Type of errors المحاضرة الأولى

Errors

Occurrence of error is unavoidable in the field of scientific computing. Instead, numerical analysts try to investigate the possible and best ways to minimise the error. The study of the error and how to estimate and minimize it are the fundamental issues in *error analysis*.

Error Analysis

In numerical analysis we approximate the exact solution of the problem by using numerical method and consequently an error is committed. The numerical error is the difference between the exact solution and the approximate solution.

Definition (Numerical Error): Let x be the exact solution of the underlying problem and x^* its approximate solution, then the error (denoted by e) in solving this problem is

$$e = x - x^*$$

Absolute and Relative Errors

Definition (Absolute Error): The absolute error $\hat{e}$ of the error e is defined as the absolute value of the error e

$$\hat{e} = |x - x^*|.$$

Definition (Relative Error): The relative error $\tilde{e}$ of the error e is defined as the ratio between the absolute error $\hat{e}$ and the absolute value of the exact solution x

$$\tilde{e} = \frac{\hat{e}}{|x|} = \frac{|x - x^*|}{|x|}, \quad x \neq 0.$$

Example: Let $x = 3.141592653589793$ is the value of the constant ratio π correct to 15 decimal places and $x^* = 3.14159265$ be an approximation of x.

Compute the following quantities:

a. The error.

b. The absolute error.

c. The relative error.

Solution:

a. The error

$$e = x - x^* = 3.141592653589793 - 3.14159265 = 3.589792907376932e - 09$$
$$= 3.589792907376932 \times 10^{-9} = 0.000000003589792907376932.$$

b. The absolute error

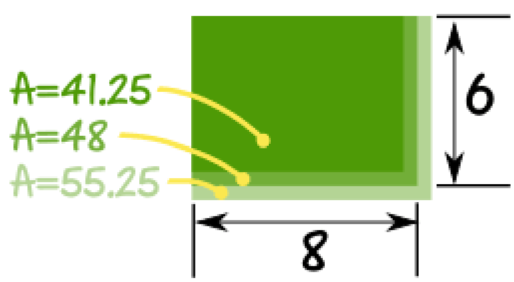
$$\hat{e} = |x - x^*| = |3.141592653589793 - 3.14159265| = 3.589792907376932\mathrm{e} - 09.$$

c. The relative error

$$\tilde{e} = \frac{\hat{e}}{|x|} = \frac{|x - x^*|}{|x|} = \frac{3.141592653589793 - 3.14159265}{3.141592653589793}$$
$$= \frac{3.589792907376932\mathrm{e} - 09}{3.141592653589793} = 1.142666571770530\mathrm{e} - 09.$$

Example: Alex measured the field to the nearest meter, and got a width of 6 m and a length of 8 m.

Measuring to the nearest meter means the true value could be up to **half a meter** smaller or larger.



The width (w) could be from 5.5m to 6.5m: $5.5 \le w < 6.5$

The length (l) could be from 7.5m to 8.5m: $7.5 \le l < 8.5$

The area is width × length: $A = w \times l$

The smallest possible area is: $5.5m \times 7.5m = \mathbf{41.25\ m2}$

The measured area is: $6m \times 8m = \mathbf{48\ m2}$

And the largest possible area is: $6.5m \times 8.5m = \mathbf{55.25\ m2}$

$$41.25 \le A < 55.25$$

Absolute, Relative and Percentage Error

The only tricky thing here is ... **which** is the absolute error?

- From 41.25 to 48 = **6.75**
- From 48 to 55.25 = **7.25**

Answer: pick the biggest one! So:

$$Absolute\ Error\ =\ 7.25\ m2$$

$$Relative\ Error = \frac{7.25m^2}{48m^2} =\ 0.151\ldots$$

$$Percentage\ Error\ =\ 15.1\%$$

Roundoff and Truncation Errors

Computers represent numbers in finite number of digits and hence some quantities cannot be represented exactly. The error caused by replacing a number a by its closest machine number is called the **roundoff error** and the process is called **correct rounding**.

Truncation errors also sometimes called **chopping errors** are occurred when chopping an infinite number and replaced it by a finite number or by truncated a series after finite number of terms.

Example: Approximate the following decimal numbers to three digits by using rounding and chopping (truncation) rules:

1. $x_1 = 1.34579.$
2. $x_2 = 1.34679.$
3. $x_3 = 1.34479.$
4. $x_4 = 3.34379.$
5. $x_5 = 2.34579.$

Solution:

i. Rounding:

a) $x_1 = 1.35.$

b) $x_2 = 1.35.$

c) $x_3 = 1.34.$

d) $x_4 = 3.34.$

e) $x_5 = 2.35.$

ii. Chopping:

a) $x_1 = 1.34.$

b) $x_2 = 1.34.$

c) $x_3 = 1.34.$

d) $x_4 = 3.34.$

e) $x_5 = 2.34.$

Example: $\pi = 3.141592654$

مثلا نحسب خطأ التدوير لل π عند اربع مراتب $\pi = 3.1416$

ونحسب خطأ القطع او البتر لل π عند اربع مراتب $\pi = 3.1415$

Example:

x	Round-by-chop	Roundoff Error	Round-to-nearest	Roundoff Error
1.649	1.6	0.049	1.6	0.049
1.650	1.6	0.050	1.6	0.050
1.651	1.6	0.051	1.7	-0.049
1.699	1.6	0.099	1.7	-0.001
1.749	1.7	0.049	1.7	0.049
1.750	1.7	0.050	1.8	-0.050

Solution of Nonlinear Equations

Bisection method

Let $f(x)$ be a continuous function and defined on $[a,b]$, let $f(a)$ and $f(b)$ have opposite signs $f(a)\cdot f(b)<0$. Then there exist on p such that $f(p)=0$ and $p\in(a,b)$.

The steps of solve bisection method

١. نجد قيم $p=\frac{a+b}{2}$

٢. نحسب قيمة $f(p)$ و هناك ثلاث أحتمالات

أ.أذا كانت $f(p)=0$ فان p هو الجذر المطلوب.

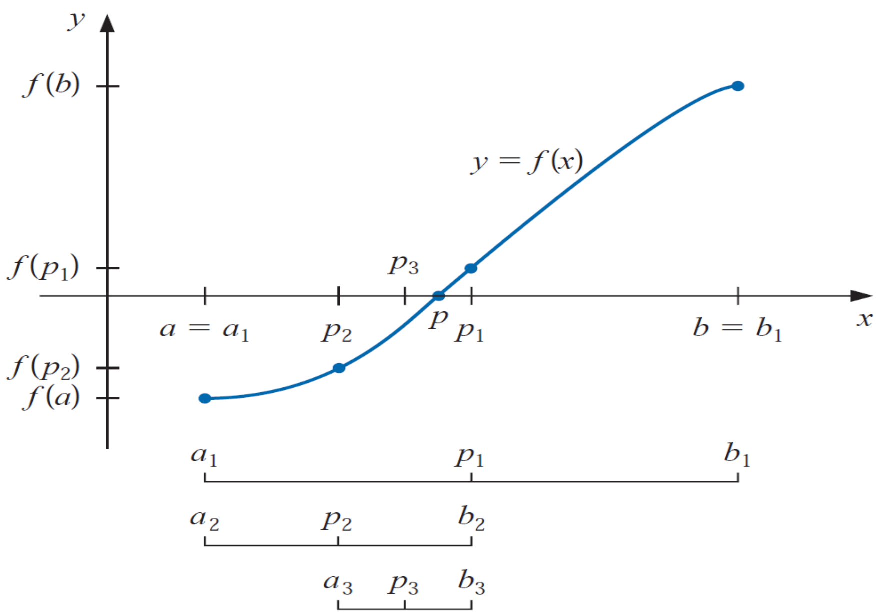
ب. أذا كانت $f(a)f(p)<0$ فان للمعادلة جذر يقع ضمن الفترة $[a,p]$.

ج. أذا كانت $f(b)f(p)<0$ فان للمعادلة جذر يقع ضمن الفترة $[p,b]$.

٣. تكرر الخطوة (١,٢) الى أن نصل للجذر المطلوب ونسبة خطأ لاتتجاوز ε.

* شرط التوقف: يمكن أستخدام الصيغ الأتية لغرض التوقف عن التكرار.

1. $|p_n-p_{n-1}|\le\varepsilon$
2. $|f(p_n)|\le\varepsilon$



Ex. (1):- Find the root of the equation $f\ (x) = x^3, x \in [-1,1]$

Solu. $f(-1) = -1, f(1) = 1$

$$p = \frac{a+b}{2} = \frac{1-1}{2} = 0$$

$$f(0) = 0$$

$P = 0$ is a root

n	a	b	p	$f(p)$
1	-1	1	0	0

Ex. (2):- Find the root of the equation $f\ (x) = 2x - 1, x \in [0,2]$

Solu. $f(0) = -1\ f(2) = 3$

$$p = \frac{a+b}{2} = \frac{2+0}{2} = 1$$

$$f(1) = 2 - 1 = 1$$

1. $f(0)f(1) < 0$
2. $f(2)f(1) > 0$

$$p = \frac{1+0}{2} = 0.5$$

$$f(0.5) = 2 * 0.5 - 1 = 0$$

$p = 0.5$ is a root

n	a	b	p	$f(p)$
1	0	2	1	1
2	0	1	0.5	0

Ex. (3):- Find the approximate positive value of the root of the equation $f\ (x) = cosx - x^2, x \in [0,1]$ by using bisection method with error $\varepsilon = 0.01$

Solu. first, we must determine a & b

$f(x) = 0 \Rightarrow \cos x = x^2$

$a = 0, b = 1$

$$f\ (0) = cos0 - 0 = 1$$

$$f(1) = cos(1 * \frac{180}{\pi}) - 1 = -0.46$$

$$f(0)f(1) < 0$$

المحاضرة الثانية Bisection method

n	A	b	p	f(p)
1	0	1	0.5	0.628
2	0.5	1	0.75	0.169
3	0.75	1	0.875	-0.125
4	0.75	0.875	0.813	0.026
5	0.813	0.875	0.844	-0.048
6	0.813	0.844	0.829	-0.011
7	0.813	0.829	0.821	0.007

* $p = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$

$$f(0.5) = cos(0.5 * \frac{180}{\pi}) - (0.5)^2 = 0.628$$

$$f(1)f(0.5) < 0$$

* $p = \frac{0.5+1}{2} = 0.75$

$$f(0.75) = cos(0.75 * \frac{180}{\pi}) - (0.75)^2 = 0.169$$

$$f(1)f(0.75) < 0$$

* $p = \frac{0.75+1}{2} = 0.875$

$$f(0.875) = cos(0.875 * \frac{180}{\pi}) - (0.875)^2 = -0.125$$

$$f(0.75)f(0.875) < 0$$

* $p = \frac{0.75+0.875}{2} = 0.813$

$$f(0.813) = cos(0.813 * \frac{180}{\pi}) - (0.813)^2 = 0.026$$

$$f(0.813)f(0.875) < 0$$

* $p = \frac{0.813+0.875}{2} = 0.844$

$$f(0.844) = cos(0.844 * \frac{180}{\pi}) - (0.844)^2 = -0.048$$

$$f(0.813)f(0.844) < 0$$

$* \ p = \frac{0.813 + 0.844}{2} = 0.829$

$$f(0.829) = cos(0.829 * \frac{180}{\pi}) - (0.829)^2 = -0.011$$

$$f(0.813)f(0.829) < 0$$

$* \ p = \frac{0.813 + 0.829}{2} = 0.821$

$$f(0.821) = cos(0.821 * \frac{180}{\pi}) - (0.821)^2 = 0.007$$

$$|0.821 - 0.829| = 0.008 < 0.01$$

$\therefore$ the root is $p = 0.821$

Theorem: Let $f \in c[a,b]$ and suppose $f(a)f(b) < 0$, the bisection method generates a sequence $\{p_n\}_{n=1}^{\infty} \to p$ with property $|p_n - p| \le \frac{b-a}{2^n}, n \ge 1$

Problem: Determine approximates how many iterations are needed (necessary) to solve $f(x) = cosx - x2 = 0, x \in [0,1]$ with accuracy $\varepsilon = 0.01$

$|p_n - p| \le \frac{b-a}{2^n}, \ |p_n - p| \le \varepsilon$

$$\Rightarrow \frac{b-a}{2^n} < \varepsilon \Rightarrow \frac{1-0}{2^n} = \frac{1}{2^n} = 2^{-n} < 10^{-2}$$

$$\Rightarrow \log_{10} 2^{-n} < \log_{10} 10^{-2}$$

$$\Rightarrow -n \log_{10} 2 < -2 \log_{10} 10$$

$$\Rightarrow -n \log_{10} 2 < -2$$

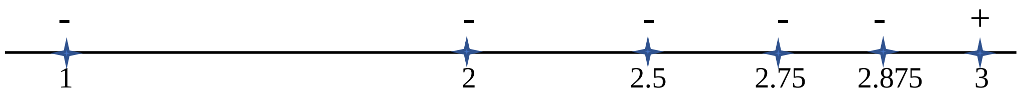
$$n > \frac{2}{\log_{10} 2} = 6.6$$

$$n \cong 7$$

H.W

1. Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-1}, $x \in [1,3]$.

Solution:



n	A	b	p	f(p)
1	1	3	2	-17
2	2	3	2.5	-9.375
3	2.5	3	2.75	-4.203
4	2.75	3	2.875	-1.236
5	2.875	3	2.938	0.36

2. Find an approximation to $\sqrt{3}$ correct to within 10^{-1}, $x \in [1,2]$.

Solution:

n	A	b	p	f(p)
1	1	2	1.5	-0.75
2	1.5	2	1.75	0.063
3	1.5	1.75	1.625	-0.359
4	1.625	1.75	1.688	-0.151

3. Find the root $f(x) = x^3 + 4x^2 - 10$, $x \in [1,2]$ by using bisection method with error $\varepsilon = 10^{-3}$.

4. Find the approximation value of the root of

a. $f(x) = xe^x - 1$ with $\varepsilon = 0.005$

b. $f(x) = e^x - 2$, $I = [0,1]$, $\varepsilon = 0.01$

Algorithm (Bisection Method)

```
F=@(x)cos(x)-exp(x)*x^2+2;
a=0; b=1; imax=35; tol=0.0005;
Fa=F(a); Fb=F(b);
disp ('Bisection Method:');
if Fa*Fb>0
   disp('Error:The function has the same sing at points a and b.')
else
   disp('iteration    a           b           (xNS)        f(xNS)     Tolerance')
   for i=1:imax
      xNS=(a+b)/2;
      toli=(b-a)/2;
      FxNS=F(xNS);
      fprintf('%3i   %11.6f   %11.6f   %11.6f   %11.6f   %11.6f\n',i,a,b,xNS,FxNS,toli)
      if FxNS == 0
        fprintf('An exact solution x=%11.6f was found',xNS)
        break
      end
      if toli<tol
        break
      end
      if i == imax
        fprintf('Solution was not obtained in %i iterations',imax)
        break
      end
      if F(a)*FxNS<0
        b=xNS;
      else
        a=xNS
      end
   end
end
```

Method of False Position

The method is also called linear interpolation method or chord method or regula-falsi method. At the start of all iterations of the method, we require the interval in which the root lies. Let the root of the equation $f(x) = 0$, lie in the interval (x_{k-1}, x_k), that is, $f_{k-1} f_k < 0$, where $f(x_{k-1}) = f_{k-1}$, and $f(x_k) = f_k$. Then, $P(x_{k-1}, f_{k-1}), Q(x_k, f_k)$ are points on the curve $f(x) = 0$. Draw a straight line joining the points P and Q. We approximate the curve in this interval by the chord, that is, $f(x) \approx ax + b$. The next approximation to the root is given by $x = -b/a$. Since the chord passes through the point P and Q, we get

$$f_{k-1} = ax_{k-1} + b, \text{ and } f_k = ax_k + b.$$

Subtracting the two equations, we get

$$f_k - f_{k-1} = a(x_k - x_{k-1}), \text{or } a = \frac{f_k - f_{k-1}}{x_k - x_{k-1}}.$$

The second equation givens $b = f_k - ax_k$.

