set x=0 $\Rightarrow$ y=0
y-int. is:(0,0)

<u>Symm.</u>

$y=x^3-x$

about x-axis: $-y=x^3-x \Rightarrow y=-x^3+x$

No symm. w.r.t x-axis

about y-axis: $y=(-x)^3-(-x)=-x^3+x$

about origin:

2. $f(x)=\dfrac{1}{x^2-1}$

D: $x \neq \pm 1$

$yx^2-y=1 \Rightarrow yx^2=y+1 \Rightarrow x^2=\dfrac{y+1}{y}$

$x=\pm\sqrt{\dfrac{y+1}{y}} \Rightarrow \dfrac{y+1}{y} \geq 0$

R: $y \leq -1$ or $y>0$

Applications of Derivatives

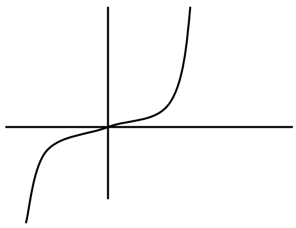
[1] Curve Sketching

Suppose that y=f(x) is a cont. function on interval I and has derivative at every point $x \in I$, then

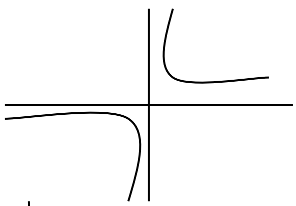
1. f(x) increases on I if $f'(x)>0, \forall x \in I$

2. f(x) decreases on I if $f'(x)<0, \forall x \in I$

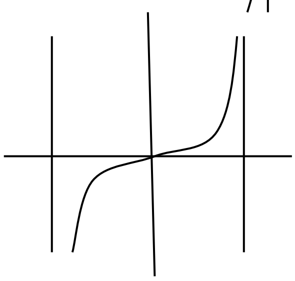
Ex(1):- $y=x^3$ increases on $(-\infty,\infty)$



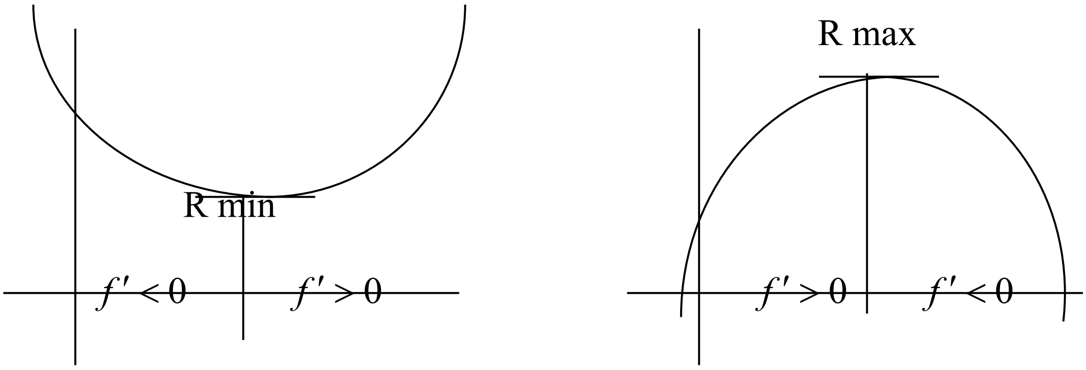
Ex(2):- $y=\dfrac{1}{x}$ decreases on $(-\infty,0)$



Ex(3):- $y=\tan x$ increases on $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

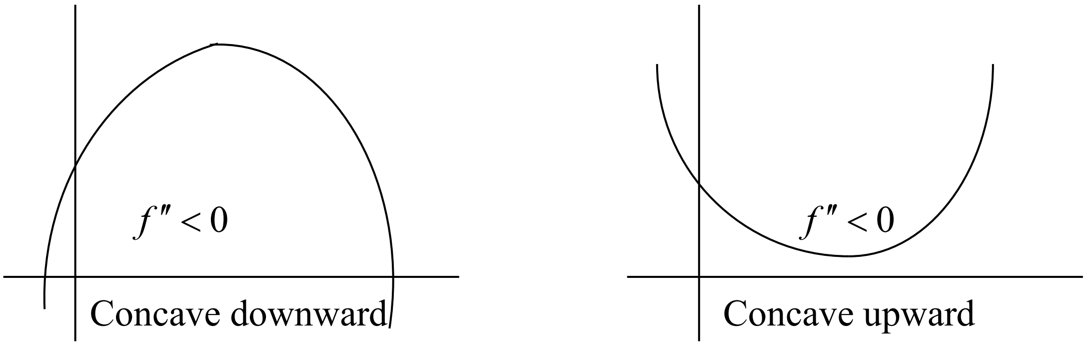


Defn. A critical point x of a function f(x) is the value of x where $f'(x_0) = 0$
Suppose that f(x) is cont. at x_0
If f(x) is **increasing (decreasing)** in an interval I, with x_0 as aright endpoint and **decreasing (increasing)** in an interval I_2 with x_0 as left endpoint then f(x) has relative maximum (minimum) at x_0.



