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# On The Category of Graded Module

**Omar Adnan[*1], Abdulsatar J. AL-Juburie[2] and Lieth A. Majed[1]**

University of Diyala – College of science, Iraq,
University of Diyala – College of education for pure science, Iraq

omar.adnan.199784@gmail.com

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## Abstract

In this article, we introduce some results on category theory and construct some propositions and examples for the category of R-module, and graded R-module, differential graded R-module. Also we gave the proof for results.

**Keywords:** Category theory**,** graded ring, graded module, differential graded module.

## نظرية الفئة للمقاسات التفاضلية المدرجة

**عمر عدنان[1] ، عبد الستار جمعة ذياب[2] و ليث عبد اللطيف [1]**

كلية العلوم – جامعة ديالى
كلية التربية للعلوم الصرفة – جامعة ديالى

## الخلاصة

قمنا في هذا البحث ببناء بعض النتائج حول نظرية الفئة، وكذلك قمنا بإنشاء بعض القضايا مع الامثلة حول نظرية الفئة للمقاسات بشكل عام وللمقاسات المدرجة والمقاسات التفاضلية المدرجة بشكل خاص. ايضا اعطينا برهان للنتائج.

**الكلمات المفتاحية:** نظرية الفئة، الحلقات المدرجة، المقاسات المدرجة، المقاسات التفاضلية المدرجة.

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## Introduction

Let $\mathbb{G}$ be a group together with the identity $e$ and $\mathcal{R}$ institute a ring with the unit 1 vary various from 0. Then $\mathcal{R}$ is said to be $\mathbb{G}$-graded ring if found an collective subgroup $\mathcal{R}_g$ of $R$ s.t $\mathcal{R} = \oplus \sum_{g \in \mathbb{G}} \mathcal{R}_g$ and $\mathcal{R}_g \mathcal{R}_h \subseteq \mathcal{R}_{gh}$ for each $g, h \in \mathbb{G}$.

A differential graded category (DG category) over the commutative ring $\mathcal{R}$ is $\mathcal{R}$-category $\mathcal{F}$ whose morphism spaces are differential graded $\mathcal{R}$-modules and whose compositions.

$$\mathcal{F}(\mathbb{Y}, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{F}(\mathbb{Y}, Z), (f, g) \mapsto fg$$

Are morphisms of differential graded R-modules.

DG categories can be found in [11]. In the 1970, they started to be utilized in the representation theory of finite –dimensional algebras (see [13] and [7]). According to B. Keller [10], the DG category improves our comprehension of triangulated categories that appear in algebra and geometry. Since then, DG categories have been the subject of in-depth research. See [10] for a description of the DG category theory

A differential graded algebra (DG algebra) over the commutative ring $\mathcal{R}$ is a graded algebra, $A = \oplus_{i \in z} A_i$ over R together with a differential, that is a $\mathcal{R}$-linear map $d: A \to A$ of degree -1 with $d2 = 0$ satisfying the Leibniz rule $d(rs) = d(r)s + (-1)^{|r|} r d(s)$, where $r, s \in \mathcal{R}$ and $r$ is a graded elements of degree $|r|$. We can think of DG algebras, a thorough understanding of their properties has been developed via the work of many different scholars as they have been the subject of extensive investigation in recent years. For example, D. Dugger and B. Shipley [8] have investigated the relationship between DG algebras and topological ring spectra. M. Angel and R. Dlaz [4] have introduced N-DG algebras (abbreviated as N-dga), and they have studied the module space of deformations of the differential of an N-differential graded algebra. J. Jardine [9] has constructed a closed model structure for the category of non-commutative DG algebra can be found in [3], [5], [6], and [12]. We will introduce some results about category theory and construct some propositions and examples for the category of $\mathcal{R}$-module graded $\mathcal{R}$-module, differential graded $\mathcal{R}$-module.

## Prerequisites and Results

In this section, we introduce some of the definitions and the fundamental concepts of graded algebras differential graded modules and category theory that play significant role in our work. Also, we will introduce some facts and example. The proofs for these facts and examples are also given. For more information of differential graded module and category theory you can see [1], [2], [3], [5], [6], [8]and [13] is given in this part.

