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Direct and alternative methods for parabolic problem containing an unknown Dirichlet boundary condition

A Thesis
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بسم الله الرحمن الرحيم

﴿هُوَ ٱلَّذِىٓ أَنزَلَ عَلَيۡكَ ٱلۡكِتَٰبَ مِنۡهُ ءَايَٰتٞ مُّحۡكَمَٰتٌ هُنَّ
أُمُّ ٱلۡكِتَٰبِ وَأُخَرُ مُتَشَٰبِهَٰتٞۖ فَأَمَّا ٱلَّذِينَ فِي قُلُوبِهِمۡ زَيۡغٞ
فَيَتَّبِعُونَ مَا تَشَٰبَهَ مِنۡهُ ٱبۡتِغَآءَ ٱلۡفِتۡنَةِ وَٱبۡتِغَآءَ تَأۡوِيلِهِۦۖ وَمَا يَعۡلَمُ
تَأۡوِيلَهُۥٓ إِلَّا ٱللَّهُۗ وَٱلرَّٰسِخُونَ فِي ٱلۡعِلۡمِ يَقُولُونَ ءَامَنَّا بِهِۦ كُلّٞ
مِّنۡ عِندِ رَبِّنَاۗ وَمَا يَذَّكَّرُ إِلَّآ أُوْلُواْ ٱلۡأَلۡبَٰبِ﴾

صدق الله العظيم

(سورة ال عمران / الآية 7)

Chapter One

Introduction

1.1 Introduction

In 1993, Chen & Lin have analysed two-dimensional hyperbolic heat conduction problems using a hybrid numerical approach. In such problems, thermal waves propagate at a finite speed, often resulting in numerical oscillations near the thermal front. To mitigate these oscillations, the authors have proposed a hybrid numerical method. This technique eliminates the time-dependent terms in the governing differential equations through the application of the Laplace transform. Subsequently, the control volume method is employed to discretize the spatial domain within the transformed space. The effectiveness of the proposed method is attributed to the careful selection of shape functions. Several examples involving irregular geometries are presented to demonstrate the method's capability [9]. In 2009, Ching-Yu introduced a sequential method for estimating the boundary conditions in two-dimensional hyperbolic heat conduction problems. The inverse solution is obtained through a combination of the finite difference method, the concept of future time levels, and a modified Newton–Raphson iteration. The resulting system of equations is solved using an iterative procedure. Numerical results confirm that the proposed approach is both accurate and stable in determining the unknown boundary conditions for two-dimensional inverse hyperbolic heat conduction problems [11]. In 2013 Pourgholi et. al., proposed a numerical approach that integrates the Haar wavelet method with Tikhonov regularization. This combined method was employed to solve inverse problems related to two-dimensional parabolic and hyperbolic equations using noisy data. The authors demonstrated that their approach yields stable numerical solutions. The technique involves a sensor positioned at an interior point of the domain, specifically at $x = a$ where $0 < a < 1$, to measure the temperature function v. Additionally, they established that the proposed method exhibits exponential convergence [36].

Su & Silva Neto, (2001), have estimated the transient radial and circumferential (azimuthal) variation of a volumetric heat source within a cylindrical rod using Alifanov's iterative regularization method. The authors have formulated the inverse problem as an optimization problem, where a squared residual functional is minimized employing the conjugate gradient method. The step size in the descent direction is determined through a sensitivity analysis, while the gradient is obtained by solving the corresponding adjoint problem. To assess the accuracy of the proposed estimation technique, two test scenarios are analyzed: the first involves radial and temporal dependence, while the second incorporates radial, azimuthal, and temporal variations. The influence of sensor quantity and measurement noise on the solution is also examined [42].