Ex:- $y = x^2$ **decreases** on $(-\infty,0)$ where the derivative $y' = 2x$ is negative and **increases** on $(0,\infty)$ where the derivative is positive.

Defn. The graph of a differentiable function y=f(x) is concave up on an interval I where y' is **increasing** and concave down on an interval I where y' is **decreasing**.



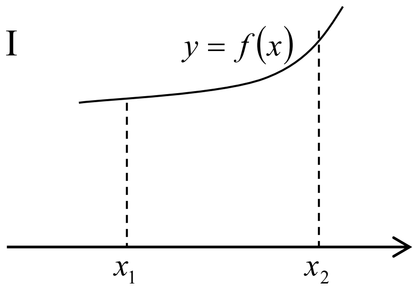
The second derivative test for concavity
The graph of y=f(x) is concave down on any interval I where $y'' < 0$ and concave up on any interval I where $y'' > 0$.

Defn. A point on the curve y=f(x) where the concavity changes from upward to downward (or vice versea) is called a point of inflection.
At a point of inflection on the graph of a twice differentiable function, $y'' = 0$

Defn,

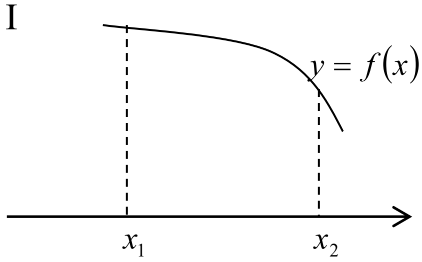
1. if, for any two points x_1 and x_2 in I

$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$



2. if, for any two points x_1 and x_2 in I

$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$



Steps in Graphing y=f(x)

1. Find y', y''.

2. Find where y' is positive, negative, and zero. This will show where the curve may have local maxima and minima. And where the curve is rising and falling.

3. Find where y'' is positive, negative, and zero. This will tell us about concavity and possible inflection points.

4. Making a summary table, we include the values of y, y' and y'' at the intercepts and at the other important points, we summarize what we have learned about the curves behavior.

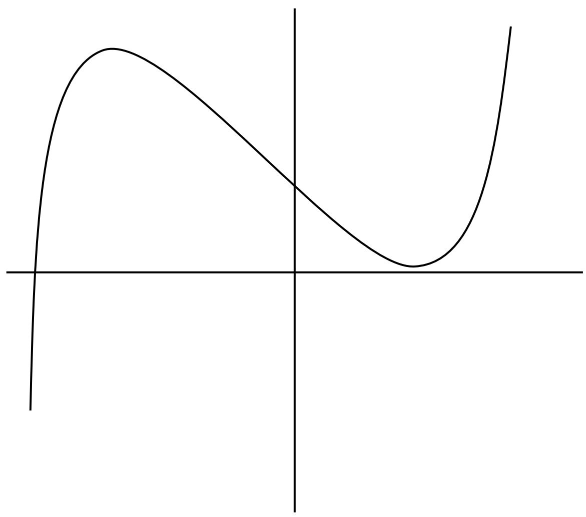
5. Draw the graph; in order to do this, we plot the points from the table and sketch the tangents at these points. Then we draw the curve by using information about rise, fall, and concavity.

Ex:- Discuss and sketch the curve of the following functions:

a. $y=x^3-3x+2$

D: all x

R: all y



Intercepte :

Set $x=0 \Rightarrow y=2$

$\therefore$ (0,2) is y-int.

Set $y=0 \Rightarrow x^3-3x+2=0$

$(x-1)(x^2+x-2)=0$

$(x-1)^2(x+2)=0$

$\therefore$ x=1,-2

$\therefore$ (1,0),(-2,0) are x-int.