**Definition2.1** *A* GK-algebra $\mathbb{A}$ is a sequence of K-vector spaces $\{\mathbb{A}_j\}_{j\in\mathbb{Z}}$, with vector space homomorphisms:

$\rho$: $\mathbb{A}_i \otimes_K \mathbb{A}_j \rightarrow \mathbb{A}_{i+j}$ for all $i, j \in \mathbb{Z}$ *and*

$\mu: K \rightarrow \mathbb{A}_0$, s.t. the following diagrams

$$\begin{array}{ccc} \mathbb{A}_i \otimes \mathbb{A}_j \otimes \mathbb{A}_m & \xrightarrow{\rho\otimes 1} & A_{i+j} \otimes \mathbb{A}_m \\ {\scriptstyle 1\otimes\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{A}_i \otimes \mathbb{A}_{j+m} & \xrightarrow{\rho} & \mathbb{A}_{i+j+m} \end{array}$$

$$\begin{array}{ccc} \mathrm{K} \otimes \mathbb{A}_j = \mathbb{A}_j \otimes K & \xrightarrow{1\otimes\mu} & \mathbb{A}_j \otimes \mathbb{A}_0 \\ {\scriptstyle 1\otimes\mu}\downarrow & \searrow & \downarrow{\scriptstyle \rho} \\ \mathbb{A}_0 \otimes \mathbb{A}_j & \xrightarrow[\rho]{} & \mathbb{A}_j \end{array}$$

Commute for all $i, j, m \in \mathbb{Z}$

**Definition 2.2** Let $\mathbb{A}$ be a graded K-algebra and $\Psi: \mathbb{A}_j \otimes_K \mathbb{A}_j \rightarrow \mathbb{A}_j \otimes_K \mathbb{A}_i$ formation the K-vector space isomorphism which takes $p \otimes q$ *into* $q \otimes p$. then $\mathbb{A}$ is commutative iff the following diagram :

$$\begin{array}{ccc} \mathbb{A}_i \otimes_K \mathbb{A}_j & \xrightarrow{\Psi} & \mathbb{A}_j \otimes_K \mathbb{A}_i \\ & \searrow \quad \swarrow & \end{array}$$

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$\rho$ $\rho$

$\mathbb{A}_{i+j}$

Commutes for all $i, j, m, \in \mathbb{Z}$.

**Definition 2.3** Let $\mathcal{R}$ formation a GK-algebra. *A left graded* $\mathcal{R}$-module $\mathcal{M}$ is a graded K-module, together with a sequence $\varphi: \mathcal{R}_i \otimes \mathcal{M}_j \to \mathcal{M}_{i+j}$ of K-homomorphism, for all $i, j \in \mathbb{Z}$ s.t the following diagram :

$$\begin{array}{ccc} \mathcal{R}_i \otimes \mathcal{R}_j \otimes \mathcal{M}_m & \xrightarrow{\rho \otimes 1} & \mathcal{R}_{i+j} \otimes \mathcal{M}_m \\ {\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \mathcal{R}_i \otimes \mathcal{M}_{j+m} & \xrightarrow{\varphi} & \mathcal{M}_{i+j+m} \end{array}$$

$$\begin{array}{ccc} \mathrm{K} \otimes \mathcal{M}_j & \longrightarrow & \mathcal{M}_j \\ {\scriptstyle \mu \otimes 1} \downarrow & & \| \\ \mathcal{R}_0 \otimes \mathcal{M}_j & \xrightarrow[\varphi]{} & \mathcal{M}_j \end{array}$$

Commute for all $i, j, m \in \mathbb{Z}$ where $\mathcal{U}: K \to \mathcal{R}_0$ $here\ \mathcal{R}_0 = K$ $, (k \otimes m) \mapsto km \mapsto km\ and\ (k \otimes m) \mapsto (\mathcal{U}(k) \otimes m) \longmapsto \varphi(\mathcal{U}(k) \otimes m) = km$ is the map presenter by the definition. We indicate to this by $\mathcal{M} = \oplus \sum_{i=-\infty}^{\infty} \mathcal{M}_i$. In a Similar way, we can know the right graded$\mathcal{R}$-modules. If $\mathcal{R}$ is commutative, we may count left $\mathcal{R}$-modules as right $\mathcal{R}$-modules and invers is true. If $m \in \mathcal{M}_j$ we know dim= $j$.