Hence, the next approximation is given by

$$x_{k+1} = -\frac{b}{a} = -\frac{f_k - ax_k}{a} = x_k - \frac{f_k}{a} = x_k - \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right) f_k$$

Therefore, in this case, in actual computations, the method behaves like

$$x_{k+1} = \frac{x_k f_1 - x_1 f_k}{f_1 - f_k}, k = 1, 2, \ldots$$

False Position method (regula falsi method) Steps (Rule)

Step 1: Find points x_0 and x_1 such that $x_0 < x_1$ and $f(x_0).f(x_1) < 0$.

Step 2: Take the interval $[x_0, x_1]$ and find next value

$$x_2 = x_0 - f(x_0).\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

Step 3: If $f(x_2) = 0$ then x_2 is an exact root,

else if $f(x_0).f(x_2) < 0$ then $x_1 = x_2$,

else if $f(x_2).f(x_1) < 0$ then $x_0 = x_2$.

Step 4: Repeat steps 2 & 3 until $f(x_i) = 0$ or $|f(x_i)| \leq$ Accuracy.

Example 1: Locate the intervals which contain the positive real roots of the equation $x^3 - 3x + 1 = 0$. Obtain these roots correct to three decimal places, using the method of false position.

Solution We form the following table of values for the function $f(x)$.

There is one positive real root in the interval (1, 2).

$$x_0 = 1, x_1 = 2, f_0 = f(x_0) = f(1) = -1, f_1 = f(x_1) = f(2) = 3.$$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{3 - 2(-1)}{3 - (-1)} = 1.25, f(x_2) = f(1.25) = -0.796875.$$

Since $f(1.25)f(2) < 0$, the root lies in the interval (1.25,2).

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{1.25(3) - 2(-0.796875)}{3 - (-0.796875)} = 1.407407,$$

$$f(x_3) = f(1.407407) = -0.434437.$$

Since $f(1.407407)f(2) < 0$, the root lies in the interval (1.407407,2).

$$x_4 = \frac{x_3 f_1 - x_1 f_3}{f_1 - f_3} = \frac{1.407407(3) - 2(-0.434437)}{3 - (-0.434437)} = 1.482367,$$

$$f(x_4) = f(1.482367) = -0.189730.$$

Since $f(1.482367)f(2) < 0$, the root lies in the interval (1.482367,2).

$$x_5 = \frac{x_4 f_1 - x_1 f_4}{f_1 - f_4} = \frac{1.482367(3) - 2(-0.18973)}{3 - (-0.18973)} = 1.513156,$$

$$f(x_5) = f(1.513156) = -0.074884.$$

Since $f(1.513156)f(2) < 0$, the root lies in the interval (1.513156,2).

$$x_6 = \frac{x_5 f_1 - x_1 f_5}{f_1 - f_5} = \frac{1.513156(3) - 2(-0.074884)}{3 - (-0.74884)} = 1.525012.$$

$$f(x_6) = f(1.525012) = -0.028374.$$

Since $f(1.525012)f(2) < 0$, the root lies in the interval (1.525012,2).

$$x_7 = \frac{x_6 f_1 - x_1 f_6}{f_1 - f_6} = \frac{1.525012(3) - 2(-0.028374)}{3 - (-0.028374)} = 1.529462.$$

$$f(x_7) = f(1.529462) = -0.010584.$$

Since $f(1.529462)f(2) < 0$, the root lies in the interval (1.529462,2).

$$x_8 = \frac{x_7 f_1 - x_1 f_7}{f_1 - f_7} = \frac{1.529462(3) - 2(-0.010586)}{3 - (-0.10586)} = 1.531116.$$

$$f(x_8) = f(1.531116) = -0.003928.$$

Since $f(1.531116)f(2) < 0$, the root lies in the interval (1.531116,2).

$$x_9 = \frac{x_8 f_1 - x_1 f_8}{f_1 - f_8} = \frac{1.531116(3) - 2(-0.003928)}{3 - (-0.003928)} = 1.531729.$$

$$f(x_9) = f(1.531729) = -0.001454.$$

Since $f(1.531729)f(2) < 0$, the root lies in the interval (1.531729,2).

$$x_{10} = \frac{x_9 f_1 - x_1 f_9}{f_1 - f_9} = \frac{1.531729(3) - 2(-0.001454)}{3 - (-0.001454)} = 1.531956.$$

Now, $|x_{10} - x_9| = |1.531956 - 1.531729| \approx 0.000227 < 0.0005$.

The root has been computed correct to three decimal places. The required root can be taken as $x \approx x_{10} = 1.531956$. Note that the left end point *x* = 2 is fixed for all iterations.

Example 2: Find the root correct to two decimal places of the equation $xe^x = cos\, x$, using the method of false position.

Solution: Define $f(x) = cosx - xe^x = 0$. There is no negative root for the equation. We have $f(0) = 1, f(1) = cos1 - e = -2.17798$.

A root of the equation lies in the interval (0,1). Let $x_0 = 0, x_1 = 1$. Using the method of false position, we obtain the following results.

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{0 - 1(1)}{-2.17798 - 1} = 0.31467, f(x_2) = f(0.31467) = 0.51986.$$

Since $f(0.31467)f(1) < 0$, the root lies in the interval (0.31467,1).

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{0.31467(-2.17798) - 1(0.51986)}{-2.17798 - 0.51986} = 0.44673,$$

$$f(x_3) = f(0.44673) = 0.20354.$$

Since $f(0.44673)f(1) < 0$, the root lies in the interval (0.44673,1).

$$x_4 = \frac{x_3 f_1 - x_1 f_3}{f_1 - f_3} = \frac{0.44673(-2.17798) - 1(0.20354)}{-2.17798 - 0.20354} = 0.49402,$$

$$f(x_4) = f(0.49402) = 0.07079.$$

Since $f(0.49402)f(1) < 0$, the root lies in the interval (0.49402,1).

$$x_5 = \frac{x_4 f_1 - x_1 f_4}{f_1 - f_4} = \frac{0.49402(-2.17798) - 1(0.07079)}{-2.17798 - 0.07079} = 0.50995,$$

$$f(x_5) = f(0.50995) = 0.02360.$$

Since $f(0.50995)f(1) < 0$, the root lies in the interval (0.50995,1).

$$x_6 = \frac{x_5 f_1 - x_1 f_5}{f_1 - f_5} = \frac{0.50995(-2.17798) - 1(0.0236)}{-2.17798 - 0.0236} = 0.51520,$$

$$f(x_6) = f(0.51520) = 0.00776.$$

Since $f(0.51520)f(1) < 0$, the root lies in the interval (0.51520,1).

$$x_7 = \frac{x_6 f_1 - x_1 f_6}{f_1 - f_6} = \frac{0.5152(-2.17798) - 1(0.00776)}{-2.17798 - 0.00776} = 0.51692.$$

Now, $|x_7 - x_6| = |0.51692 - 0.51520| \approx 0.00172 < 0.005.$

The root has been computed correct to two decimal places. The required root can be taken as $x \approx x_7 = 0.51692$. Note that the left end point $x = 1$ is fixed for all iterations.

H.W

Find a root of an equation $f(x) = x^3 - x - 1$ and the interval (1, 2) using false position method.

Algorithm (False Position)

```
f=@(c)(c)^3-3*(c)+1;
a=0;
b=1;
delta=0.0005;
epsilon=0.0005;
max1=10;
ya=feval(f,a);
yb=feval(f,b);
if ya*yb>0
  disp('Note: f(a)*f(b)>0'),break,end
for k=1:max1
   dx=yb*(b-a)/(yb-ya);
   c=b-dx;
   ac=c-a;
   yc=feval(f,c);
   if yc==0,break;
   elseif yb*yc>0
          b=c;
          yb=yc;
       else
          a=c;
          ya=yc;
        end
      dx=min(abs(dx),ac);
   if abs(dx)<delta,break,end
   if abs(yc)<delta,break,end
end
disp('Regula Falsi Method');
c;
err=abs(b-a)/2;
yc=feval(f,c);
disp(c)
```

Newton-Raphson method

Suppose f(x) be twice continuously differentiable function on [a,b], i.e $f \in c^2[a,b]$. Let $f'(x) \neq 0$, let $\bar{x}$ be any point in [a,b] such that $f'(\bar{x}) \neq 0$, let p be the root of f(x), i.e $f(p) = 0$ and $|\bar{x} - p|$ be small enough, i.e $|\bar{x} - p| < \varepsilon$. Then (Taylor expansion for f(x) about $x = \bar{x}$)

$$f(x) \cong f(\bar{x}) + (x - \bar{x}) f'(\bar{x}) + \frac{(x - \bar{x})^2}{2!} f''(\eta), \eta \in (x, \bar{x})$$

since $f(p) = 0$ with $x = p$ gives

$$0 = f(p) \cong f(\bar{x}) + (p - \bar{x}) f'(\bar{x}) + \frac{(p - \bar{x})^2}{2!} f''(\eta), \eta \in (p, \bar{x})$$

$$\Rightarrow -(p - \bar{x}) f'(\bar{x}) = f(\bar{x})$$

$\Rightarrow -(p - \bar{x}) = \dfrac{f(\bar{x})}{f'(\bar{x})} \Rightarrow p = \bar{x} - \dfrac{f(\bar{x})}{f'(\bar{x})}$ Newton Raphson p is the approximate root for f(x), i.e $f(p) = 0$

$\bar{x}$ is any initial point $\in [a,b]$ and $\bar{x}$ closed to p.

The steps of solve Newton Raphson

$n = 1$

$p_1 = p_0 - \dfrac{f(p_0)}{f'(p_0)}, p_0 \in [a,b], p_0$ is initial point

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)}$$

$\vdots$

$p_n = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}, n \geq 1$ Newton Raphson method

Stopping criterion

1. $|p_n - p_{n-1}| < \varepsilon$

2. $|f(p_n)| < \varepsilon$

3. $\dfrac{|p_n - p_{n-1}|}{|p_n|} < \varepsilon,\ p_n \neq 0$

Ex(1):- Find the root of the eq. $x^2 + 2.1x - 1 = 0$ by N.R with $x_0 = 0.5$

Solu. $f'(x) = 2x + 2.1$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})},\ n \geq 1$$

$$x_1 = 0.5 - \frac{0.3}{3.1} = 0.4032$$

$$x_2 = 0.4032 - \frac{0.0093}{2.9064} = 0.4000$$

$$x_3 = 0.4000 - \frac{0}{2.9} = 0.4$$

$\therefore x_3 = 0.4$ is a root

Ex(2):- Use N.R to estimate $\sqrt{7}$ take $x_0 = 2.5$

Solu. Suppose $x = \sqrt{7} \Rightarrow x^2 = 7 \Rightarrow x^2 - 7 = 0$

$$f(x) = x^2 - 7 \Rightarrow f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} = x_n - \frac{x_n^2 - 7}{2x_n} = x_n - \left(\frac{x_n^2}{2x_n} - \frac{7}{2x_n}\right) = x_n - \left(\frac{x_n}{2} - \frac{7}{2x_n}\right) = x_n - \frac{x_n}{2}$$

$$-\frac{7}{2x_n} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

$$x_1 = \frac{1}{2}\left(2.5 + \frac{7}{2.5}\right) = 2.65$$

$$x_2 = \frac{1}{2}\left(2.65 + \frac{7}{2.65}\right) = 2.64576$$

$$x_3 = \frac{1}{2}\left(2.64576 + \frac{7}{2.64576}\right) = 2.64575$$

$$x_4 = \frac{1}{2}\left(2.64575 + \frac{7}{2.64575}\right) = 2.64575$$

$\therefore \sqrt{7} = 2.64575$ متطابقتان لخمس مراتب عشرية x_3, x_4

باستخدام طريقة نيوتن رافسون أوجد

١. صيغه عامة لإيجاد الجذر التربيعي لأي عدد

٢. صيغه عامة لإيجاد مقلوب العدد a بدون أي قسمة

1. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

2. $x = \frac{1}{a} \Rightarrow a = \frac{1}{x} \Rightarrow a - \frac{1}{x} = 0$

$$\therefore f(x) = a - \frac{1}{x} \Rightarrow f'(x) = \frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{a - \frac{1}{x_n}}{\frac{1}{x_n^2}} = x_n - \frac{\frac{ax_n - 1}{x_n}}{\frac{1}{x_n^2}} = x_n - x_n(ax_n - 1) = x_n\left(1 - (ax_n - 1)\right) = x_n\left(2 - ax_n\right)$$

Ex(3):-Find $\frac{1}{7}, x_0 = 0.1$

Solu. $x_{n+1} = x_n\left(2 - ax_n\right)$

$$x_1 = x_0(2 - 7x_0) = 0.1(2 - 0.7) = 0.13$$
$$x_2 = 0.13(2 - 7 * 0.13) = 0.13(2 - 0.91) = 0.1417$$
$$x_3 = 0.14285$$
$$x_4 = 0.14286$$
$$x_5 = 0.14286$$
$$\therefore \frac{1}{7} = 0.14286$$

H.W

1. $f(x) = x^3 - 4x^2 + 10, x_0 = -1$

2. Calculate $\sqrt{5}, x_0 = 2$

$f(x) = 2x - 1 - 2\sin x, x_0 = 2$

Secant Method

Since $(p-\bar{x})^2$ is deleted then unless p is sufficiently close to $\bar{x}$ N-R method may not converge to the root p ($\bar{x}$ be an initial point $p_0 = \bar{x}$).

N-R is extremely powerful technique, but has a major difficulty, the need to know the value of $f'(x)$ at each iterative point.

$$\because\ f'(p_{n-1}) = \lim_{x \to p_{n-2}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}$$

$$\therefore\ f'(p_{n-1}) \cong \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$

using this approximation for $f'(p_{n-1})$ in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

The technique using this formula is called the secant method.

Ex (1):- use secant method to find the smallest root of $f(x) = x\ln x - 1$ with error $\varepsilon = 0.002$.