In [43] Tadi, (1997), has focused on the estimation of the diffusion coefficient using measurements taken at the boundary. Specifically, the problem involves recovering the spatially dependent diffusion coefficient of a rod. The proposed approach introduces a time-dependent formulation of the unknown parameter. With suitable measurement data, the system parameters are iteratively adjusted from arbitrary initial guesses toward their true values over time. An explicit evolution equation for the parameter is derived by minimizing the error along the trajectory. The resulting method forms the basis of an iterative algorithm. In [47] Zhou et. al., (2010), have studied the inverse heat conduction problem (IHCP) in a one-dimensional composite slab accounting for rate-dependent pyrolysis reactions and outgassing flow effects. The thermal properties of the composite material are assumed to vary with temperature. To solve the IHCP, the authors have employed an iterative regularization strategy through a nonlinear conjugate gradient formulation. The analysis focuses on scenarios where the front surface of the composite slab is exposed to high-intensity periodic laser heating. In 1910, Haar has interested by studying the Timing channels (TCs). Indeed, the TCs remain a significant security threat, enabling unauthorized information leakage through the manipulation of event timing or ordering. Traditional detection techniques either rely on signature-based methods to identify known TCs or anomaly-based models that detect deviations from legitimate traffic patterns. However, these approaches are often ineffective in software-defined networking (SDN) environments due to factors such as fluctuating traffic behavior, imprecise time synchronization, and dynamic network topologies. Additionally, stealthy TCs can mimic legitimate traffic, evading anomaly-based detection. To address these challenges, Haar has introduced a novel detection framework that leverages the flexible resources of cloud environments. The proposed system dynamically reconfigures SDN to perform differential analysis on outbound flows from multiple virtual machines (VMs). It employs a wavelet-based multi-resolution decomposition of flow timing data, followed by the application of the Kullback-Leibler divergence (KLD) to quantify dissimilarities among flows. Unlike traditional anomaly detection methods, the proposed approach does not require prior modeling of legitimate traffic and remains robust even under noisy conditions and timing inaccuracies. The framework is implemented as a prototype system named OBSERVER, designed for deployment in SDN environments. Experimental results demonstrate its effectiveness in detecting both known and novel stealthy TCs, offering improved detection accuracy, reduced latency, and minimal performance overhead compared to existing methods [18]. In[21] Hariharan et. al., (2009), have proposed an accurate and efficient Haar wavelet method for solving the well-known Cahn–Allen equation. The developed scheme is versatile and applicable to a broad class of nonlinear partial differential equations. The effectiveness and robustness of the method are validated through its application. The use of Haar wavelets proves to be a reliable and computationally attractive approach, offering simplicity, flexibility, rapid convergence, and low computational cost. In

[8] Chen & Hsiao, (1997), have presented a state analysis framework for time-delay systems using Haar wavelets. Leveraging key properties of Haar functions, the method employs a specialized product matrix along with an associated coefficient matrix to address the complexities of time-delayed systems. The unknown Haar coefficient matrix is determined using the Kronecker product technique. The effectiveness, accuracy, and broad applicability of the proposed Haar wavelet approach are demonstrated through several numerical examples. In [24] Hsiao & Wang, (2001), have introduced a simple and efficient algorithm based on the Haar wavelet method for solving nonlinear stiff problems. Simulation results demonstrate that the proposed method significantly reduces computational time—requiring only about one-tenth of that needed by the widely used Runge–Kutta–Fehlberg method—while maintaining comparable accuracy. In[28] Kalpana & Balachandar, (2007), have proposed a Haar wavelet-based analytical method for observer design in generalized state-space or singular systems, specifically within transistor circuit models. The method is formulated from both incremental and multi-resolution perspectives. By adjusting the time scale parameter, the approach allows for accurate solutions while preserving the essential features of the system's response. The technique is straightforward to implement computationally and demonstrates notable advantages in terms of speed, flexibility, and efficiency.