**Definition2.4:** A **category theory** consists of three pieces:

1) Combination of objects obj ($\mathcal{C}$);

2) For any objects $A_1$, $A_2$ in the Combination of objects obj($\mathcal{C}$), we have a set of the morphisms (Hom ($A_1$, $A_2$));

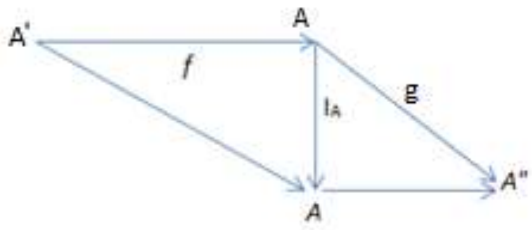
3) There is a composition

$\circ : Hom(A_1, A_2) \times Hom(A_2, A_3) \rightarrow Hom(A_1, A_3)$ for $A_1$ $A_2$, $A_3$ in obj($\mathcal{C}$).

This composition has associative and identity properties.

By the next diagram we explain identity property for the composition. Such that:

$g = g \circ I_A$ and $f = I_A \circ f$.




To explain associative property for the composition operation, let

$$f: A \to B, Hom(A,B)$$

$$g: B \to C, Hom(B,C)$$

$$h: C \to D, Hom(C,D)$$

$$Hom(A,C) \circ Hom(C,D) \to \mathrm{Hom(A,D)}$$

$$Hom(A,B) \circ Hom(B,D) \to \mathrm{Hom(A,D)}.$$

So, if we have objects, arrows, the composition exists and must be associative, then we can construct the category for these structures.

**Definition 2.5:** Let $\mathcal{C}$ and $\mathcal{D}$ are two created categories. A **functor** is a map from $\mathcal{C}$ and $\mathcal{D}$ whach is denoted by f

i) Associates for every object $X$ in $\mathcal{C}$ and object $F(X)$ in $\mathcal{D}$;

ii) Associates for every morphism $f: X \to Y$ in $\mathcal{C}$ a morphism

$$F(f): F(X) \longrightarrow F(Y)$$

in $\mathcal{D}$ s.t following two the conditions hold:

i) $F(id_x) = id_{f(x)}$ for each object $X$ in $\mathcal{C}$;

ii) $F(g \circ f) = F(g) \circ F(f)$ for every morphisms

$$f: X \longrightarrow Y$$

and,

$$g: Y \longrightarrow Z \text{ in } \mathcal{C}.$$

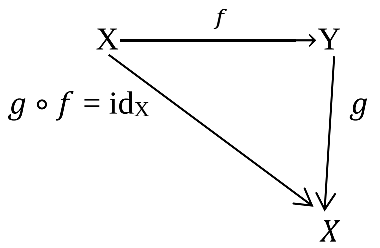
A forgetful functor is a functor which is defined by forgetting some thoughts. For example, the forgetful from category of groups to category of sets, forgets the group structure remembering only the underlying set. Then, a forgetful (also called **underlying functor**) is a mapping from the category of algebraic (groups, abelian groups, modules, rings, vector spaces, ect) into category of sets.

The objects and arrows are left intact by a forgetful functor, except that they are ultimately only taken into account as a sets and maps. Furthermore, The category $\mathcal{S}$ is a subcategory of the category $\mathcal{C}$, sharing the same identities and morphism composition with the category $\mathcal{C}$ objects. There is an obvious faithful functor $I: \mathcal{S} \longrightarrow \mathcal{C}$, called the **inclusion functor** which takes objects and morphisms to themselves.

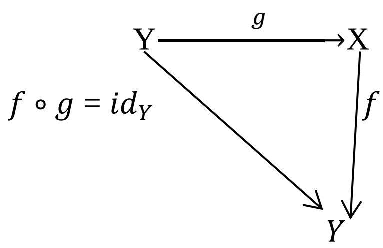
**Proposition 2.6:** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $f: X \to Y$ is an equivalence in $\mathcal{C}$, then F(f) is an equivalence in $\mathcal{D}$. It is possible that F(f) is an evenness without f being an evenness.