Solu. $x_0 = 1,\ f(x_0) = 1\ln 1 - 1 = 1$

$x_1 = 2,\ f(x_1) = 2\ln 2 - 1 = 0.3863$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_2 = 2 - \frac{0.3863(2-1)}{0.3863+1} = 2 - 0.2787 = 1.7213$$

$$x_3 = 1.7213 - \frac{-0.0652(-0.2787)}{-0.0652 - 0.3863} = 1.7213 - \frac{-0.0182}{-0.4515} = 1.7616$$

$$x_4 = 1.7616 - \frac{-0.0025(0.0403)}{-0.025 + 0.0652} = 1.7616 - \frac{-0.0001}{0.0627} = 1.7632$$

$$|x_4 - x_3| = 0.0016$$

$\therefore x_4 = 1.7632$ is a root.

Ex(2):- Find the smallest positive roots of the eq. $f(x) = 3x - 1 - \cos x$ with $\varepsilon = 0.003$.

Solu. $3x - 1 - \cos x = 0 \Rightarrow 3x - 1 = \cos x$

$$x_0 = 0,\ f(0) = -2$$
$$x_1 = 1,\ f(1) = 2 - 0.541 = 1.459$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_2 = 1 - \frac{1.459(1-0)}{1.459 + 2} = 1 - 0.4218 = 0.5782$$

$$x_3 = 0.5782 - \frac{-0.1028(0.5782 - 1)}{-0.1028 - 1.459} = 0.5782 - 0.0278 = 0.606$$

$$x_4 = 0.606 - \frac{0.0039(0.0278)}{0.0039 + 0.1028} = 0.606 - 0.001 = 0.605$$

$|x_4 - x_3| = 0.001 < 0.003$

$\therefore x_4 = 0.605$ is a root.

H.W

1. $f(x) = \cos x - x^2, \varepsilon = 0.003$

2. $f(x) = \cos x + x^2, \varepsilon = 0.003$

Algorithm (Secant Method)

```
function [p1,err,k,y]=secant(f,p0,p1,delta,epsilon,max1)
clear;clc;
for k=1:max1
   p2=p1-feval(f,p1)*(p1-p0)/(feval(f,p1)-feval(f,p0));
   err=abs(p2-p1);
   relerr=2*err/(abs(p2)+delta);
   p0=p1;
   p1=p2;
   y=feval(f,p1);
   if (err<delta)|(relerr<delta)|(abs(y)<epsilon),break,end
end
```

Algorithm (Newton Raphson)

```
f=@(p0) (p0)^4+(p0)^2+1;
df=@(p0) 4*(p0)^3+2*(p0);
p0=0.5;
delta=0.0005;
epsilon=0.0005;
max1=10;
for k=1:max1
   p1=p0-feval(f,p0)/feval(df,p0);
   err=abs(p1-p0);
   relerr=2*err/(abs(p1)+delta);
   p0=p1;
   y=feval(f,p0);
   if (err<delta)|(relerr<delta)|(abs(y)<epsilon),break,end
end
disp('Newton Raphson Method');
disp(p0)
```

Numerical solution of set of equation

Consider a system of n linear algebraic equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

where a_{ij}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$, are the known coefficients, b_i, $i = 1, 2, \dots, n$, are the known right hand side values and x_i, $i = 1, 2, \dots, n$ are the unknowns to be determined.

In matrix notation we write the system as

$$\mathbf{Ax} = \mathbf{b} \qquad (1)$$

where $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}.$$

The matrix $[\mathbf{A} \mid \mathbf{b}]$, obtained by appending the column $\mathbf{b}$ to the matrix $\mathbf{A}$ is called the *augmented matrix*. That is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$$

Example: How to Write a System in Matrix Form

$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$
$$10x_1 - x_2 + 2x_3 = 4$$

Solution: We have the matrices as $Ax = B$

$$A = \begin{bmatrix} 1 & 10 & -1 \\ 2 & 3 & 20 \\ 10 & -1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 10 & -1 \\ 2 & 3 & 20 \\ 10 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

We have the augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 7 \\ 10 & -1 & 2 & 4 \end{array}\right]$$

We define the following.

(i) The system of equations (1) is *consistent* (has at least one solution), if

$$\text{rank}(\mathbf{A}) = \text{rank}[\mathbf{A} \mid \mathbf{b}] = r.$$

If $r = n$, then the system has unique solution.

If $r < n$, then the system has $(n - r)$ parameter family of infinite number of solutions.

(ii) The system of equations (1) is *inconsistent* (has no solution) if

$$\text{rank}(\mathbf{A}) \neq \text{rank}[\mathbf{A} \mid \mathbf{b}].$$

We assume that the given system is consistent.

The methods of solution of the linear algebraic system of equations (1) may be classified as direct and iterative methods.

(*a*) *Direct methods* produce the exact solution after a finite number of steps (disregarding the round-off errors). In these methods, we can determine the total number of operations (additions, subtractions, divisions and multiplications). This number is called the *operational count* of the method.

(*b*) *Iterative methods* are based on the idea of successive approximations. We start with an initial approximation to the solution vector $\mathbf{x} = \mathbf{x}_0$, and obtain a sequence of approximate vectors $\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_k, ...$, which in the limit as $k \to \infty$, converge to the exact solution vector $\mathbf{x}$.

Now, we derive some direct methods.

Direct Methods

If the system of equations has some special forms, then the solution is obtained directly. We consider two such special forms.

(*a*) Let $\mathbf{A}$ be a diagonal matrix, $\mathbf{A} = \mathbf{D}$. That is, we consider the system of equations $\mathbf{Dx} = \mathbf{b}$ as

$$\begin{aligned} a_{11}x_1 \qquad\qquad &= b_1 \\ a_{22}x_2 \qquad &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad &\cdots \\ a_{n-1,\, n-1}\, x_{n-1} &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned} \qquad (2)$$

This system is called a *diagonal system of equations*. Solving directly, we obtain

$$x_i = \frac{b_i}{a_{ii}}, \; a_{ii} \neq 0, \; i = 1, 2, ..., n. \qquad (3)$$

(*b*) Let $\mathbf{A}$ be an upper triangular matrix, $\mathbf{A} = \mathbf{U}$. That is, we consider the system of equations $\mathbf{Ux} = \mathbf{b}$ as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots \quad \dots \quad &+ a_{1n}\,x_n = b_1 \\ a_{22}x_2 + \dots \quad \dots \quad &+ a_{2n}\,x_n = b_2 \\ \dots \quad \dots \quad \dots \quad &\dots \\ a_{n-1,\,n-1}\,x_{n-1} &+ a_{n-1,\,n}\,x_n = b_{n-1} \\ &a_{nn}x_n = b_n \end{aligned} \tag{4}$$

This system is called an *upper triangular system of equations.* Solving for the unknowns in the order $x_n, x_{n-1}, \dots, x_1$, we get

$$x_n = b_n/a_{nn},$$

$$x_{n-1} = (b_{n-1} - a_{n-1,\,n}x_n)/a_{n-1,\,n-1},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$x_1 = \frac{\left(b_1 - \sum_{j=2}^{n} a_{1,j}x_j\right)}{a_{11}} = \left(b_1 - \sum_{j=2}^{n} a_{1,j}\,x_j\right)\Big/ a_{11} \tag{5}$$

The unknowns are obtained by back substitution and this procedure is called the *back substitution* method.

Therefore, when the given system of equations is one of the above two forms, the solution is obtained directly.

Before we derive some direct methods, we define elementary row operations that can be performed on the rows of a matrix.

Gauss Elimination Method

The method is based on the idea of reducing the given system of equations **Ax = b,** to an upper triangular system of equations **Ux = z,** using elementary row operations. We know that these two systems are equivalent. That is, the solutions of both the systems are identical. This reduced system **Ux = z,** is then solved by the back substitution method to obtain the solution vector **x**.

We illustrate the method using the 3 × 3 system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (1)$$

We write the augmented matrix [**A** | **b**] and reduce it to the following form

$$[\mathbf{A}|\mathbf{b}] \xrightarrow{\text{Gauss elimination}} [\mathbf{U}|\mathbf{z}]$$

The augmented matrix of the system (1) is

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \qquad (2)$$

First stage of elimination

We assume $a_{11} \neq 0$. This element a_{11} in the 1 × 1 position is called the *first pivot*. We use this pivot to reduce all the elements below this pivot in the first column as zeros. Multiply the first row in (2) by a_{21}/a_{11} and a_{31}/a_{11} respectively and subtract from the second and third rows. That is, we are performing the elementary row operations $R_2 - (a_{21}/a_{11})R_1$ and $R_3 - (a_{31}/a_{11})R_1$ respectively. We obtain the new augmented matrix as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right] \qquad (3)$$

where
$$a_{22}^{(1)} = a_{22} - \left(\frac{a_{21}}{a_{11}}\right)a_{12},\ a_{23}^{(1)} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)a_{13},\ b_2^{(1)} = b_2 - \left(\frac{a_{21}}{a_{11}}\right)b_1,$$

$$a_{32}^{(1)} = a_{32} - \left(\frac{a_{31}}{a_{11}}\right)a_{12},\ a_{33}^{(1)} = a_{33} - \left(\frac{a_{31}}{a_{11}}\right)a_{13},\ b_3^{(1)} = b_3 - \left(\frac{a_{31}}{a_{11}}\right)b_1.$$

Second stage of elimination

We assume $a_{22}^{(1)} \neq 0$. This element $a_{22}^{(1)}$ in the 2 × 2 position is called the *second pivot*. We use this pivot to reduce the element below this pivot in the second column as zero. Multiply the second row in (3) by $a_{32}^{(1)}/a_{22}^{(1)}$ and subtract from the third row. That is, we are performing the elementary row operation $R_3 - (a_{32}^{(1)}/a_{22}^{(1)})R_2$. We obtain the new augmented matrix as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right] \tag{4}$$

where $$a_{33}^{(2)} = a_{33}^{(1)} - \left(\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right)a_{23}^{(1)},\ b_3^{(2)} = b_3^{(1)} - \left(\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right)b_2^{(1)}.$$

The element $a_{33}^{(2)} \neq 0$ is called the *third pivot*. This system is in the required upper triangular form $[\mathbf{U}|\mathbf{z}]$. The solution vector $\mathbf{x}$ is now obtained by back substitution.

From the third row, we get $x_3 = b_3^{(2)}/a_{33}^{(2)}$.

From the second row, we get $x_2 = (b_2^{(1)} - a_{23}^{(1)} x_3)/a_{22}^{(1)}$.

From the first row, we get $x_1 = (b_1 - a_{12}x_2 - a_{13} x_3)/a_{11}$.

Procedure

if n is large it is difficult to find the solution by hand.

Augmented matrix $[A{:}\,B]$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Read $A(i,j), i = 1,\dots,n\ j = 1,\dots,n+1$

We will eliminate the first element in the ith row (except the first row)

$$a_{21} = 0 \Rightarrow \left(Row2 - \frac{a_{21}}{a_{11}}Row1\right) \Rightarrow a_{21} - \frac{a_{21}}{a_{11}}a_{11} = 0$$

$$Row3 - \frac{a_{31}}{a_{11}}Row1$$

In general, let a_{i1} be the element of row i

$$Rowi - \frac{a_{i1}}{a_{11}}Row1, i = 2,\dots,n$$

Ex.1: Solve the following linear system using Gaussian elimination:

$$x + 2y = 3$$

$$2x + 3y = 1$$

Transcribing the system of linear equations into an augmented matrix we obtain the matrix equation:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array}\right] \xrightarrow[2R_1 \to R_1]{} \left[\begin{array}{cc|c} 2 & 4 & 6 \\ 2 & 3 & 1 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_1]{} \left[\begin{array}{cc|c} 2 & 4 & 6 \\ 0 & -1 & -5 \end{array}\right]$$

We can clearly see from the resulting matrix that $-y = -5$ meaning that $y = 5$. We substitute this value into the equation equivalent to the first row of the resulting augmented matrix to solve for the variable x:

$$2x + 4y = 6 \to 2x + 4\,(5) = 2x + 20 = 6 \to 2x = -14 \to x = -7$$

Therefore, the final solution for the system of linear equations is:

$$x = -7 \quad and \quad y = 5$$

Ex.2: If we were to have the following system of linear equations containing three equations for three unknowns:

$$x + y + z = 3$$

$$x + 2y + 3z = 0$$

$$x + 3y + 2z = 3$$

We know from our lesson on representing a linear system as a matrix that we can represent such system as an augmented matrix like the one below:

$$\begin{array}{r} x + y + z = 3 \\ x + 2y + 3z = 0 \\ x + 3y + 2z = 3 \end{array} \quad \to \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right]$$

Let us row-reduce (use Gaussian elimination) so we can simplify the matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[2R_2 \to R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & -6 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_3]{}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & -3 & 6 \end{array}\right] \begin{array}{r} x + y + z = 3 \\ 2y + 4z = -6 \\ -3z = 6 \end{array}$$

$$having : z = -2$$

$$2y + 4z = -6 \to 2y + 4\left(-2\right) = 2y - 8 = -6$$

$$2y = 2 \to y = 1$$

Applying the values of y and z to the first equation

$$x + y + z = 3 \to x + (1) + (-2) = x - 1 = 3 \to x = 4$$

And the final solution for the system is:

$$x = 4, \;\; y = 1, \;\; z = -2$$

Ex.3: Solve the following linear system using Gaussian elimination.

$$x + 3y + 3z = 2$$

$$3x + 9y + 3z = 3$$

$$3x + 6y + 6z = 4$$

We transcribe the linear system as an augmented matrix and then we start the Gaussian elimination process:

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 3 & 9 & 3 & 3 \\ 3 & 6 & 6 & 4 \end{array}\right] \xrightarrow[R_2 - R_3 \to R_2]{} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & 3 & -3 & -1 \\ 3 & 6 & 6 & 4 \end{array}\right] \xrightarrow[\frac{1}{3}R_3 \to R_3]{}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & 3 & -3 & -1 \\ 1 & 2 & 2 & 4/3 \end{array}\right] \xrightarrow[R_1 - R_3 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 1 & 1 & 2/3 \end{array}\right] \xrightarrow[3R_3 \to R_3]{}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 3 & 3 & 2 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 6 & 3 \end{array}\right]$$

From which we can see that the last row provides the equation: $6z = 3$ and therefore $z = 1/2$. We substitute this in the equations resulting from the second and first row (in that order) to calculate the values of the variables x and y:

$$Having\ \ z = \tfrac{1}{2}$$

$$\mathbf{3y - 3z = -1 \to 3y - 3\left(\tfrac{1}{2}\right) = 3y - \tfrac{3}{2} = -1}$$

$$\mathbf{3y = \tfrac{1}{2} \to y = \tfrac{1}{6}}$$

Applying the values of y and z to the first equation:

$$x + 3y + 3z = 2 \to x + 3\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{2}\right) = x + \tfrac{3}{6} + \tfrac{3}{2} = x + \tfrac{1}{2} + \tfrac{3}{2}$$

$$x + 2 = 2 \to x = 0$$

Therefore, the final solution to the system of linear equations is:

$$x = 0,\ \ y = \tfrac{1}{6}\ \ and\ \ z = \tfrac{1}{2}$$

Gauss-Jordan Method

The method is based on the idea of reducing the given system of equations **Ax = b,** to a diagonal system of equations **Ix = d,** where **I** is the identity matrix, using elementary row operations. We know that the solutions of both the systems are identical. This reduced system gives the solution vector **x**. This reduction is equivalent to finding the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

$$[\mathbf{A}|\mathbf{b}] \xrightarrow{\text{Gauss-Jordan method}} [\mathbf{I} \mid \mathbf{X}]$$

In this case, after the eliminations are completed, we obtain the augmented matrix for a 3 × 3 system as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right] \qquad (1)$$

and the solution is $x_i = d_i,\ i = 1, 2, 3.$

Elimination procedure The first step is same as in Gauss elimination method, that is, we make the elements below the first pivot as zeros, using the elementary row transformations. From the second step onwards, we make the elements below and above the pivots as zeros using the elementary row transformations. Lastly, we divide each row by its pivot so that the final augmented matrix is of the form (1). Partial pivoting can also be used in the solution. We may also make the pivots as 1 before performing the elimination.

