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Department of Mathematics

A Meshless Method for solving interface Problems in heat transfer

A Thesis

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بِسْمِ اللَّـهِ الرَّحْمَٰنِ الرَّحِيمِ

قُلْ هَلْ يَسْتَوِي الَّذِينَ يَعْلَمُونَ وَالَّذِينَ لَا يَعْلَمُونَ ۗ
إِنَّمَا يَتَذَكَّرُ أُولُو الْأَلْبَابِ

صدق الله العظيم

(سورة الزمر: الاية 9)

Abstract

This thesis presents an advanced meshless method based on a polynomial expansion to solve an inverse Cauchy problem in heat transfer within a multi-region domain with a free-interface boundary. The computational domain, defined as rectangle in $\mathbb{R}^2$, is subdivided into two distinct sub-regions (Ω_1, Ω_2), each satisfying Laplace's equation with unknown boundary and interface conditions that rend the problem severely ill-posed and unstable. Traditional mesh-dependent numerical approaches often struggle with geometric complexity and instability inherent in such problems. The proposed method approximates the temperature distribution $T(x,y)$ in each sub-region through a polynomial expansion, enabling the transformation of partial differential equations and interface conditions into a solvable linear system $AC = B$, where the coefficient vector C encapsulates the polynomial terms. Numerical implementation employs a conjugate gradient least squares algorithm to compute the expansion coefficients, ensuring convergence and computational stability. The method's accuracy and robustness are verified for different cases of exact Dirichlet data, demonstrating excellent agreement and resilience to data perturbations. By eliminating the need for mesh generation, this approach offers a highly flexible, efficient, and stable framework for inverse heat transfer problems, with broad potential applications in engineering design, thermal diagnostics, and materials science.

Chapter One

1.1 Introduction

Inverse problems represent a fundamental branch of modern applied mathematics, appeared from unknown causes of observed effects, especially when direct measurements of the basic phenomena are impossible. These problems have attracted attention in the 19th century, by efforts to reconstruct electromagnetic fields from surface measurements (Hadamard, 1953) [9]. And the theoretical work was advanced in the 20th century by presenting (regularization techniques) by (Tikhonov and Arsenin - 1977) [2], with the goal was to stabilize solutions for (ill-posed problems) [15, 17, 23]. (ill-posedness) features is no- (existence, uniqueness) solutions only depended on (input data) which makes these problems sensitive to noise and measurement errors [23].

Among inverse problems, the (inverse Cauchy problem) plays a leading role in applied mathematical [12, 16, 22, and 28]. This problem involves recovering unknown boundary conditions over specified data on given parts of the boundary, and its hypersensitivity to (noise and instability) [1, 3, 5, 8, 11, 19, 24, 33].

In the field of heat transfer, inverse problems appear in applications such as thermal conductivity, heat flux recovery on boundaries, and thermal diagnostics of materials and structures [21, 28]. These applications are important for developing efficient numerical methods for such problem like ill-posed issues [2, 27, and 29].

The applications of the numerical methods (like finite element methods [14] and finite difference methods [13]) to solve the inverse problems, possess important difficulties. Their reliance on mesh generation leads to computational inefficiencies and difficulties in handling interface conditions, especially when boundaries are irregular or data are incomplete also, these methods may require mesh refinement and stabilization strategies that increase computational cost without guaranteeing solution stability [14].

Meshless methods have recently attracted attention as promising alternatives [12, 13, 24, 38], because they eliminate the mesh generation by employing scattered nodes across the computational domain [12], and this flexibility is advance for complex geometries shapes problems [12, 37].

Merging meshless method with (polynomial expansion techniques) provides powerful approximation solutions to partial differential equations (PDEs) in inverse problems [10, 26, 31, and 32]. Polynomial expansions enable efficient solution through series of basic functions, which transform a differential equations and boundary conditions into an easy solvable algebraic system [10, 19, 25, and 32].

This thesis focuses on processing (the inverse Cauchy problem) for Laplace's equation in a multi-region domain divided into two subdomains Ω_1 , Ω_2 each subdomain is governed by Laplace's equation, subject to unknown boundary conditions and interface continuity requirements at the internal boundaries Ω_1 , Ω_2 . The problem is an ill-posed problem; which makes it more difficult missing boundary data on one of the interfaces, which demands a stable and accurate computational strategy [3, 8, 18, 27, and 29].We propose a technique (meshless polynomial expansion method) to approximate these temperature field T(x, y) in each subdomains [10, 26, 31, and 32]. The method represents as a variate polynomial, with coefficients determined by solving a linear system derived from the governing PDEs and interface conditions. This will permit to obtain a linear system AC = B for which we can use any well-known techniques to solve the linear system (like conjugate gradient method (CGM) [27], conjugate gradient least squares

(CGLS) algorithm [34], LU, Gauss elimination, …) [36], in this thesis we use CGLS which demonstrate numerical stability and convergence when it use with this kind of problems [34]. Our proposed method has the advantage that it is a kind of meshless based method [10, 26, 31, and 32]. This thesis is composed of the following chapters:

In Chapter 2, we recall some basic concepts and definitions that we will need for the rest of the thesis. In Chapter 3, we present the theoretical part that depends on the use of the polynomial expansion to approximate the solution companied with the Newton method. In Chapter 4, the numerical results applied on some different examples are presented to validate the proposed method. Finally, in Chapter 5, we present some conclusions and some suggested future works.

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