***Proof***: Let $F: C \to D$ be a functor. Let $f: X \to Y$ , an evenness in $\mathcal{C}$.

Now, since f is an equivalence in $\mathcal{C}$ this means there exists $g \in$ mor $(Y, X)$ such that $gof = id_X$ and $fog = id_y$




and




Now, $F(f): F(X) \to F(Y)$

**We claim that:** there exists $F(g): F(Y) \to F(X)$, Such that $F(g) = (F(f))^{-1}$

Now we will check this:

$F(g) o F(f) = F(gof)$ (By properties of functor )

$= F(id_X)$ (From above )

$= id_{F(X)}$ (Properties of functor )

So, $F(g)$ is a left inverse of $F(f)$

Also, $F(f)$ is right inverse of $F(g)$ for:

$$F(f) \circ F(g) = F(f \circ g)$$

$$= F(id_Y)$$

$$= id_{F(Y)}$$

therefore, $F(f)$ is an equivalence. So, it is acceptable $F(f)$ to be equivalence even if f is not.

**Definition 2.7:** The **product of objects** $A \times B$ together with the morphisms

$$\pi_A: A \times B \to A \text{ and}$$

$$\pi_B: A \times B \to B$$

that achieve the following international property. If given any other object C in category, with:

$$f: C \to A$$

$$g: C \to B$$

There exists uninque morphism function

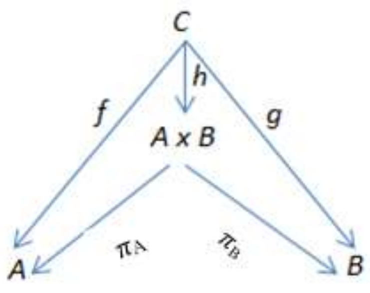
$$h: C \to A \times B$$

s.t :

$$\pi_A \circ h = f$$

$$\pi_B \circ h = g.$$

In other words there is one and only one morphism $h$ s.t both triangles in the diagram below commute.




There exists a unique morphism function .

**Example 2.8:** The product lying under the set category is Cartesian product of non-empty sets. Indeed, let $A, B$ are two sets, we can forms $A \times B = \{(a, b): a \in A, b \in B\}$ is Cartesian product and $\pi_1$, $\pi_2$ the canonical coordinate projection maps. We must prove $A \times B$ satisfies the international property of product, if we have $C$ been any set and if we have

$$f: C \to A$$

$$g: C \to B$$

Be all functions. We construct the function

$$h: C \to A \times B$$

such that

$$h(c) = \big(f(c), g(c)\big)$$

for every element $c \in C$. This satisfies the initial part of international property

$$(\pi_1 \circ h)(c) = \pi_1\left(h(c)\right) = \pi_1\big(f(c)\big)$$

$$(\pi_2 \circ h)(c) = \pi_2\left(h(c)\right) = \pi_2\big(g(c)\big).$$

**Definition 2.9:** Let $\mathcal{C}$ be a category. Given tow objects $X, Y$ in $\mathcal{C}$

their product (if it is exists) is an object $Z$ together with two morphisms,

$Pr_1: z \to X$, in G and

$Pr_2: z \to Y$,

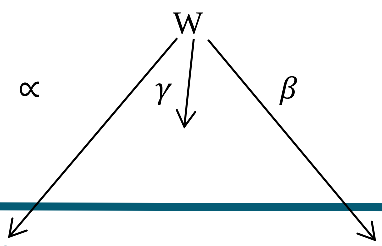
Satisfying the following international property of w is and object in G having morphisms

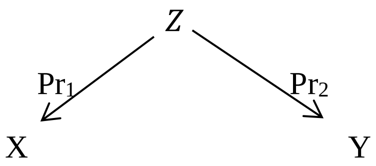
$\propto: w \to x$ and

$\beta: w \to y$

Then there occur a unique morphism

$\gamma: w \to z$ making the diagram commute



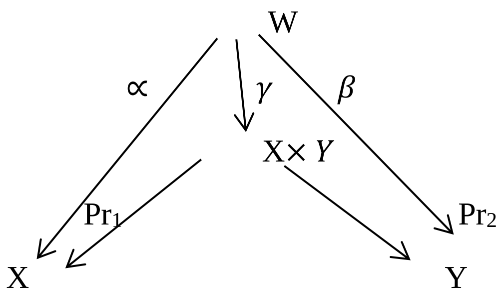



**Example 2.10:** Let $\mathcal{C}$ be a category.

Given two sets $X, Y$. Their product is $X \times Y$, (the Cartesian product). Considering of pairs $(x, y),\ x \in X,\ y \in Y$