Let us illustrate the method.

Example *Solve the following system of equations*

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

using the Gauss-Jordan method (i) without partial pivoting, (ii) with partial pivoting.

Solution We have the augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right]$$

We perform the following elementary row transformations and do the eliminations.

$$R_2 - 4R_1, R_3 - 3R_1 : \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 2 \\ 0 & 2 & 0 & 1 \end{array}\right].$$

$$R_1 + R_2, R_3 + 2R_2 : \left[\begin{array}{ccc|c} 1 & 0 & -4 & 3 \\ 0 & -1 & -5 & 2 \\ 0 & 0 & -10 & 5 \end{array}\right].$$

$$R_1 - (4/10)R_3, R_2 - (5/10)\,R_3 \ : \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1/2 \\ 0 & 0 & -10 & 5 \end{array}\right].$$

Now, making the pivots as 1, $((-R_2), (R_3/(-10)))$ we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right].$$

Therefore, the solution of the system is $x_1 = 1$, $x_2 = 1/2$, $x_3 = -1/2$.

Example Solve the following system by elimination method.

$$x + 3y = 7$$

$$3x + 4y = 11$$

Solution

We multiply the first equation by -3, and add it to the second equation.

$$\begin{array}{r} -3x - 9y = -21 \\ 3x + 4y = 11 \\ \hline -5y = -10 \end{array}$$

By doing this we transformed our original system into an equivalent system:

$$x + 3y = 7$$

$$-5y = -10$$

We divide the second equation by -5, and we get the next equivalent system.

$$x + 3y = 7$$

$$y = 2$$

Now we multiply the second equation by -3 and add to the first, we get

$$x = 1$$

$$y = 2$$

Example Solve the following system by the Gauss-Jordan method.

$$x + 3y = 7$$

$$3x + 4y = 11$$

Solution

The augmented matrix for the system is as follows.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \begin{bmatrix} x + 3y = 7 \\ 3x + 4y = 11 \end{bmatrix}$$

We multiply the first row by -3, and add to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & 10 \end{array}\right] \quad \begin{bmatrix} x + 3y = 7 \\ -5y = 10 \end{bmatrix}$$

We divide the second row by -5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \begin{bmatrix} x + 3y = 7 \\ y = 2 \end{bmatrix}$$

Finally, we multiply the second row by -3 and add to the first row, and we get,

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \begin{bmatrix} x = 1 \\ y = 2 \end{bmatrix}$$

Example Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x+y+z=5 \\ 2x+3y+5z=8 \\ 4x+5z=2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3}\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} A + B + 2C = 1 \\ 2A - B + D = -2 \\ A - B - C - 2D = 4 \\ 2A - B + 2C - D = 0 \end{cases}$$

Solution: We will perform row operations on the augmented matrix of the system until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right] \xrightarrow{R_4 - R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1/2 & 4 & -17/2 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array}\right] \xrightarrow{-\frac{1}{18}R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-8R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2-R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-\frac{3}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-2R_3,\ R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{A = 1, \quad B = 2, \quad C = -1, \quad D = -2.}$$

Iterative Methods

As discussed earlier, iterative methods are based on the idea of successive approximations. We start with an initial approximation to the solution vector $\mathbf{x} = \mathbf{x}_0$, to solve the system of equations $\mathbf{Ax} = \mathbf{b}$, and obtain a sequence of approximate vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots$, which in the limit as $k \to \infty$, converges to the exact solution vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. A general linear iterative method for the solution of the system of equations $\mathbf{Ax} = \mathbf{b}$, can be written in matrix form as

$$\mathbf{x}^{(k+1)} = \mathbf{H}\mathbf{x}^{(k)} + \mathbf{c}, \quad k = 0, 1, 2, \ldots \tag{1}$$

where $\mathbf{x}^{(k+1)}$ and $\mathbf{x}^{(k)}$ are the approximations for $\mathbf{x}$ at the $(k + 1)$th and kth iterations respectively. $\mathbf{H}$ is called the iteration matrix, which depends on $\mathbf{A}$ and $\mathbf{c}$ is a column vector, which depends on $\mathbf{A}$ and $\mathbf{b}$.

When to stop the iteration We stop the iteration procedure when the magnitudes of the differences between the two successive iterates of all the variables are smaller than a given accuracy or *error tolerance* or an error bound ε, that is,

$$\left| x_i^{(k+1)} - x_i^{(k)} \right| \le \varepsilon, \quad \text{for all } i. \tag{2}$$

For example, if we require two decimal places of accuracy, then we iterate until $\left| x_i^{(k+1)} - x_i^{(k)} \right| < 0.005$, for all i. If we require three decimal places of accuracy, then we iterate until $\left| x_i^{(k+1)} - x_i^{(k)} \right| < 0.0005$, for all i.

Convergence property of an iterative method depends on the iteration matrix $\mathbf{H}$.

Now, we derive two iterative methods for the solution of the system of algebraic equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (3)$$

Gauss-Jacobi Iteration Method

Sometimes, the method is called *Jacobi method*. We assume that the pivots $a_{ii} \neq 0$, for all i. Write the equations as

$$a_{11}x_1 = b_1 - (a_{12}x_2 + a_{13}x_3)$$
$$a_{22}x_2 = b_2 - (a_{21}x_1 + a_{23}x_3)$$
$$a_{33}x_3 = b_3 - (a_{31}x_1 + a_{32}x_2)$$

The Jacobi iteration method is defined as

$$x_1^{(k+1)} = \frac{1}{a_{11}}[b_1 - (a_{12}x_2^{(k)} + a_{13}x_3^{(k)})]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}[b_2 - (a_{21}x_1^{(k)} + a_{23}x_3^{(k)})]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}[b_3 - (a_{31}x_1^{(k)} + a_{32}x_2^{(k)})], \quad k = 0, 1, 2, \ldots \tag{4}$$

Since, we replace the complete vector $\mathbf{x}^{(k)}$ in the right hand side of (4) at the end of each iteration, this method is also called the *method of simultaneous displacement*.

Remark *A sufficient condition for convergence of the Jacobi method is that the system of equations is diagonally dominant*, that is, the coefficient matrix **A** is diagonally dominant. We can verify that $| a_{ii} | \geq \sum_{j=1, i \neq j}^{n} |a_{ij}|$. This implies that convergence may be obtained even if the system is not diagonally dominant. If the system is not diagonally dominant, we may exchange the equations, if possible, such that the new system is diagonally dominant and convergence is guaranteed. However, such manual verification or exchange of equations may not be possible for large systems that we obtain in application problems. The necessary and sufficient condition for convergence is that the spectral radius of the iteration matrix **H** is less than one unit, that is, $\rho(\mathbf{H}) < 1$, where $\rho(\mathbf{H})$ is the largest eigen value in magnitude of **H.** Testing of this condition is beyond the scope of the syllabus.

Remark *How do we find the initial approximations to start the iteration*? If the system is diagonally dominant, then the iteration converges for any initial solution vector. If no suitable approximation is available, we can choose $\mathbf{x} = \mathbf{0}$, that is $x_i = 0$ for all i. Then, the initial approximation becomes $x_i = b_i/a_{ii}$, for all i.

Example *Solve the system of equations*

$$4x_1 + x_2 + x_3 = 2$$

$$x_1 + 5x_2 + 2x_3 = -6$$

$$x_1 + 2x_2 + 3x_3 = -4$$

using the Jacobi iteration method. Use the initial approximations as

(*i*) $x_i = 0, i = 1, 2, 3,$ (*ii*) $x_1 = 0.5, x_2 = -0.5, x_3 = -0.5.$

Perform five iterations in each case.

Solution Note that the given system is diagonally dominant. Jacobi method gives the iterations as

$$x_1^{(k+1)} = 0.25 [2 - (x_2^{(k)} + x_3^{(k)})]$$

$$x_2^{(k+1)} = 0.2 [-6 - (x_1^{(k)} + 2x_3^{(k)})]$$

$$x_3^{(k+1)} = 0.33333 [-4 - (x_1^{(k)} + 2x_2^{(k)})], \quad k = 0, 1, ...$$

We have the following results.

(*i*) $x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0.$

First iteration

$$x_1^{(1)} = 0.25\ [2 - (x_2^{(0)} + x_3^{(0)})] = 0.5,$$

$$x_2^{(1)} = 0.2\ [-6 - (x_1^{(0)} + 2x_3^{(0)})] = -1.2,$$

$$x_3^{(1)} = 0.33333\ [-4 - (x_1^{(0)} + 2x_2^{(0)})] = -1.33333.$$

Second iteration

$$x_1^{(2)} = 0.25\ [2 - (x_2^{(1)} + x_3^{(1)})] = 0.25\ [2 - (-1.2 - 1.33333)] = 1.13333,$$

$$x_2^{(2)} = 0.2\ [-6 - (x_1^{(1)} + 2x_3^{(1)})] = 0.2\ [-6 - (0.5 + 2(-1.33333))] = -0.76668,$$

$$x_3^{(2)} = 0.33333\ [-4 - (x_1^{(1)} + 2x_2^{(1)})] = 0.33333\ [-4 - (0.5 + 2(-1.2))] = -0.7.$$

Third iteration

$$x_1^{(3)} = 0.25\ [2 - (x_2^{(2)} + x_3^{(2)})] = 0.25\ [2 - (-0.76668 - 0.7)] = 0.86667,$$

$$x_2^{(3)} = 0.2\ [-6 - (x_1^{(2)} + 2x_3^{(2)})] = 0.2\ [-6 - (1.13333 + 2(-0.7))] = -1.14667,$$

$$x_3^{(3)} = 0.33333\ [-4 - (x_1^{(2)} + 2x_2^{(2)})]$$

$$= 0.33333\ [-4 - (1.13333 + 2(-0.76668))] = -1.19998.$$

Fourth iteration

$$x_1^{(4)} = 0.25\,[2 - (x_2^{(3)} + x_3^{(3)})] = 0.25\,[2 - (-\,1.14667 - 1.19999)] = 1.08666,$$

$$x_2^{(4)} = 0.2\,[-\,6 - (x_1^{(3)} + 2x_3^{(3)})] = 0.2\,[-\,6 - (0.86667 + 2(-\,1.19998))] = -\,0.89334,$$

$$x_3^{(4)} = 0.33333\,[-\,4 - (x_1^{(3)} + 2x_2^{(3)})]$$

$$= 0.33333\,[-\,4 - (0.86667 + 2(-\,1.14667))] = -\,0.85777.$$

Fifth iteration

$$x_1^{(5)} = 0.25\,[2 - (x_2^{(4)} + x_3^{(4)})] = 0.25\,[2 - (-\,0.89334 - 0.85777)] = 0.93778,$$

$$x_2^{(5)} = 0.2\,[-\,6 - (x_1^{(4)} + 2x_3^{(4)})] = 0.2\,[-\,6 - (1.08666 + 2(-\,0.85777))] = -\,1.07422,$$

$$x_3^{(5)} = 0.33333\,[-\,4 - (x_1^{(4)} + 2x_2^{(4)})]$$

$$= 0.33333\,[-\,4 - (1.08666 + 2(-\,0.89334))] = -\,1.09998.$$

It is interesting to note that the iterations oscillate and converge to the exact solution $x_1 = 1.0,\ x_2 = -\,1,\ x_3 = -\,1.0.$

(*ii*) $x_1^{(0)} = 0.5, x_2^{(0)} = -0.5, x_3^{(0)} = -0.5.$

First iteration

$$x_1^{(1)} = 0.25\ [2 - (x_2^{(0)} + x_3^{(0)})] = 0.25\ [2 - (-0.5 - 0.5)] = 0.75,$$

$$x_2^{(1)} = 0.2\ [-6 - (x_1^{(0)} + 2x_3^{(0)})] = 0.2\ [-6 - (0.5 + 2(-0.5))] = -1.1,$$

$$x_3^{(1)} = 0.33333\ [-4 - (x_1^{(0)} + 2x_2^{(0)})] = 0.33333\ [-4 - (0.5 + 2(-0.5))] = -1.16667.$$

Second iteration

$$x_1^{(2)} = 0.25\ [2 - (x_2^{(1)} + x_3^{(1)})] = 0.25\ [2 - (-1.1 - 1.16667)] = 1.06667,$$

$$x_2^{(2)} = 0.2\ [-6 - (x_1^{(1)} + 2x_3^{(1)})] = 0.2\ [-6 - (0.75 + 2(-1.16667))] = -0.88333,$$

$$x_3^{(2)} = 0.33333\ [-4 - (x_1^{(1)} + 2x_2^{(1)})] = 0.33333\ [-4 - (0.75 + 2(-1.1))] = -0.84999.$$

Third iteration

$$x_1^{(3)} = 0.25\ [2 - (x_2^{(2)} + x_3^{(2)})] = 0.25\ [2 - (-0.88333 - 0.84999)] = 0.93333,$$

$$x_2^{(3)} = 0.2\ [-6 - (x_1^{(2)} + 2x_3^{(2)})] = 0.2\ [-6 - (1.06667 + 2(-0.84999))] = -1.07334,$$

$$x_3^{(3)} = 0.33333\ [-4 - (x_1^{(2)} + 2x_2^{(2)})]$$

$$= 0.33333\ [-4 - (1.06667 + 2(-0.88333))] = -1.09999.$$

Fourth iteration

$$x_1^{(4)} = 0.25\,[2 - (x_2^{(3)} + x_3^{(3)})] = 0.25\,[2 - (-\,1.07334 - 1.09999)] = 1.04333,$$

$$x_2^{(4)} = 0.2\,[-\,6 - (x_1^{(3)} + 2x_3^{(3)})] = 0.2\,[-\,6 - (0.93333 + 2(-\,1.09999))] = -\,0.94667,$$

$$x_3^{(4)} = 0.33333\,[-\,4 - (x_1^{(3)} + 2x_2^{(3)})]$$

$$= 0.33333\,[-\,4 - (0.93333 + 2(-\,1.07334))] = -\,0.92887.$$

Fifth iteration

$$x_1^{(5)} = 0.25\,[2 - (x_2^{(4)} + x_3^{(4)})] = 0.25\,[2 - (-\,0.94667 - 0.92887)] = 0.96889,$$

$$x_2^{(5)} = 0.2\,[-\,6 - (x_1^{(4)} + 2x_3^{(4)})] = 0.2\,[-\,6 - (1.04333 + 2(-\,0.92887))] = -\,1.03712,$$

$$x_3^{(5)} = 0.33333\,[-\,4 - (x_1^{(4)} + 2x_2^{(4)})]$$

$$= 0.33333\,[-\,4 - (1.04333 + 2(-\,0.94667))] = -\,1.04999.$$

Example *Solve the system of equations*

$$26x_1 + 2x_2 + 2x_3 = 12.6$$

$$3x_1 + 27x_2 + x_3 = -14.3$$

$$2x_1 + 3x_2 + 17x_3 = 6.0$$

using the Jacobi iteration method. Obtain the result correct to three decimal places.