Symm.: No

Asymptote: No

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$

$$\frac{dy}{dx} = 0 \Rightarrow 3(x-1)(x+1) = 0$$

x=1, -1 are critical pts

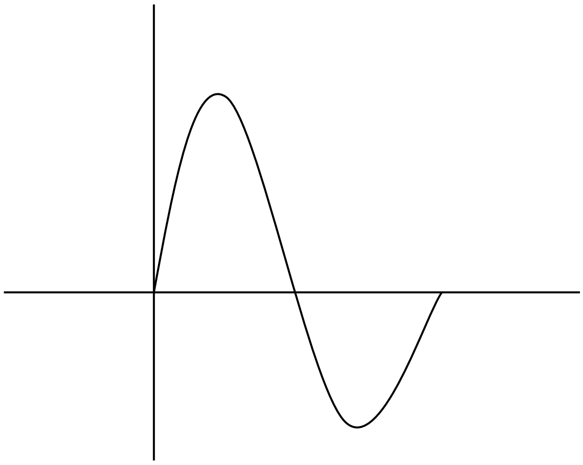
$$\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$$

b. $y = \sin x$

D: $0 \le x \le 2\pi$

R: $-1 \le y \le 1$



Interceptes:

Set $y = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

(0,0),(π,0),(2π,0) are x-int.

Set x=0 $\Rightarrow$ y=sin0=0

$\therefore$ (0,0) is y-int.

Symm.: w.r.t origin only

Asymptotes: No

$$\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = 0 \Rightarrow \cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ are critical pts}$$

$$\frac{d^2y}{dx^2} = -\sin x$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow -\sin x = 0$$

$$x = 0, \pi, 2\pi$$

c. $y = \tan x$ for $0 \le x \le 2\pi$

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$D: 0 \le x < \frac{\pi}{2} \, or \, \frac{\pi}{2} < x < \frac{3\pi}{2} \, or \, \frac{3\pi}{2} < x < 2\pi$$

$$R: -\infty < y < \infty$$

Intercepts:

Set y=0 $\Rightarrow$ tanx=0 $\Rightarrow$ x=0,π,2π

$\therefore$ (0,0),(π,0),(2π,0) are x-int.

Set x=0 $\Rightarrow$ y=tan0=0

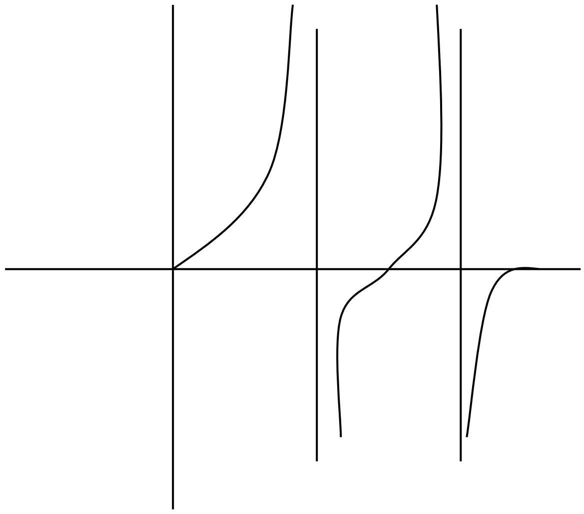
$\therefore$(0,0) is y-int.

Symm.: w.r.t origin

Asymptotes:

$x=\frac{\pi}{2},\frac{3\pi}{2}$ are V.Asy.

No H.Asy.



$\frac{dy}{dx}=\sec^2 x$

$\frac{dy}{dx}=0\Rightarrow\sec^2 x=0$ impossible

$\frac{d^2y}{dx^2}=2\sec^2 x\cdot\tan x$

$\frac{d^2y}{dx^2}=0\Rightarrow 2\sec^2 x\cdot\tan x=0$

$\frac{\sin x}{\cos x}=0$

$\sin x=0\Rightarrow x=0,\pi,2\pi$

d. $y=\ln x$

D: x>0

x=e^y R: all y

Intercepts:

Set y=0 $\Rightarrow$ ln x=0

x=e^0=1

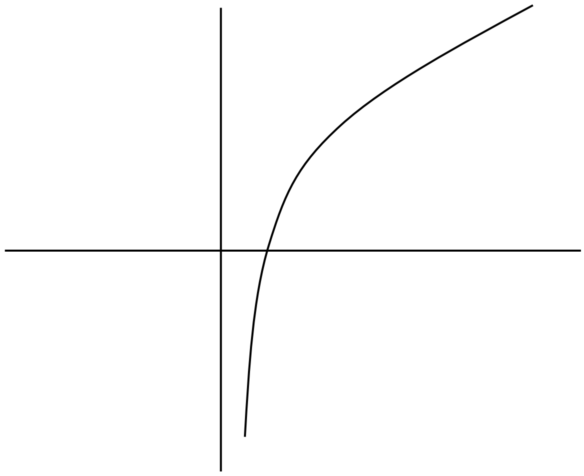
$\therefore$(1,0)is x-int.

Set x=0 $\Rightarrow$ y=ln 0=∞

No y-int.