$Pr_1 : X \times Y \to X, such\ that\ (x, y) \mapsto x$

$Pr_2 : X \times Y \to Y, such\ that\ (x, y) \mapsto y$. For

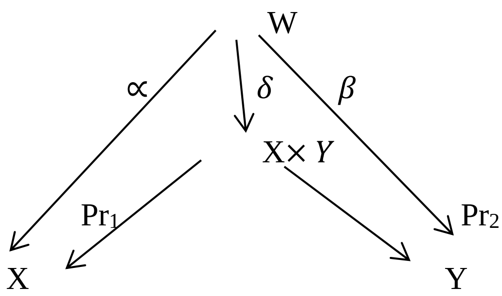



Define $\gamma : W \longrightarrow X \times Y$, by $\gamma(w) = (\propto(w), \beta(w))$

Let $\delta = W \longrightarrow X \times Y$

So $\delta(w) = (\delta_1(w), \delta_2(w))$ with $\delta_1 = w \to x$ and $\delta_2 = w \to y$

But, the diagram is commute




$(Pr_1 \circ \delta)(w) = \propto(w)$

$Pr_1(\delta(w)) = \propto (w)$

$Pr_1(\delta_1(w), \delta_2(w))$

$\delta_1(w)) = \propto (w)$

Similarly $Pr_2(\delta(w)) = \beta(w)$

Hence, $\delta = \gamma$.

**Definition 2.11:** The dual of product is **coproduct**. Let $\mathbb{C}$ be a category, and $A, B$ be an objects in that category. An objects $(A \sqcup B,\ A \oplus B,\ A + B)$ together with morphisms

$$p: A \rightarrow\ A \cup B$$

$$q: B \rightarrow A \cup B$$

is called coproduct of $A, B$ and satisfy universal property, i.e. for any object $C$ and morphisms

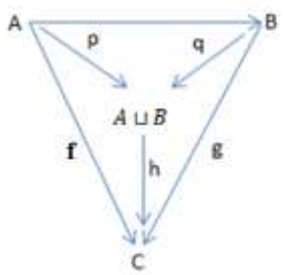
$$f: A \rightarrow C$$

$$g: B \rightarrow C$$

there exists a unique map

$$h: A \cup B \rightarrow C$$

such that $h \circ q = g$ and $h \circ p = f$.




**Category of the G_Modules**

When a ring $\mathcal{R}$ is graded by the group $\mathbb{G}$, we take into account the category $\mathbb{G}\mathcal{R}$-graded, abbreviated $\mathcal{R}$-graded if there is no possibility of ambiguity: We use the graded (left) $\mathcal{R}$-modules for the objects of $\mathcal{R}$-graded, and we know the morphisms in the G_category for the graded $\mathcal{R}$-modules M and N as: $Hom_{\mathcal{R}} - gr(M, N) = \{f \in Hom_{\mathcal{R}}(M, N), f(M_\sigma) \subset N_\sigma$, for all $\sigma \in G\}$

From the definition it is visible that $Hom_{\mathcal{R}} - gr(M, N)$ is an additive subgroup of $Hom_{\mathcal{R}}(M, N)$.

The coproducts and products that make up the category $\mathcal{R}$-gr. For a family of G_modules $\{M_i, i \in J\}$ a coproduct $S_J = \oplus\ \sigma \in GS_\sigma$ may be given by taking $S_\sigma = \oplus_{I \in J}\ (M_i)_\sigma$, and a product $P_J$ can be acquired by taking

$P_\sigma = \prod_{i \in J} (M_i)_\sigma, so\ P_J = \oplus_{\sigma \in G} \prod_{i \in J} (M_i)_\sigma$.

Since for any $f \in Hom_{\mathcal{R}} - gr\ (M, N)$ then, there are a kernel, Ker(*f*), and an image object, Im(*f*), which are in $R$-gr and such that : $M\ /\ Ker(f) \cong Im(f\ )$ The category $\mathcal{R}$-gr is thought of as an abelian category because all objects in it are initially isomorphic. A graded morphism f is moreover regarded as a monomorphism, accordingly if and only if f is injective and is regarded as surjective, epimorphism falls within this category in the sense of set theory. Several of the fundamental possession of category of the G- $\mathcal{R}$-modules for an spot G.

This section focuses on graded rings.