Solution The given system of equations is strongly diagonally dominant. Hence, we can expect faster convergence. Jacobi method gives the iterations as

$$x_1^{(k+1)} = [12.6 - (2x_2^{(k)} + 2x_3^{(k)})]/26$$

$$x_2^{(k+1)} = [-14.3 - (3x_1^{(k)} + x_3^{(k)})]/27$$

$$x_3^{(k+1)} = [6.0 - (2x_1^{(k)} + 3x_2^{(k)})]/17 \qquad k = 0, 1, \ldots$$

Choose the initial approximation as $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$. We obtain the following results.

First iteration

$$x_1^{(1)} = \frac{1}{26}[12.6 - (2x_2^{(0)} + 2x_3^{(0)})] = \frac{1}{26}[12.6] = 0.48462,$$

$$x_2^{(1)} = \frac{1}{27}\,[-\,14.3 - (3x_1^{(0)} + x_3^{(0)})] = \frac{1}{27}\,[-\,14.3] = -\,0.52963,$$

$$x_3^{(1)} = \frac{1}{17}\,[6.0 - (2x_1^{(0)} + 3x_2^{(0)})] = \frac{1}{17}\,[6.0] = 0.35294.$$

Second iteration

$$x_1^{(2)} = \frac{1}{26}\,[12.6 - (2x_2^{(1)} + 2x_3^{(1)})] = \frac{1}{26}\,[12.6 - 2\,(-\,0.52963 + 0.35294)] = 0.49821,$$

$$x_2^{(2)} = \frac{1}{27}\,[-\,14.3 - (3x_1^{(1)} + x_3^{(1)})] = \frac{1}{27}\,[-\,14.3 - (3(0.48462) + 0.35294)] = -\,0.59655,$$

$$x_3^{(2)} = \frac{1}{17}\,[-\,6.0 - (2x_1^{(1)} + 3x_2^{(1)})] = \frac{1}{17}\,[6.0 - (2(0.48462) + 3(-\,0.52963))] = 0.38939.$$

Third iteration

$$x_1^{(3)} = \frac{1}{26}\,[12.6 - (2x_2^{(2)} + 2x_3^{(2)})] = \frac{1}{26}\,[12.6 - 2(-\,0.59655 + 0.38939)] = 0.50006,$$

$$x_2^{(3)} = \frac{1}{27}\,[-\,14.3 - (3x_1^{(2)} + x_3^{(2)})] = \frac{1}{27}\,[-\,14.3 - (3(0.49821) + 0.38939)] = -\,0.59941,$$

$$x_3^{(3)} = \frac{1}{17}\,[-\,6.0 - (2x_1^{(2)} + 3x_2^{(2)})] = \frac{1}{17}\,[6.0 - (2(0.49821) + 3(-\,0.59655))] = 0.39960.$$

Fourth iteration

$$x_1^{(4)} = \frac{1}{26}\,[12.6 - (2x_2^{(3)} + 2x_3^{(3)})] = \frac{1}{26}\,[12.6 - 2(-0.59941 + 0.39960)] = 0.50000,$$

$$x_2^{(4)} = \frac{1}{27}\,[-14.3 - (3x_1^{(3)} + x_3^{(3)})] = \frac{1}{27}\,[-14.3 - (3(0.50006) + 0.39960)] = -0.59999,$$

$$x_3^{(4)} = \frac{1}{17}\,[-6.0 - (2x_1^{(3)} + 3x_2^{(3)})] = \frac{1}{17}\,[6.0 - (2(0.50006) + 3(-0.59941))] = 0.39989.$$

We find $\left| x_1^{(4)} - x_1^{(3)} \right| = |\, 0.5 - 0.50006 \,| = 0.00006,$

$$\left| x_2^{(4)} - x_2^{(3)} \right| = |\, -0.59999 + 0.59941 \,| = 0.00058,$$

$$\left| x_3^{(4)} - x_3^{(3)} \right| = |\, 0.39989 - 0.39960 \,| = 0.00029.$$

The required solution is $x_1 = 0.5, x_2 = -0.59999, x_3 = 0.39989.$

Gauss-Seidel Iteration Method

We use the updated values of $x_1, x_2, \ldots, x_{i-1}$ in computing the value of the variable x_i. We assume that the pivots $a_{ii} \neq 0$, for all i. We write the equations as

$$a_{11}x_1 = b_1 - (a_{12}x_2 + a_{13}x_3)$$
$$a_{22}x_2 = b_2 - (a_{21}x_1 + a_{23}x_3)$$
$$a_{33}x_3 = b_3 - (a_{31}x_1 + a_{32}x_2)$$

The Gauss-Seidel iteration method is defined as

$$x_1^{(k+1)} = \frac{1}{a_{11}}[b_1 - (a_{12}x_2^{(k)} + a_{13}x_3^{(k)})]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}[b_2 - (a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)})] \qquad (1)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}[b_3 - (a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)})]$$

$$k = 0, 1, 2, \ldots$$

This method is also called the *method of successive displacement.*

We observe that (1) is same as writing the given system as

$$\begin{aligned} a_{11}x_1^{(k+1)} &= b_1 - (a_{12}x_2^{(k)} + a_{13}x_3^{(k)}) \\ a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} &= b_2 - a_{23}x_3^{(k)} \\ a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} + a_{33}x_3^{(k+1)} &= b_3 \end{aligned} \qquad (2)$$

Remark *A sufficient condition for convergence of the Gauss-Seidel method is that the system of equations is diagonally dominant*, that is, the coefficient matrix **A** is diagonally dominant. This implies that convergence may be obtained even if the system is not diagonally dominant. If the system is not diagonally dominant, we may exchange the equations, if possible, such that the new system is diagonally dominant and convergence is guaranteed. The necessary and sufficient condition for convergence is that the spectral radius of the iteration matrix **H** is less than one unit, that is, $\rho(\mathbf{H}) < 1$, where $\rho(\mathbf{H})$ is the largest eigen value in magnitude of **H.** Testing of this condition is beyond the scope of the syllabus.

If both the Gauss-Jacobi and Gauss-Seidel methods converge, then *Gauss-Seidel method converges at least two times faster than the Gauss-Jacobi method.*

Example *Find the solution of the system of equations*

$$45x_1 + 2x_2 + 3x_3 = 58$$

$$-3x_1 + 22x_2 + 2x_3 = 47$$

$$5x_1 + x_2 + 20x_3 = 67$$

correct to three decimal places, using the Gauss-Seidel iteration method.

Solution The given system of equations is strongly diagonally dominant. Hence, we can expect fast convergence. Gauss-Seidel method gives the iteration

$$x_1^{(k+1)} = \frac{1}{45}\,(58 - 2x_2^{(k)} - 3x_3^{(k)}),$$

$$x_2^{(k+1)} = \frac{1}{22}\,(47 + 3x_1^{(k+1)} - 2x_3^{(k)}),$$

$$x_3^{(k+1)} = \frac{1}{20}\,(67 - 5x_1^{(k+1)} - x_2^{(k+1)}).$$

Starting with $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$, we get the following results.

First iteration

$$x_1^{(1)} = \frac{1}{45}\,(58 - 2x_2^{(0)} - 3x_3^{(0)}) = \frac{1}{45}\,(58) = 1.28889,$$

$$x_2^{(1)} = \frac{1}{22}\,(47 + 3x_1^{(1)} - 2x_3^{(0)}) = \frac{1}{22}\,(47 + 3(1.28889) - 2(0)) = 2.31212,$$

$$x_3^{(1)} = \frac{1}{20}\,(67 - 5x_1^{(1)} - x_2^{(1)}) = \frac{1}{20}\,(67 - 5(1.28889) - (2.31212)) = 2.91217.$$

Second iteration

$$x_1^{(2)} = \frac{1}{45}\left(58 - 2x_2^{(1)} - 3x_3^{(1)}\right) = \frac{1}{45}\left(58 - 2(2.31212) - 3(2.91217)\right) = 0.99198,$$

$$x_2^{(2)} = \frac{1}{22}\left(47 + 3x_1^{(2)} - 2x_3^{(1)}\right) = \frac{1}{22}\left(47 + 3(0.99198) - 2(2.91217)\right) = 2.00689,$$

$$x_3^{(2)} = \frac{1}{20}\left(67 - 5x_1^{(2)} - x_2^{(2)}\right) = \frac{1}{20}\left(67 - 5(0.99198) - (2.00689)\right) = 3.00166.$$

Third iteration

$$x_1^{(3)} = \frac{1}{45}\left(58 - 2x_2^{(2)} - 3x_3^{(2)}\right) = \frac{1}{45}(58 - 2(2.00689) - 3(3.00166) = 0.99958,$$

$$x_2^{(3)} = \frac{1}{22}\left(47 + 3x_1^{(3)} - 2x_3^{(2)}\right) = \frac{1}{22}\left(47 + 3(0.99958) - 2(3.00166)\right) = 1.99979,$$

$$x_3^{(3)} = \frac{1}{20}\left(67 - 5x_1^{(3)} - x_2^{(3)}\right) = \frac{1}{20}\left(67 - 5(0.99958) - (1.99979)\right) = 3.00012.$$

Fourth iteration

$$x_1^{(4)} = \frac{1}{45}\left(58 - 2x_2^{(3)} - 3x_3^{(3)}\right) = \frac{1}{45}\left(58 - 2(1.99979) - 3(3.00012)\right) = 1.00000,$$

$$x_2^{(4)} = \frac{1}{22}\left(47 + 3x_1^{(4)} - 2x_3^{(3)}\right) = \frac{1}{22}\left(47 + 3(1.00000) - 2(3.00012)\right) = 1.99999,$$

$$x_3^{(4)} = \frac{1}{20}\left(67 - 5x_1^{(4)} - x_2^{(4)}\right) = \frac{1}{20}\left(67 - 5(1.00000) - (1.99999)\right) = 3.00000.$$

We find $\left| x_1^{(4)} - x_1^{(3)} \right| = \left| 1.00000 - 0.99958 \right| = 0.00042,$

$$\left| x_2^{(4)} - x_2^{(3)} \right| = \left| 1.99999 - 1.99979 \right| = 0.00020,$$

$$\left| x_3^{(4)} - x_3^{(3)} \right| = \left| 3.00000 - 3.00012 \right| = 0.00012.$$

Since, all the errors in magnitude are less than 0.0005, the required solution is

$$x_1 = 1.0,\ x_2 = 1.99999,\ x_3 = 3.0.$$

Example . *Computationally show that Gauss-Seidel method applied to the system of equations*

$$3x_1 - 6x_2 + 2x_3 = 23$$
$$-4x_1 + x_2 - x_3 = -8$$
$$x_1 - 3x_2 + 7x_3 = 17$$

diverges. Take the initial approximations as $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$. *Interchange the first and second equations and solve the resulting system by the Gauss-Seidel method. Again take the initial approximations as* $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$, *and obtain the result correct to two decimal places. The exact solution is* $x_1 = 1.0$, $x_2 = -3.0$, $x_3 = 1.0$.

Solution Note that the system of equations is not diagonally dominant. Gauss-Seidel method gives the iteration

$$x_1^{(k+1)} = [23 + 6x_2^{(k)} - 2x_3^{(k)})]/3$$
$$x_2^{(k+1)} = [-8 + 4x_1^{(k+1)} + x_3^{(k)}]$$
$$x_3^{(k+1)} = [17 - x_1^{(k+1)} + 3x_2^{(k+1)}]/7.$$

Starting with the initial approximations $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$, we obtain the following results.

First iteration

$$x_1^{(1)} = \frac{1}{3}\,[23 + 6x_2^{(0)} - 2x_3^{(0)}] = \frac{1}{3}\,[23 + 6(-3.1) - 2(0.9)] = 0.8667,$$

$$x_2^{(1)} = [-8 + 4x_1^{(1)} + x_3^{(0)}] = [-8 + 4(0.8667) + 0.9] = -3.6332,$$

$$x_3^{(1)} = \frac{1}{7}\,[17 - x_1^{(1)} + 3x_2^{(1)}] = \frac{1}{7}\,[17 - (0.8667) + 3(-3.6332)] = 0.7477.$$

Second iteration

$$x_1^{(2)} = \frac{1}{3}\,[23 + 6x_2^{(1)} - 2x_3^{(1)}] = \frac{1}{3}\,[23 + 6\,(-3.6332) - 2(0.7477)] = -0.0982,$$

$$x_2^{(2)} = [-8 + 4x_1^{(2)} + x_3^{(1)}] = [-8 + 4(-0.0982) + 0.7477] = -7.6451,$$

$$x_3^{(2)} = \frac{1}{7}\,[17 - x_1^{(2)} + 3x_2^{(2)}] = \frac{1}{7}\,[17 + 0.0982 + 3(-7.6451)] = -0.8339.$$

Third iteration

$$x_1^{(3)} = \frac{1}{3}\,[23 + 6x_2^{(2)} - 2x_3^{(2)}] = \frac{1}{3}\,[23 + 6\,(-\,7.6451) - 2(-\,0.8339)] = -\,7.0676,$$

$$x_2^{(3)} = [-\,8 + 4x_1^{(3)} + x_3^{(2)}] = [-\,8 + 4(-\,7.0676) - 0.8339] = -\,37.1043,$$

$$x_3^{(3)} = \frac{1}{7}\,[17 - x_1^{(3)} + 3x_2^{(3)}] = \frac{1}{7}\,[17 + 7.0676 + 3(-\,37.1043)] = -\,12.4636.$$

It can be observed that the iterations are diverging very fast.

Now, we exchange the first and second equations to obtain the system

$$-4x_1 + x_2 - x_3 = -8$$
$$3x_1 - 6x_2 + 2x_3 = 23$$
$$x_1 - 3x_2 + 7x_3 = 17.$$

The system of equations is now diagonally dominant. Gauss-Seidel method gives iteration

$$x_1^{(k+1)} = [8 + x_2^{(k)} - x_3^{(k)}]/4$$
$$x_2^{(k+1)} = -[23 - 3x_1^{(k+1)} - 2x_3^{(k)}]/6$$
$$x_3^{(k+1)} = [17 - x_1^{(k+1)} + 3x_2^{(k+1)}]/7.$$

Starting with the initial approximations $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$, we obtain the following results.