In[37] Rashid & Nachaoui, (2025), have presente a Haar wavelet-based reconstruction method for recovering missing boundary data on inaccessible regions using measurements obtained from accessible boundary segments. The method is specifically designed to address inverse Cauchy problems governed by the Poisson equation, which are known to be severely ill-posed. The proposed approach is mathematically straightforward and adaptable, making it suitable for Cauchy problems associated with a variety of partial differential equations encountered in natural sciences, engineering, and economics. To mitigate the ill-conditioning inherent in the resulting linear system, a combination of preconditioning and regularization techniques is employed. Comparative numerical experiments, including those using a meshless method based on polynomial expansion; demonstrate the superior accuracy and stability of the proposed approach. Additional numerical results further confirm its effectiveness and robustness. Berdawood, K.A et. al., (2020), the finite element method, as described [5], has been employed to solve the problem efficiently. In (2020) an efficient relaxed alternating procedure demonstrated convergence for all values of k in the Helmholtz equation and enhanced the rate of convergence for the modified Helmholtz equation. As shown in [4], the authors established that for every k , there exists an interval within which convergence is guaranteed.

In this thesis, we address an inverse Cauchy problem associated with the parabolic heat equation in two spatial dimensions, where partial boundary data are provided. To handle the ill-posedness inherent in such problems, we propose a numerical technique that reconstructs the unknown boundary data using the available information from the opposite side. The solution is approximated using a polynomial expansion, in this thesis; we apply a polynomial expansion method similar to those used in [1, 2, 16, 22, 23, 26, 33, 35] to approximate the solution of the problem. Which reduces the problem to a direct linear system solvable through iterative techniques. Remarkably, the proposed method achieves excellent convergence and stability without the need for additional regularity conditions, even for large values of the parameter k. This thesis evaluates the effectiveness of the proposed approach by comparing its performance with previous studies, particularly those by Berdawood, K.A. et al. (2020), demonstrating improved accuracy and convergence behavior. The numerical experiments are implemented using MATLAB.

We therefore explore a direct polynomial expansion approach, originally proposed in Aboud et, al. (2022), [1] to approximate solutions of Helmholtz-type Cauchy problems. Unlike iterative schemes, it avoids slow convergence, though care must be taken since ill-posedness manifests as ill-conditioning in the linear system. This difficulty is alleviated by efficient preconditioning.

The aim of this thesis is to develop and analyze a numerical method for solving an inverse Cauchy problem associated with a two-dimensional parabolic heat equation.

The proposed method is based on polynomial expansion and the obtained linear system is solved iteratively using the CGLS algorithm.

The main objective is to reconstruct the unknown Dirichlet boundary condition from partial boundary data, while ensuring accuracy, stability, and efficiency.

Thesis outline on structures is organized as follows

Chapter one: view the Introduction.

Chapter Two: provides a review of fundamental concepts and relevant literature.

Chapter Three: presents the mathematical formulation, the reformulation into a modified Helmholtz equation, and the polynomial expansion technique.

Chapter Four: describes the implementation and displays the numerical results.

Finally, Chapter Five: concludes the thesis and suggests directions for future research.

خلاصة:

تعالج هذه الرسالة مسألة عكسية لمعادلة حرارة مكافئة ببيانات حدوديه غير معروفة. يتم أولاً تحليل المسألة المباشرة باستخدام طريقة الفروقات المحددة، ثم تُحوَّل إلى معادلة هلمهولتز المعدّلة.

ولتقريب الحل، تُستخدم متسلسلة متعددة الحدود، ويُحل النظام الناتج تكرارياً باستخدام خوارزمية التدرج المترافق للمربعات الصغرى (CGLS).

حققّت الطريقة المقترحة تقريبياً دقيقاً ومستقراً حسابياً، حتى للقيم الكبيرة للمعامل الفيزيائي k، حيث تتطلب الطرق التقليدية غالبًا شروطًا مقيدة لضمان التقارب.

تُظهر التجارب العددية فعالية الطريقة دون فرض شروط إضافية على البيانات، مما يحقق خصائص تقارب ممتازة. أُجريت العديد من الاختبارات، مما يُظهر أن الخطأ المطلق يظل صغيرًا حتى مع ازدياد قيمة k، مما يؤكد استقرار الطريقة.

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