**Example 3.1:** Let $M \in \mathcal{R}$-graded and $\sigma \in G$ we know the σ-suspension $M(\sigma)$ of $M$ to be the G $\mathcal{R}$-module acquired from $M$ by position $M(\sigma)\tau = M\tau\sigma$ for all $\tau \in G$. This explains a functor $A_\sigma : R - gr \rightarrow \mathcal{R}$-graded by putting $A_\sigma(M) = M(\sigma)$. The group of functors $\{A_\sigma, N_\sigma \in G\}$ satisfies:

$A_\sigma \circ A_\tau = A_{\sigma\tau}$ for all $\sigma, \tau \in G$

$A_\sigma \circ A_{\sigma-1} = A_{\sigma-1} \circ A_\sigma =$ Id, for all $\sigma \in G$.

$F \in \mathcal{R}$-graded can be considered **graded free** when it processes an $\mathcal{R}$-basis composing of mixture elements, proportionately $F \cong \oplus\ i \in J\mathcal{R}(\sigma_i)$, where $\{\sigma_i, i \in J\}$ is a group of elements of *G*.

A grading scale due to its homogeneous alternators, M is manifestly isomorphic to a quotient of a graded free object of the $\mathcal{R}$-graded. each object in $\mathcal{R}$- graded that is graded free is unquestionably a free $\mathcal{R}$-module when it is treated as one by removing the gradation. An explanation of the forgetful with the functor U: $\mathcal{R}$-graded $\mathcal{R}$-mod, in more detail. The underlying $\mathcal{R}$-module U is connected to a G $\mathcal{R}$-module U(M). If P is graded projective, it is isomorphic to the direct sum of a graded free F; in fact, use the projectivity of P in $\mathcal{R}$-graded to identify a graded free F mapping to P epimorphically. A gr-projective in $\mathcal{R}$-graded is just a graded and projective R-module, it follows. The same property will not hold true for graded free modules! In fact, combining $\mathcal{R}$ = Z × Z with simple gradation and picking the R-module R that has the given gradation for F $F_0 = \mathbb{Z} \times \{0\}, F_1 = \{0\} \times \mathbb{Z}$ and $F_1 = 0$ for $i \neq 0, 1$ for $F$ , then it is obvious that $F$ cannot have a homogeneous basis! Hence graded free is a stronger characteristics than “graded plus free”.

**Proposition 3.2**: Consider $M, N, P$ in $\mathcal{R}$-graded with given $R$-linear maps,

$f : M \rightarrow P,\ h : M \rightarrow N,\ g : N \rightarrow P,$ such that, $f = g \circ h$ and $f$ is a morphism in $\mathcal{R}$-graded. If $g$, respect $h$ is a morphism in the category $R$-graded, then a morphism $\acute{h} : M \rightarrow N$ , respect $\acute{g} : N \rightarrow P$ R-graded s.t, respect $f = g \circ \acute{h}$ , $f = \acute{g} \circ h$ is existed.

**Proof:** To prove that in which $g$ is a morphism in $\mathcal{R}$-graded. Take an homogeneous $m \in M_\sigma$ for some $\sigma \in G$. We decompose $h(m)$ *as* $h(m) = \sum_{\tau \epsilon G} h(m)_\tau$. The supposition $f = g \circ h$ entails that $f(m) = \sum_{\tau \epsilon G} g(h(m)_\tau)$

with $g(h(m)\tau) \in P_\sigma$. Since $f(m) \in P_\sigma$ the morphism (in $R$-graded) $h$ can be defined by putting $h'(m) = h(m)_\sigma$. That $f = g \circ h'$ follows easily.

**Corollary 3.3:** M is treated as projective, resp. injective in category R-graded when $M \in R$-graded is projective (resp. injective), and considered as an not graded module.

**Proof** : The variant for injectivity is dua, which demonstrates the claim for projectivity. Regard an epimorphism $\mu: N \rightarrow N'$ *in* $\mathcal{R}$-graded and all morphism $f: M \rightarrow N'$ *in* $\mathcal{R}$-graded. Given that u is also surjective as a morphism in $\mathcal{R}$-module, is found an $\mathcal{R}$-linear s.t f = $\mu \circ$ g. Applying the aforementioned statement results in the discovery of a morphism$g'$ in $\mathcal{R}$-graded, $g': M \to N$ s.t $f = \mu \circ g'$ and this creates the projectivity of $M$ in $\mathcal{R}$-graded.