First iteration

$$x_1^{(1)} = \frac{1}{4}[8 + x_2^{(0)} - x_3^{(0)}] = \frac{1}{4}[8 - 3.1 - 0.9] = 1.0,$$
$$x_2^{(1)} = -\frac{1}{6}[23 - 3x_1^{(1)} - 2x_3^{(0)}] = -\frac{1}{6}[23 - 3(1.0) - 2(0.9)] = -3.0333,$$
$$x_3^{(1)} = \frac{1}{7}[17 - x_1^{(1)} + 3x_2^{(1)}] = \frac{1}{7}[17 - 1.0 + 3(-3.0333)] = 0.9857.$$

Second iteration

$$x_1^{(2)} = \frac{1}{4}\,[8 + x_2^{(1)} - x_3^{(1)}] = \frac{1}{4}\,[8 - 3.0333 - 0.9857] = 0.9953,$$

$$x_2^{(2)} = -\frac{1}{6}\,[23 - 3x_1^{(2)} - 2x_3^{(1)}] = -\frac{1}{6}\,[23 - 3(0.9953) - 2(0.9857)] = -3.0071,$$

$$x_3^{(2)} = \frac{1}{7}\,[17 - x_1^{(2)} + 3x_2^{(2)}] = \frac{1}{7}\,[17 - 0.9953 + 3(-3.0071)] = 0.9976.$$

Third iteration

$$x_1^{(3)} = \frac{1}{4}\,[8 + x_2^{(2)} - x_3^{(2)}] = \frac{1}{4}\,[8 - 3.0071 - 0.9976] = 0.9988,$$

$$x_2^{(3)} = -\frac{1}{6}\,[23 - 3x_1^{(3)} - 2x_3^{(2)}] = -\frac{1}{6}\,[23 - 3(0.9988) - 2(0.9976)] = -3.0014,$$

$$x_3^{(3)} = \frac{1}{7}\,[17 - x_1^{(3)} + 3x_2^{(3)}] = \frac{1}{7}\,[17 - 0.9988 + 3(-3.0014)] = 0.9996.$$

Fourth iteration

$$x_1^{(4)} = \frac{1}{4}\,[8 + x_2^{(3)} - x_3^{(3)}] = \frac{1}{4}\,[8 - 3.0014 - 0.9996] = 0.9998,$$

$$x_2^{(4)} = -\frac{1}{6}\,[23 - 3x_1^{(4)} - 2x_3^{(3)}] = -\frac{1}{6}\,[23 - 3(0.9998) - 2(0.9996)] = -3.0002,$$

$$x_3^{(4)} = \frac{1}{7}\,[17 - x_1^{(4)} + 3x_2^{(4)}] = \frac{1}{7}\,[17 - 0.9998 + 3(-3.0002)] = 0.9999.$$

We find $\left| x_1^{(4)} - x_1^{(3)} \right| = |\, 0.9998 - 0.9988 \,| = 0.0010,$

$$\left| x_2^{(4)} - x_2^{(3)} \right| = |\, -3.0002 + 3.0014 \,| = 0.0012,$$

$$\left| x_3^{(4)} - x_3^{(3)} \right| = |\, 0.9999 - 0.9996 \,| = 0.0003.$$

Since, all the errors in magnitude are less than 0.005, the required solution is

$$x_1 = 0.9998,\ x_2 = -3.0002,\ x_3 = 0.9999.$$

Interpolation and the Lagrange polynomial

In this chapter, we discuss the problem of approximating a given function by polynomials. There are two main uses of these approximating polynomials. The first use is to reconstruct the function $f(x)$ when it is not given explicitly and only values of $f(x)$ and/ or its certain order derivatives are given at a set of distinct points called *nodes* or *tabular points*. The second use is to perform the required operations which were intended for $f(x)$, like determination of roots, differentiation and integration etc. can be carried out using the approximating polynomial $P(x)$. The approximating polynomial $P(x)$ can be used to predict the value of $f(x)$ at a non-tabular point. The deviation of $P(x)$ from $f(x)$, that is $f(x) - P(x)$, is called the *error of approximation*.

Let $f(x)$ be a continuous function defined on some interval $[a, b]$, and be prescribed at $n + 1$ distinct tabular points $x_0, x_1,..., x_n$ such that $a = x_0 < x_1 < x_2 < ... < x_n = b$. The distinct tabular points $x_0, x_1,..., x_n$ may be non-equispaced or equispaced, that is $x_{k+1} - x_k = h$, $k = 0, 1, 2,..., n - 1$. The problem of polynomial approximation is to find a polynomial $P_n(x)$, of degree $\leq n$, which fits the given data exactly, that is,

$$P_n(x_i) = f(x_i), \quad i = 0, 1, 2,..., n. \tag{1}$$

The polynomial $P_n(x)$ is called the *interpolating polynomial*. The conditions given in (1) are called the *interpolating conditions*.

Lagrange Interpolation

Let the data

x	x_0	x_1	x_2	...	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_n)$

be given at distinct unevenly spaced points or non-uniform points $x_0, x_1,..., x_n$. This data may also be given at evenly spaced points.

For this data, we can fit a unique polynomial of degree $\leq n$. Since the interpolating polynomial must use all the ordinates $f(x_0), f(x_1),... f(x_n)$, it can be written as a linear combination of these ordinates. That is, we can write the polynomial as

$$\begin{aligned} P_n(x) &= l_0(x)\, f(x_0) + l_1(x)\, f(x_1) + ... + l_n(x)\, f(x_n) \\ &= l_0(x)\, f_0 + l_1(x)\, f_1 + ... + l_n(x)\, f(x_n) \end{aligned} \tag{2}$$

where $f(x_i) = f_i$ and $l_i(x)$, $i = 0, 1, 2, ...,n$ are polynomials of degree n. This polynomial fits the data given in (1) exactly.

At $x = x_0$, we get

$$f(x_0) \equiv P_n(x_0) = l_0(x_0)\, f(x_0) + l_1(x_0)\, f(x_1) + ... + l_n(x_0)\, f(x_n).$$

This equation is satisfied only when $l_0(x_0) = 1$ and $l_i(x_0) = 0$, $i \neq 0$.

At a general point $x = x_i$, we get

$$f(x_i) \equiv P_n(x_i) = l_0(x_i)\, f(x_0) + \ldots + l_i(x_i)\, f(x_i) + \ldots + l_n(x_i)\, f(x_n).$$

This equation is satisfied only when $l_i(x_i) = 1$ and $l_j(x_i) = 0$, $i \neq j$.

Therefore, $l_i(x)$, which are polynomials of degree n, satisfy the conditions

$$l_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \tag{3}$$

We seek a polynomial of degree $\leq n$ which at all the x_i with exception $x_k (x_i \neq x_n)$

Let $g_k(x) = \prod_{i=0, i\neq k}^{n} (x - x_i) = (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)$

- $g_k(x_k) = \prod_{i=0, i\neq k}^{n} (x_k - x_i) = (x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n) \neq 0$

$L_k(x) = \dfrac{g_k(x)}{g_k(x_k)}, x_k \neq 0$ (Lagrange function)

$$= \prod_{i=0, i\neq k}^{n} (x - x_i) \Big/ \prod_{i=0, i\neq k}^{n} (x_k - x_i) = \prod_{i=0, i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}, \quad L_k(x_i) = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}$$

For any arbitrary function f(x),

Define $p(x) = \sum_{k=0}^{n} f(x_k) l_k(x)$ ------------ (1) (Lagrange form)

$L_k = \prod_{i=0, i\neq k}^{n} \dfrac{(x - x_i)}{(x_k - x_i)} \Rightarrow p(x) = \sum_{k=0}^{n} \prod_{i=0, i\neq k}^{n} f(x_k) \dfrac{(x - x_i)}{(x_k - x_i)}$ is a polynomial of degree $\leq$n and interpolate f(x) at $x_0, x_1, \ldots, x_n$.

Let $n = 2 \Rightarrow x_0, x_1, x_2 \Rightarrow 3$

$$L_0(x) = \prod_{i=0, i\neq 0}^{2} \frac{(x - x_i)}{(x_0 - x_i)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, L_1(x) = \prod_{i=0, i\neq 1}^{2} \frac{(x - x_i)}{(x_1 - x_i)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)},$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}, p(x) \cong f(x_0) l_0(x) + f(x_1) l_1(x) + f(x_2) l_2(x)$$

Ex (1): (x, f(x)) = {(0,0), (1,1), (2,4)}, find f(1.5) using Lagrange function

$$n=2,(x_0,f(x_0)),(x_1,f(x_1)),(x_2,f(x_2))\equiv(0,0),(1,1),(2,4)$$

$$p(x)=\sum_{k=0}^{2}\prod_{i=0,i\neq k}^{2}f(x_k)\frac{(x-x_i)}{(x_k-x_i)}=f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}+f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$+f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$p(x)\cong 0+1\frac{(x-0)(x-2)}{(1)(-1)}+4\frac{(x-0)(x-1)}{(2)(1)}=\frac{x^2-2x}{-1}+2(x^2-x)=-x^2+2x+2x^2$$

$$-2x=x^2$$

$$\therefore p(x)=f(x)=x^2\Rightarrow f(1.5)=(1.5)^2=2.25$$

Ex (2): By Lagrange formula find f(3), f(5).

x	0	1	2	4
f(x)	1	1	2	5

$$p_3(x)=f(x)=\sum_{i=0}^{3}l_i(x)f(x_i)=l_0(x)f(x_0)+l_1(x)f(x_1)+l_2(x)f(x_2)+l_3(x)f(x_3)$$

$$f(3)=\frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)}(1)+\frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)}(1)+\frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)}(2)$$

$$+\frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}(5)=\frac{-2}{-8}+\frac{-3}{3}+\frac{-6}{-4}\cdot 2+\frac{6}{24}\cdot 5=0.25-1+3+1.25=3.5$$

$$f(5)=\frac{(5-1)(5-2)(5-4)}{(0-1)(0-2)(0-4)}(1)+\frac{(5-0)(5-2)(5-4)}{(1-0)(1-2)(1-4)}(1)+\frac{(5-0)(5-1)(5-4)}{(2-0)(2-1)(2-4)}(2)$$

$$+\frac{(5-0)(5-1)(5-2)}{(4-0)(4-1)(4-2)}(5)=\frac{12}{-8}+\frac{15}{3}+\frac{20}{-4}\cdot 2+\frac{60}{24}\cdot 5=-1.5+5-10+12.5=6$$

Ex (3): Using the numbers $x_0 = 2, x_1 = 2.5, x_2 = 4$ to find the second-degree interpolating polynomial for $f(x) = \frac{1}{x}$.

x	2	2.5	4
f(x)	0.5	0.4	0.25

$$p_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$l_0(x) = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = \frac{x^2 - 6.5x + 10}{(-0.5)(-2)} = x^2 - 6.5x + 10$$

$$l_1(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{x^2 - 6x + 8}{(0.5)(-1.5)} = \frac{-1}{3}(4x^2 - 24x + 32)$$

$$l_2(x) = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \frac{x^2 - 4.5x + 5}{(2)(1.5)} = \frac{1}{3}(x^2 - 4.5x + 5)$$

$$\because f(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$= 0.5(x^2 - 6.5x + 10) + \frac{0.4}{3}(-4x^2 + 24x - 32) + \frac{0.25}{3}(x^2 - 4.5x + 5)$$

$$= 0.05x^2 - 0.425x + 1.15$$

Divided Differences

Let $x_0, x_1, \ldots, x_n$ be distinct points, we define the 0^{th} divided difference at x_i by $f[x_i] = f(x_i)$, the first divided difference at $[x_i, x_j]$ is,

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j} = \frac{f[x_i] - f[x_j]}{x_i - x_j}$$

in general, the k^{th} divided difference

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_0, x_1, \ldots, x_{k-1}] - f[x_1, \ldots, x_k]}{x_0 - x_k}$$

Ex (1): Find the d. d. $f[x_0, x_1], f[x_0, x_1, x_2], f[x_0, x_2, x_1], f[x_1, x_0]$

i	0	1	2
x_i	1	2	3
$f(x_i)$	3	5	7

Solu.

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{3-5}{1-2} = 2$$

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{5-3}{2-1} = 2$$

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2} = \frac{2-2}{1-3} = 0$$

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{5-7}{2-3} = 2$$

$$f[x_0, x_2, x_1] = \frac{f[x_0, x_2] - f[x_2, x_1]}{x_0 - x_1} = 0$$

First divided difference

$$f[x_i, x] = \frac{f(x_i) - f(x)}{x_i - x}$$

$$f(x) = f(x_i) + (x - x_i) f[x_i, x]$$

$$f(x) = f(x_0) + (x - x_0) f[x_0, x]$$

$$p_1(x) = f(x_0) + (x - x_0) f[x_0, x]$$

Approximate $f(x)$ and interpolation $f(x)$ at (x_0, x_1), $p_1(x_0) = f(x_0)$

$$p_1(x_1) = f(x_0) + (x_1 - x_0)\frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_0) + (x_1 - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= f(x_0) + f(x_1) - f(x_0) = f(x_1)$$

Ex (2): Let $(x_i, f(x_i))$ be given in the following table.

i	x_i	$f(x_i)$	1st d. d.	2nd d. d.	3rd d. d.
0	1	2			
			3		
1	2	5		1	
			5		0
2	3	10		1	
			7		
3	4	17			

Estimate $f(1.5)$ using Newton d.d. interpolation poly.

$$f[x_0, x_1] = \frac{f[x_0] - f[x_1]}{x_0 - x_1} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{2-5}{1-2} = 3$$

$$f[x_1, x_2] = \frac{f[x_1] - f[x_2]}{x_1 - x_2} = \frac{5-10}{-1} = 5$$

$$f[x_2, x_3] = \frac{f[x_2] - f[x_3]}{x_2 - x_3} = \frac{10-17}{-1} = 7$$

$$f[x_0,x_1,x_2]=\frac{f[x_0,x_1]-f[x_1,x_2]}{x_0-x_2}=\frac{3-5}{1-3}=1$$

$$f[x_1,x_2,x_3]=\frac{f[x_1,x_2]-f[x_2,x_3]}{x_1-x_3}=\frac{5-7}{2-4}=1$$

$$f[x_0,x_1,x_2,x_3]=\frac{f[x_0,x_1,x_2]-f[x_1,x_2,x_3]}{x_0-x_3}=\frac{1-1}{1-4}=0$$

$$p(x)=f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]$$

$p_2(x)\cong f(x)$ then to estimate $f(x=1.5)$, we evaluate $p(x=1.5)$

$$\Rightarrow p(x=1.5)=f(x_0)+(1.5-x_0)f[x_0,x_1]+(1.5-x_0)(1.5-x_1)f[x_0,x_1,x_2]$$

$$=2+(0.5)3+(0.5)(-0.5)1=2+\frac{3}{2}-\frac{1}{4}=\frac{8}{4}+\frac{6}{4}-\frac{1}{4}=\frac{13}{4}=3.25$$

Newton Forward divided difference

$x_0, x_1, \ldots, x_n$

$f(x) \cong p_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + \ldots + f[x_0, \ldots, x_n](x - x_0) \cdots (x - x_{n-1})$

$x_0, x_1, \ldots, x_n$ are arranged consecutively with equal spacing.