Symm.: No

Asymptotes: x=0 is V.Asy



$\frac{dy}{dx}=\frac{1}{x}$

$\frac{dy}{dx}=0\Rightarrow\frac{1}{x}=0\Rightarrow 1=0$

impossible

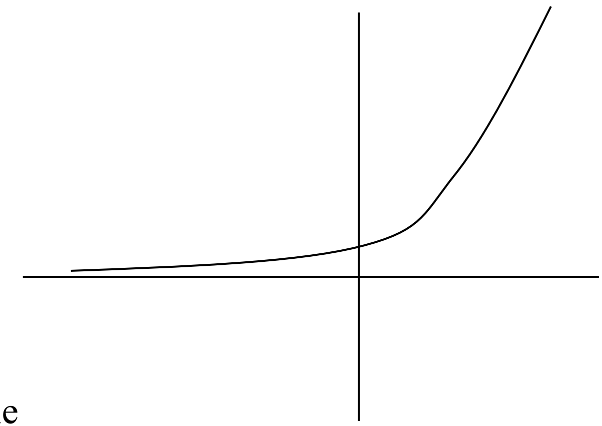
$\frac{d^2y}{dx^2}=-\frac{1}{x^2}$

$$\frac{d^2y}{dx^2}=0\Rightarrow -\frac{1}{x^2}=0\Rightarrow -1=0 \qquad \text{impossible}$$

e. $y=e^x$

D: all x

x=ln y   R: y>0



Interceptes:

Set x=0 $\Rightarrow$ y=e^0=1

$\therefore$(0,1) is y-int.

Set y=0 $\Rightarrow$ e^x=0

x=ln 0 impossible

No x-int.

Symm.: No

Asymptotes:

y=0 is H-Asy.

$y'=e^x \Rightarrow y'=0\Rightarrow e^x=0$ impossible

$y''=e^x$

$y''=0\Rightarrow e^x=0$ Impossible

[2] Maxima and Minima Problems

Ex(1):- Find two positive numbers whose sum is 20 and whose product is as large as possible.

Solu. Let the two no's be x, 20-x and their product is $f(x)=x(20-x)=20x-x^2$

$f'(x)=20-2x$

$f'(x)=0\Rightarrow 20-2x=0$

20=2x $\Rightarrow$ x=10 is critical point

$f''(x)=-2$

$f(10)=-2<0$

$\therefore$ the two numbers are: 10, 20-x=10

Ex(2):- Find the maximum height of the curve $y=4\sin^2 x-3\cos^2 x$

Solu.

$y'=8\sin x\cos x-6\cos x(-\sin x)=8\sin x\cos x+6\cos x\sin x$

$=14\sin x\cos x=7(2\sin x\cos x)=7\sin 2x$

$f'(x)=0\Rightarrow 7\sin 2x=0\Rightarrow \sin 2x=0$

$2x=0,\pm\pi,\pm 2\pi,\pm 3\pi$

$x=0,\pm\frac{\pi}{2},\pm\pi,\pm\frac{3\pi}{2}$

are critical points

$f(0)$=4sin^2 (0)-3cos^2 (0)=-3

$f\left(\pm\frac{\pi}{2}\right)$=4sin^2 $\left(\pm\frac{\pi}{2}\right)$-3cos^2 $\left(\pm\frac{\pi}{2}\right)$=4

$f(\pm\pi)$=4sin^2 $(\pm\pi)$-3cos^2 $(\pm\pi)$=-3

$f\left(\pm\frac{3\pi}{2}\right)$=4sin^2 $\left(\pm\frac{3\pi}{2}\right)$-3cos^2 $\left(\pm\frac{3\pi}{2}\right)$=4

$\therefore$ the max. height of the function is y=4

Ex(3):- If $ax+\frac{b}{x}-c\geq 0$ for (x>0) where a,b,c are constants, show that $4ab\geq c^2$

Solu. let

$$f(x)=ax+\frac{b}{x}-c\Rightarrow f(x)\geq 0$$

$$f'(x)=a-\frac{b}{x^2}\Rightarrow f'(x)=0\Rightarrow a-\frac{b}{x^2}=0\Rightarrow a=\frac{b}{x^2}\Rightarrow x^2=\frac{b}{a}\Rightarrow x=\sqrt{\frac{b}{a}}$$

is critical point

$$f\left(\sqrt{\frac{b}{a}}\right)=a\left(\sqrt{\frac{b}{a}}\right)+\frac{b}{\sqrt{\frac{b}{a}}}-c=\sqrt{ab}+\sqrt{ab}-c=2\sqrt{ab}-c\geq 0\Rightarrow 2\sqrt{ab}\geq c\Rightarrow 4ab\geq c^2$$