**In the following example we will construct and proof the category of R-module:**

**Example 3.4**: consider two rings $\mathcal{R}, S$. Let U = ${}_R U_S$ be an ($\mathcal{R}$, S)-bimodule, this is, left $\mathcal{R}$-mod and right S-mod s.t $r(us) = (ru)s$, for an $r \in \mathcal{R}, s \in S, u \in U$. fix afield K. let $\mathcal{R} = M_{p\times p}(K) = \{p \times p \; matrices \; over \; K\}$

$S = M_q(K)$ and $U = M_{p,q}(K) = \{p \times q \; matrices \; over \; K\}$

If we have $\mathcal{C}$ the category of left $\mathcal{R}$-module and $\mathcal{R}$-module homomorphism.

$T: M \to M$, such that $T(m_1 + m_2) = T(m_1) + T(m_2)$

and $T(rm) = rT(m) \; for \; each \; r \in R, m \in M$

Let $\mathcal{D}$ be the category of right S-module with right S-module homomorphism.

**We claim that**: the bimoduls U define functors $F, G$ between $\mathcal{C}$ and $\mathcal{D}$.

To define $F: \mathcal{C} \to \mathcal{D}$,

F(${}_R$M) = $Hom_R$(${}_R$M , ${}_R U_S$)

${}_R$M ∈ Ob($\mathcal{C}$).

**Claim that**: F(${}_R$M) = $Hom_R$(M , U) is a right S-module .

Let $\beta \in Hom_R({}_R M, {}_R U_S)$ , $s \in S$.

$(\beta s)$ means that, $(\beta s)(m) = \beta(m).s$ and the morphisms are that :

If ${}_R$M , ${}_R$N ∈ Ob($\mathcal{C}$) and f: ${}_R$M → ${}_R$N left R-homomorphisms.

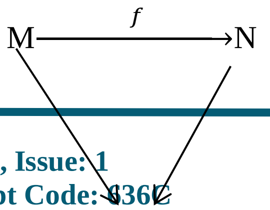
Now, to define $F(f)$:

$$F(M) = Hom_R(M, U) \xleftarrow{F(f)} F(N) = Hom_R(N, U)$$

For $T \in F(N) = Hom_R(N, U)$, define

$F(T) = T \circ f \in Hom_R(M, U)$

Now, why $F(f)$ a morphism in $\mathcal{D}$?



$T \circ f$ $T$

U

If $F(f\ )$ is a right S-module homomorphism.

To check scalar multiplication:

$$F(\ M) \xleftarrow{F(f)} F(N)$$

Take $T \in\ F(N)\ =\ Hom_R(N\ , U)\ , m \in M\ , s \in\ S$

$$(F(f\ )\ (Ts))(m) \overset{?}{=} ((Ts) \circ\ f\ )\ (m)$$

$$= (Ts)(f(m))\ , f(m)\ in\ N$$

$$= T(f(m)).\ s$$

$$= (F(f\ )\ (T))\ (m).\ s$$

$$= (F(f\ )\ (T)\ s\ )\ (m).$$

(Not that, for $f \in mor_{\mathcal{C}}(M, N)$ need $F(f) \in\ mor_{\mathcal{D}}\big(F(N), F(M)\big). F(N) \rightarrow$ $F(m)\ Hom_R(M, U)$. So, $\big((F(f)T)s\big)(m) = \big((F(f)T)(m)\big)s;\ T \in F(N.)$

**Definition 3.5:** Let $\mathcal{R}$ be a ring . A G_category $\mathcal{F}$ through R is category where each morphism set is given the temple of a GR-module and $X, Y\ , Z\ \in Ob(\mathcal{F})$ composition is R-bilinear and encourage a homomorphism

$$Hom_{\mathcal{F}}(Y, Z) \otimes_R Hom_{\mathcal{F}}(\mathrm{X}, \mathrm{Y})\ \rightarrow Hom_{\mathcal{F}}(\mathrm{X}, \mathrm{Z})$$

of graded R-module

**Definition 3.6:** Let $\mathcal{R}$ be a ring. A **Gfunctor** is a functor $f\colon \mathcal{F} \rightarrow \mathcal{B}$ where every obj $X\ , Y\ \in \mathcal{F}$ the map $f\ :\ Hom_{\mathcal{F}}(X\ , Y)\ \rightarrow Hom_{\mathcal{F}}(f(X),\ f(Z))$ is a homomorphism of graded R-module