Let $h = x_{i+1} - x_i \Rightarrow x_1 - x_0 = h$

$x_2 - x_1 = h$

$x_3 - x_2 = h$

$x_0 = 0, h = 1$

$x_1 = 1, x_2 = 2, x_3 = 3$ equal spacing points

$x = x_0 + sh$

$x - x_0 = sh + x_0 - x_0 = sh \Rightarrow (x - x_1) = (s - 1)h$

$(x - x_0)(x - x_1) = s(s - 1)h^2$

Let $x - x_i = (s - i)h, i = 0, 1, \ldots, n$

$x - x_0 = (s - 0)h = sh$

$x - x_1 = (s - 1)h$

$x - x_2 = (s - 2)h$

$x - x_3 = (s - 3)h$

$\Rightarrow p_n(x) = p_n(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s - 1)h^2 f[x_0, x_1, x_2] + \cdots + s(s - 1)$

$\cdots (s - n + 1)h^n f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} s(s - 1) \cdots (s - k + 1)h^k f[x_0, x_1, \ldots, x_k]$

$$\because \binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$$

$$p_n(x) = \sum_{k=0}^{n} \binom{s}{k} k!\, h^k f[x_0, x_1, \ldots, x_k] \text{ (N. F. d. d. formula)}$$

Newton Backward divided difference

$x_0, x_1, \ldots, x_n$

The interpolating point $x_0, x_1, \ldots, x_n$ are reordered as $x_n, x_{n-}, \ldots, x_0$ a formula.

$$p_n(x) = f[x_n] + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1}) + \ldots +$$
$$f[x_0, x_1, \ldots, x_n](x - x_n)(x - x_{n-1})\cdots(x - x_1)$$

$x = x_n + sh$ and $x = x_i + (s + n - i)h$

$$p_n(x) = p_n(x_n + sh) = f[x_n] + shf[x_{n-1}, x_n] + s(s+1)h^2 f[x_{n-2}, x_{n-1}, x_n] + \ldots + s(s+1)$$
$$(s+2)\cdots(s+n-1)h^n f[x_0, x_1, \ldots, x_n] \text{(N. B. d. d. formula)}$$

Ex (3): estimate $f(2.5)$

i	0	1	2	3
x_i	0	1	2	3
$f(x_i)$	2	6	12	20

i	x_i	$f(x_i)$	1st d. d.	2nd d. d.	3rd d. d.
0	0	2			
			4		
1	1	6		1	
			6		0
2	2	12		1	
			8		
3	3	20			

N. f. $f(2.5) = 2 + 4(x-0) + 1(x-0)(x-1) = 2 + 4(2.5) + (2.5)(2.5-1) = 15.75$

N. b. $f(2.5) = 20 + 8(2.5-3) + 1(2.5-3)(2.5-2) = 15.75$

Ex (4):

i	x_i	$f(x_i)$	1st d. d.	2nd d. d.	3rd d. d.	4th d. d.
0	1.0	0.76519				
			-0.4837			
1	1.3	0.62008		-0.108733		
			-0.54894		0.06587	
2	1.6	0.4554		-0.049443		0.0018
			-0.57861		0.06806	
3	1.9	0.281		0.01181		
			-0.57152			
4	2.2	0.11036				

1. If an approximation of $f(1.1)$ is required using N. F. d. d. formula.

$$x_0, x_1, x_2, x_3, x_4, h = x_{i+1} - x_i = 0.3, x = x_0 + sh \Rightarrow x = 1.5 + s(0.3)$$

$$\Rightarrow 1.1 = 1.0 + s(0.3) \Rightarrow \frac{(1.1 - 1.0)}{0.3} = s = \frac{1}{3}$$

$$p_4(x)=p_4(x_0+sh)=p_4\left(1.0+\frac{1}{3}0.3\right)=f[x_0]+shf[x_0,x_1]+s(s-1)h^2 f[x_0,x_1,x_2]$$

$$+s(s-1)(s-2)h^3 f[x_0,x_1,x_2,x_3]+s(s-1)(s-2)(s-3)h^4 f[x_0,x_1,x_2,x_3,x_4]$$

$$=0.765+\left(\frac{1}{3}\right)(0.3)(-0.4837)+\frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.10873)+\frac{1}{3}\left(-\frac{2}{3}\right)\left(\frac{-5}{3}\right)(0.3)^3$$

$$(0.06587)+\frac{1}{3}\left(-\frac{2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{-8}{3}\right)(0.3)^4(0.0018)=0.719698$$

2. If an approximation of $f(2)$ is required using N. B. d. d. formula reared the points $x_n, x_{n-1}, \ldots, x_0 \Rightarrow [x_4, x_3, x_2, x_1, x_0]$.

$$x=x_4+sh \Rightarrow 2.0=2.2+s(0.3) \Rightarrow s=\frac{-0.2}{0.3}=\frac{-2}{3}=-0.6666$$

$$p_4(x) = p_4(x_4 + sh) = p_4\left(2.2 + \left(\frac{-2}{3}\right)0.3\right) = f[x_4] + shf[x_4, x_3] + s(s+1)h^2$$

$$f[x_4, x_3, x_2] + s(s+1)(s+2)h^3 f[x_4, x_3, x_2, x_1] + s(s+1)(s+2)(s+3)h^4$$

$$f[x_4, x_3, x_2, x_1, x_0] = 0.11036 + \left(\frac{-2}{3}\right)(0.3)(-0.57152) + \left(\frac{-2}{3}\right)\left(\frac{1}{3}\right)(0.3)^2(0.01181)$$

$$+\left(\frac{-2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.06806) + \left(\frac{-2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{5}{3}\right)(0.3)^4(0.0018) = 0.2238$$

Numerical Integration

The need often arises for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain. The basic method involved in approximating $\int_a^b f(x)dx$ is called numerical quadrature. It uses a sum $\sum_{i=0}^{n} a_i f(x_i)$ to approximate $\int_a^b f(x)dx$.

The methods of quadrature in this section are based on the interpolation polynomials. The basic idea is to select a set of distinct nodes $\{x_0, \dots, x_n\}$ from the interval $[a, b]$. Then integrate the Lagrange interpolating polynomial,

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

and its truncation error term over $[a, b]$ to obtain,

$$\int_a^b f(x)\,dx = \int_a^b \sum_{i=0}^{n} f(x_i) L_i(x)\,dx + \int_a^b \prod_{i=0}^{n} (x - x_i) \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\,dx$$

$$= \sum_{i=0}^{n} a_i f(x_i) + \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x))\,dx,$$

Where $\xi(x)$ is in $[a, b]$ for each x and

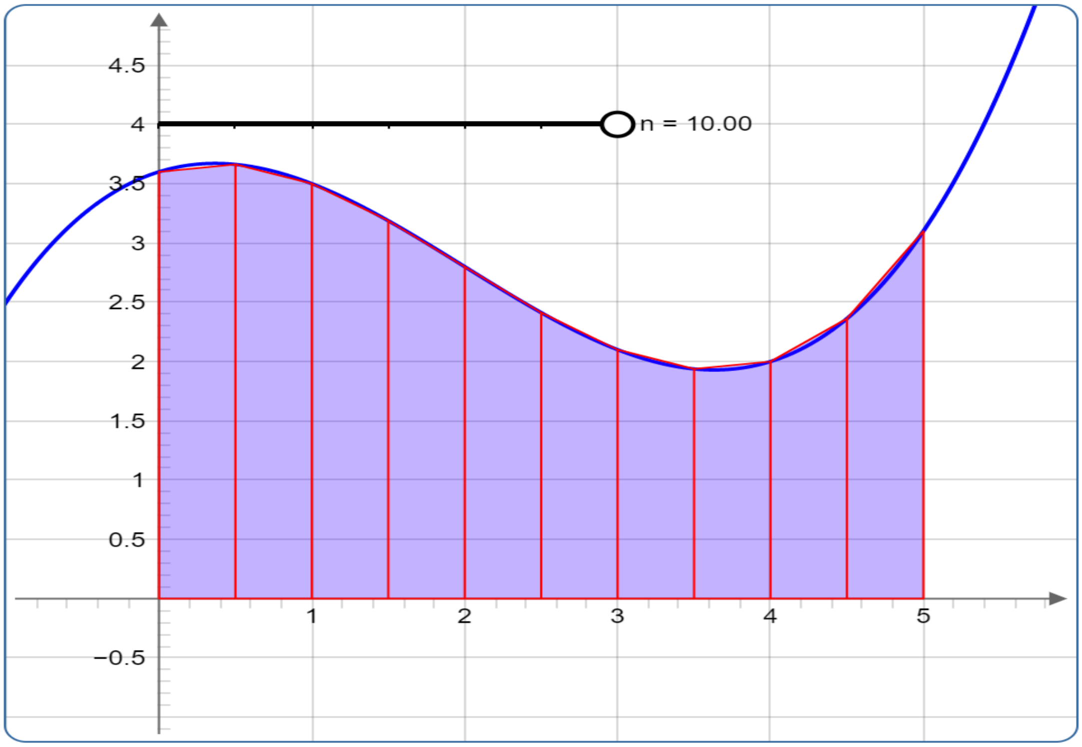
$$a_i = \int_a^b L_i(x)dx, \quad \text{for each } i = 0,1,2 \dots, n.$$

The quadrature formula is, therefore,

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i),$$

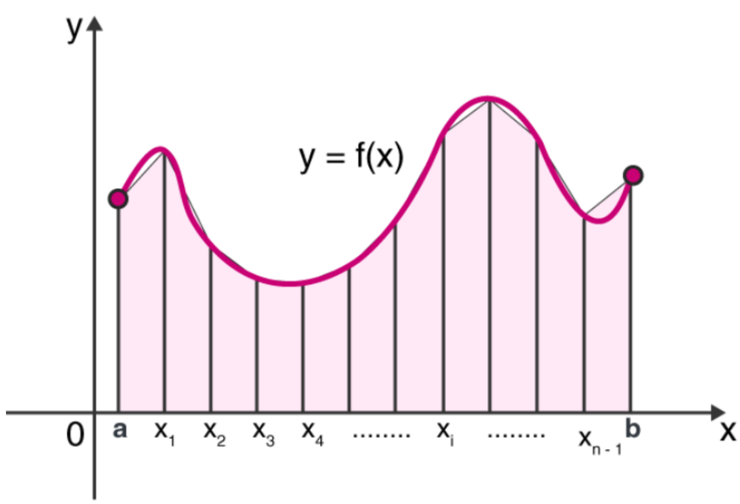
With error given by

$$E(f) = \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x))\,dx.$$



The Trapezoidal Rule

This technique is a much more accurate way to approximate area beneath a curve. To construct the trapezoids, you mark the height of the function at the beginning and end of the width interval, then connect the two points. However, this method requires you to memorize a formula.



Definition

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\frac{b-a}{2n}[f(a)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(b)]$$

The Trapezoidal Rule A Second Glimpse

$$\int_a^b f(x)\,dx=\frac{h}{2}[y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_n]$$

where $[a, b]$ is partitioned into n subintervals of equal length.

$$\Delta x=h=\frac{b-a}{n}$$

Ex 1: Approximate the area beneath $f(x)=x^2+1$ on the interval $[0,3]$ using the Trapezoidal Rule with $n=5$ trapezoids.

Solution:

$$\Delta x=h=\frac{b-a}{n}=\frac{3-0}{5}=\frac{3}{5}$$

x	0	$\frac{3}{5}$	$\frac{6}{5}$	$\frac{9}{5}$	$\frac{12}{5}$	3
$f(x)=y$	1	$\frac{34}{25}$	$\frac{61}{25}$	$\frac{106}{25}$	$\frac{169}{25}$	10

$$I=\frac{b-a}{2n}[y_0+2y_1+2y_2+2y_3+2y_4+y_5]$$

$$I=\frac{3}{2(5)}\left[1+2\left(\frac{34}{25}\right)+2\left(\frac{61}{25}\right)+2\left(\frac{106}{25}\right)+2\left(\frac{169}{25}\right)+10\right]$$

$$I=\frac{3}{10}\left[1+\frac{68}{25}+\frac{122}{25}+\frac{212}{25}+\frac{338}{25}+10\right]$$

$$I=\frac{3}{10}\left[\frac{25+68+122+212+338+250}{25}\right]$$

$$I=\frac{3}{10}\left[\frac{1015}{25}\right]=\frac{609}{50}=12.18$$

Ex 2: Approximate the area beneath $y=sinx$ on the interval $[0,\pi]$ using the Trapezoidal Rule with $n=4$ trapezoids.

Solution:

$$\Delta x=h=\frac{b-a}{n}=\frac{\pi-0}{4}=\frac{\pi}{4}$$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f(x) = y$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0

$$I = \frac{b-a}{2n}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$I = \frac{\pi}{2(4)}\left[0 + 2\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 2\left(\frac{\sqrt{2}}{2}\right) + 0\right]$$

$$I = \frac{\pi}{8}[0 + \sqrt{2} + 2 + \sqrt{2} + 0]$$

$$I = \frac{\pi}{8}[2 + 2\sqrt{2}]$$

$$I \approx 1.894$$

Ex 3: Given the following experiment data determine the approximate area enclosed by a curve passing through the data and the x-axis by the trapezoidal rule.

x	1	2	3	5
$f(x) = y$	2	4	5	7

Solution:

$$I = I_1 + I_2$$

$$I_1 = \frac{h_1}{2}[f(1) + 2f(2) + f(3)], h_1 = 2 - 1 = 1$$

$$I_2 = \frac{h_2}{2}[f(3) + f(5)], h_2 = 5 - 3 = 2$$

$$I = \frac{1}{2}[2 + 2(4) + 5] + \frac{2}{2}[5 + 7] = 19.5$$

Ex 4: Approximate the area under the curve $y = f(x)$ between $x = 0$ and $x = 8$ using Trapezoidal Rule with $n = 4$ subintervals. A function $f(x)$ is given in the table of values.

x	0	2	4	6	8
$f(x)$	3	7	11	9	3

Solution:

The Trapezoidal Rule formula for $n = 4$ subintervals is given as:

$$I = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

Here the subinterval width $\Delta x = 2$.

Now, substitute the values from the table, to find the approximate value of the area under the curve.

$$A \approx I = \frac{2}{2}[3 + 2(7) + 2(11) + 2(9) + 3]$$

$$A \approx I = 3 + 14 + 22 + 18 + 3 = 60$$

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 60.

Ex 5: Approximate the area under the curve $y = f(x)$ between $x = -4$ and $x = 2$ using Trapezoidal Rule with $n = 6$ subintervals. A function $f(x)$ is given in the table of values.

x	-4	-3	-2	-1	0	1	2
$f(x)$	0	4	5	3	10	11	2

Solution:

The Trapezoidal Rule formula for $n = 6$ subintervals is given as:

$$I = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6)]$$

Here the subinterval width $\Delta x = 1$.