**Definition 3.7:** Let $\mathcal{R}$ had become a ring, make $\mathcal{F}$ be a **graded category** $\mathcal{F}$ through $\mathcal{R}$. we suppose $\mathcal{F}^{\circ}$ be category forgather with the sum objects as $\mathcal{F}$ and with $Hom_{\mathcal{F}^{\circ}}(X, Y) =$ $Hom_{\mathcal{F}}^{\circ}(X, Y)$ the degree $\circ$ graded piece of G_module of morphism of $\mathcal{F}$. Let A be a graded category $\mathcal{F}$ through R, and make $\mathcal{R}$ be ring. Assume that $\mathcal{F}^{\circ}$ is the category with the total number of objects as $\mathcal{F}$ and that $Hom_{\mathcal{F}^{\circ}}(X,Y) = Hom_{\mathcal{F}}^{\circ}(X,Y)$ the degree-graded component of the A morphism's graded module.

**Proposition 3.8:[9]** Let A be a K-gr algebra through ring $\mathcal{R}$. Here, we will founder a G_category $Mod_A^{gr}$ through R whose associated category $(Mod_A^{gr})^0$ is category of the graded A-modules . As object of $Mod_A^{gr}$ we make right graded A-modules. Given graded A-modules $\mathcal{N}$ and$\mathcal{M}$, we suppose

$$Hom_{Mod_A^{gr}}(\mathcal{N},\mathcal{M}) = \bigoplus_{n\in K} Hom^n(\mathcal{N},\mathcal{M})$$

Where $Hom^n(\mathcal{N},\mathcal{M})$ is set of the right A-modules map $\mathcal{N} \rightarrow \mathcal{M}$ who are homogenous of the degree n,

i.e, $f(\mathcal{N}^i) \subset \mathcal{M}^{i+n}$ for all $i \in K$ . In terms of components, we have

$$Hom^n(\mathcal{N},\mathcal{M}) \subset \prod_{p+q=n} Hom(\mathcal{N}^{-q},\mathcal{M}^p)$$

Is the subset consisting of those $f = (f_{p,q})$ such that

$f_{p,q}(ma) = f_{p-i,q+i}(m)a$ for $a \in A^i$ and $m \in \mathcal{N}^{-q-i}$. For GA-modules

$Z$ , $\mathcal{N}$ , $\mathcal{M}$ we know composition in $Mod_A^{gr}$ via the maps

$$Hom^n(\mathcal{N},\mathcal{M}) \times Hom^n(Z,\mathcal{N}) \longrightarrow Hom^{n+m}(Z,\mathcal{M})$$

By simple composition of right A-module map $f: (g,f) \longmapsto g \circ f$.

**Definition 3.9:** Let $\mathcal{R}$ a ring. Every morphism set in a differential graded category $\mathcal{F}$ over $\mathcal{R}$ has the temple of the DG $\mathcal{R}$-module, and its composition for $X, Y, Z \in Ob(\mathcal{F})$ is $\mathcal{R}$-bilinear and induces a homomorphism.

$Hom_{\mathcal{F}}(Y,Z) \otimes_{\mathcal{R}} Hom_{\mathcal{F}}(X,Y) \rightarrow Hom_{\mathcal{F}}(X,Z)$

Of DG R-module

Finally, if $F \in Hom_{\mathcal{F}}^n(X,Y)$ and $g \in Hom_{\mathcal{F}}^m(Y,Z)$ are homogeneous of degree $n, m$ respectively, then

$$\partial(g \circ F) = \partial(g) \circ F + (-1)^m g \circ \partial(F)$$

In $Hom_{\mathcal{F}}^{n+m+1}(X,Z)$

**Definition 3.10:** Let $\mathcal{R}$ a ring. The functor $F{:}\mathcal{F} \rightarrow \mathcal{B}$ is a functor of the DG category through $\mathcal{R}$, where for each objects $X, Y \in \mathcal{F}$ the map F: $Hom_{\mathcal{F}}(X, Y) \rightarrow Hom_{\mathcal{F}}(F(X), F(Z))$ is a homomorphism of the DG $\mathcal{R}$-module.

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