Now, substitute the values from the table, to find the approximate value of the area under the curve.

$$A \approx I = \frac{1}{2}[0 + 2(4) + 2(5) + 2(3) + 2(10) + 2(11) + 2]$$

$$A \approx I = \frac{1}{2}[8 + 10 + 6 + 20 + 22 + 2] = \frac{68}{2} = 34$$

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 34.

Simpson's Rule

Another area approximating tool is Simpson's Rule.
Geometrically, it creates tiny parabolas to wrap closer around the function we are approximating.
The formula is similar to the Trapezoidal Rule, with a small catch...
you can only use an even number of subintervals.

Definition

Let f be continuous on $[a, b]$.
Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\frac{b-a}{3n}[f(a)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(b)]$$

Simpson's Rule A Second Glimpse

$$\int_a^b f(x)\,dx=\frac{h}{3}[y_0+4y_1+2y_2+4y_3+\cdots+2y_{n-2}+4y_{n-1}+y_n]$$

where $[a, b]$ is partitioned into n even subintervals of equal length.

$$\Delta x=h=\frac{b-a}{n}$$

Ex. 1: Approximate the area beneath $f(x)=x^2+1$ on the interval $[0,3]$ using Simpson's Rule with $n=6$ subintervals.

Solution:

$$\Delta x=h=\frac{b-a}{n}=\frac{3-0}{6}=\frac{1}{2}$$

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$f(x)=y$	1	$\frac{5}{4}$	2	$\frac{13}{4}$	5	$\frac{29}{4}$	10

$$I=\frac{h}{3}[y_0+4y_1+2y_2+4y_3+2y_4+4y_5+2y_6]$$

$$I=\frac{1}{2(3)}\left[1+4\left(\frac{5}{4}\right)+2(2)+4\left(\frac{13}{4}\right)+2(5)+4\left(\frac{29}{4}\right)+10\right]$$

$$I=\frac{1}{6}[1+5+4+13+10+29+10]$$

$$I=\frac{1}{6}[72]=12$$

Ex. 2: Approximate the area beneath $y = sinx$ on the interval $[0, \pi]$ using the Simpson's Rule with $n = 6$ subintervals.

Solution:

$$\Delta x = h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$f(x) = y$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

$$I = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6]$$

$$I = \frac{\pi}{6(3)}\left[0 + 4\left(\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) + 4(1) + 2\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right) + 0\right]$$

$$I = \frac{\pi}{18}[0 + 2 + \sqrt{3} + 4 + \sqrt{3} + 2 + 0]$$

$$I = \frac{\pi}{18}[8 + 2\sqrt{3}]$$

$$I \approx 2.001$$

Ex. 3: Find approximate value of the following integrations by Simpson's rule $I = \int_0^4 x\, ln\, x\, dx$ where $n = 8, a = 0, b = 4$.

$$h = \frac{b-a}{n} = \frac{4-0}{8} = 0.5$$

$f(x) = x\, ln\, x$

x_i	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x_i) = y_i$	0	-0.347	0	0.608	1.386	2.291	3.296	4.385	5.545

$$I = \frac{h}{3}(y_0 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_8)$$

$$I = \frac{0.5}{3}(0 + 4(-0.347 + 0.608 + 2.291 + 4.385) + 2(0 + 1.386 + 3.296) + 5.545)$$

$$= 0.167(0 + 4(6.937) + 2(4.682) + 5.545) = 0.167(27.748 + 9.364 + 5.545)$$

$$= 0.167(42.657) = 7.124$$

Ex. 4: Use the Simpson's rule of integration to determinate the value of the following integrals $I = \int_1^2 \frac{\cos x}{x} dx$, where $n = 8, a = 1, b = 2$.

$$h = \frac{b-a}{n} = \frac{2-1}{8} = 0.125$$

$$f(x) = \frac{\cos x}{x}$$

x_i	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
$f(x_i) = y_i$	0.5403	0.3833	0.2523	0.1415	0.0472	-0.0333	-0.1019	-0.1598	-0.2081

$$I = \frac{h}{3}\left(y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right)$$

$$I = \frac{0.125}{3}(0.5403 - 0.2081 + 4(0.3833 + 0.1415 - 0.0333 - 0.1598) + 2(0.2523 +$$
$$0.0472 - 0.1019)) = 0.0417\left(0.3322 + 4(0.3317) + 2(0.1976)\right) = 0.0417(2.0542)$$
$$= 0.0856$$

Euler Method

Taylor series method is the fundamental numerical method for the solution of the initial value problem

Expanding $y(x)$ in Taylor series about any point x_i, with the Lagrange form of remainder, we obtain

$$y(x) = y(x_i) + (x - x_i)\, y'(x_i) + \frac{1}{2!}(x - x_i)^2\, y''(x_i) + \ldots + \frac{1}{p!}(x - x_i)^p\, y^{(p)}(x_i)$$

$$+ \frac{1}{(p+1)!}(x - x_i)^{p+1}\, y^{(p+1)}(x_i + \theta h) \qquad (1)$$

where $0 < \theta < 1$, $x \in [x_0, b]$ and b is the point up to which the solution is required.

We denote the numerical solution and the exact solution at x_i by y_i and $y(x_i)$ respectively.

Now, consider the interval $[x_i, x_{i+1}]$. The length of the interval is $h = x_{i+1} - x_i$.

Substituting $x = x_{i+1}$ in (1) , we obtain

$$y(x_{i+1}) = y(x_i) + h\, y'(x_i) + \frac{h^2}{2!}\, y''(x_i) + \ldots + \frac{h^p}{p!}\, y^{(p)}(x_i) + \frac{h^{p+1}}{(p+1)!}\, y^{(p+1)}(x_i + \theta h).$$

Neglecting the error term, we obtain the *Taylor series method* as

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \dots + \frac{h^p}{p!} y_i^{(p)}. \tag{2}$$

Note that Taylor series method is an explicit single step method.

the truncation error of the method is given by

$$T_{i+1} = \frac{h^{p+1}}{(p+1)!} y^{(p+1)} (x_i + \theta h). \tag{3}$$

Using the definition of the order , we say that the Taylor series method (2) is of order p.

For $p = 1$, we obtain the first order Taylor series method as

$$y_{i+1} = y_i + hy_i' = y_i + hf(x_i, y_i). \tag{4}$$

This method is also called the *Euler method*. The truncation error of the Euler's method is

$$\text{T.E.} = \frac{h^2}{2!} y_i''(x_i + \theta h). \tag{5}$$

Example *Solve the initial value problem $yy' = x$, $y(0) = 1$, using the Euler method in $0 \le x \le 0.8$, with $h = 0.2$ and $h = 0.1$. Compare the results with the exact solution at $x = 0.8$.*

Solution We have $$y' = f(x, y) = (x/y).$$

Euler method gives $$y_{i+1} = y_i + h\, f(x_i, y_i) = y_i + \frac{hx_i}{y_i}.$$

Initial condition gives $$x_0 = 0,\ y_0 = 1.$$

When $h = 0.2$, we get $$y_{i+1} = y_i + \frac{0.2\, x_i}{y_i}.$$

We have the following results.

$$y(x_1) = y(0.2) \approx y_1 = y_0 + \frac{0.2\, x_0}{y_0} = 1.0.$$

$$y(x_2) = y(0.4) \approx y_2 = y_1 + \frac{0.2\, x_1}{y_1} = 1.0 + \frac{0.2(0.2)}{1.0} = 1.04.$$

$$y(x_3) = y(0.6) \approx y_3 = y_2 + \frac{0.2\,x_2}{y_2} = 1.04 + \frac{0.2(0.4)}{1.04} = 1.11692$$

$$y(x_4) = y(0.8) \approx y_4 = y_3 + \frac{0.2\,x_3}{y_3} = 1.11692 + \frac{0.2(0.6)}{1.11692} = 1.22436.$$

When $h = 0.1$, we get $y_{i+1} = y_i + \frac{0.1\,x_i}{y_i}$.

We have the following results.

$$y(x_1) = y(0.1) \approx y_1 = y_0 + \frac{0.1x_0}{y_0} = 1.0.$$

$$y(x_2) = y(0.2) \approx y_2 = y_1 + \frac{0.1x_1}{y_1} = 1.0 + \frac{0.1(0.1)}{1.0} = 1.01.$$

$$y(x_3) = y(0.3) \approx y_3 = y_2 + \frac{0.1x_2}{y_2} = 1.01 + \frac{0.1(0.2)}{1.01} = 1.02980.$$

$$y(x_4) = y(0.4) \approx y_4 = y_3 + \frac{0.1x_3}{y_3} = 1.0298 + \frac{0.1(0.3)}{1.0298} = 1.05893.$$

$$y(x_5) = y(0.5) \approx y_5 = y_4 + \frac{0.1x_4}{y_4} = 1.05893 + \frac{0.1(0.4)}{1.05893} = 1.09670.$$

$$y(x_6) = y(0.6) \approx y_6 = y_5 + \frac{0.1x_5}{y_5} = 1.0967 + \frac{0.1(0.5)}{1.0967} = 1.14229.$$

$$y(x_7) = y(0.7) \approx y_7 = y_6 + \frac{0.1x_6}{y_6} = 1.14229 + \frac{0.1(0.6)}{1.14229} = 1.19482.$$

$$y(x_8) = y(0.8) \approx y_8 = y_7 + \frac{0.1x_7}{y_7} = 1.19482 + \frac{0.1(0.7)}{1.19482} = 1.25341.$$

The exact solution is $y = \sqrt{x^2 + 1}$.

At $x = 0.8$, the exact value is $y(0.8) = \sqrt{1.64} = 1.28062$.

The magnitudes of the errors in the solutions are the following:

$$h = 0.2:\ |\ 1.28062 - 1.22436\ | = 0.05626.$$

$$h = 0.1:\ |\ 1.28062 - 1.25341\ | = 0.02721.$$

Example *Consider the initial value problem* $y' = x(y + 1)$, $y(0) = 1$. *Compute* $y(0.2)$ *with* $h = 0.1$ *using (i) Euler method (ii) Taylor series method of order two*

Solution We have $f(x, y) = x(y + 1)$, $x_0 = 0$, $y_0 = 1$.

(*i*) Euler's method: $y_{i+1} = y_i + h\, f(x_i, y_i) = y_i + 0.1[x_i\,(y_i + 1)]$.

With $x_0 = 0$, $y_0 = 1$, we get

$$y(0.1) \approx y_1 = y_0 + 0.1[x_0\,(y_0 + 1)] = 1 + 0.1[0] = 1.0.$$

With $x_1 = 0.1$, $y_1 = 1.0$, we get

$$y(0.2) \approx y_2 = y_1 + 0.1\,[x_1(y_1 + 1)]$$

$$= 1.0 + 0.1[(0.1)(2)] = 1.02.$$

(*ii*) Taylor series second order method.

$$y_{i+1} = y_i + h\, y_i' + \frac{h^2}{2!}\, y_i'' = y_i + 0.1\, y_i' + 0.005 y_i''.$$

We have $\qquad y'' = xy' + y + 1.$

With $x_0 = 0$, $y_0 = 1$, we get

$$y_0' = 0,\ y_0'' = x_0 y_0' + y_0 + 1 = 0 + 1 + 1 = 2.$$

$$y(0.1) \approx y_1 = y_0 + 0.1 y_0' + 0.005 y_0''$$

$$= 1 + 0 + 0.005\,[\,2] = 1.01.$$

With $x_1 = 0.1$, $y_1 = 1.01$, we get

$$y_1' = 0.1(1.01 + 1) = 0.201.$$

$$y_1'' = x_1 y_1' + y_1 + 1 = (0.1)(0.201) + 1.01 + 1 = 2.0301.$$

$$y(0.2) \approx y_2 = y_1 + 0.1\, y_1' + 0.005 y_1'$$

$$= 1.01 + 0.1\,(0.201) + 0.005(2.0301)] = 1.04025.$$

Runge-Kutta method

The Runge-Kutta method is a self-starting numerical method which is comely use in solving ODE.

There are many types of this method depending on the order of the drifter.

1. Second-order R-K method is

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_1)$$

Where $k_1 = hf(x_n, y_n)$

$$k_2 = hf(x_n + h, y_n + k_1)$$

2. Third order R-K method is

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf(x_n + h, y_n - k_1 + 2k_2)$$

3. Forth order R-K

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Which is generally used in solving ODE numerically.

Ex (1):- solve the following differential eq. using the 2nd order R-K method

$\frac{dy}{dx}=\frac{1}{2}y, y(0)=1, h=1$

Solu.

$y_1=y_0+\frac{1}{2}(k_1+k_2)$

$$k_1=hf(x_0,y_0)=1f(0,1)=1\left(\frac{1}{2}\cdot 1\right)=0.5$$

$$k_2=hf(x_0+h,y_0+k_1)=1f(1,1.5)=1\cdot\frac{1.5}{2}=0.75$$

$$\therefore y_1=1+\frac{1}{2}(0.5+0.75)=1+\frac{1.25}{2}=1+0.625=1.625$$

Ex (2):- by R-K fourth order solve the following eq. $y'=x^2-2xy$, when $y(1)=0, h=0.2, n=2$

Solu.

$f(x,y)=y'=x^2-2xy$

$k_1=hf(x_0,y_0)=0.2\big(1^2-2(1)(0)\big)=0.2$

$k_2=hf(x_0+0.1,y_0+0.1)=0.2f(1.1,0.1)=0.2\big((1.1)^2-2(1.1)(0.1)\big)$

$=0.198$

$k_3=0.2f(1.1,0.099)=0.2\big((1.1)^2-2(1.1)(0.099)\big)=0.2(1.21-0.2178)$

$=0.1984$

$k_4=0.2f(1.2,0.1984)=0.2\big((1.2)^2-2(1.2)(0.1984)\big)=0.2(1.44-0.4762)$

$=0.1928$

$y_1=y_0+\frac{1}{6}(k_1+2(k_2+k_3)+k_4)$

$=0+\frac{1}{6}(0.2+2(0.198+0.1984)+0.1928)=0+0.1976=0.1976$

$x_1=x_0+h=1+0.2=1.2$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

$$k_1 = hf(x_1, y_1) = 0.2f(1.2{,}0.1976) = 0.2\big((1.2)^2 - 2(1.2)(0.1976)\big) = 0.1932$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f\left(1.3{,}0.1976 + \frac{0.1932}{2}\right)$$

$$= 0.2\big((1.3)^2 - 2(1.3)(0.2942)\big)$$

$$= 0.2(1.69 - 0.76492) = 0.1850$$

$$k_3 = 0.2f(1.3{,}0.2901) = 0.2\big((1.3)^2 - 2(1.3)(0.2901)\big) = 0.2(1.69 - 0.7543)$$

$$= 0.1871$$

$$k_4 = 0.2f(1.4{,}0.3847) = 0.2\big((1.4)^2 - 2(1.4)(0.3847)\big) = 0.2(1.96 - 1.0772)$$

$$= 0.1766$$

$$y_2 = 0.1976 + \frac{1}{6}(0.1932 + 2(0.185 + 0.1871) + 0.1766) = 0.1976 + 0.1857$$

$$= 0.3833$$