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## Abstract

We know properties and structure of topological spaces are very important. In this case, one of the powerful tool which help us to figure out structure of topological spaces is (Leray- Serre) spectral sequence. One of the special spaces is Eilenberg-Maclane space which plays an important role in topology. Then finding cohomology groups of this space can be useful for classifying of structure of this space as a topological space. In this paper, we want to compute cohomology groups of Eilenberg-Maclane space K (Z, 4).

**Keywords:** Cohomology group of K (Z, 4), Spectral sequences of K (Z, 4), Eilenberg maclane space of K (Z, 4), K (Z, 4) structure, homology group of K (Z, 4).

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## مجموعة كوهومولوجي K (Z, 4)

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**ألخلاصة**

نحن نعرف خصائص وهيكل المساحات الطوبولوجيا مهمة جدا. في هذه الحالة، فإن التسلسل الطيفي (Leray- Serre) هو أحد الأدوات القوية التي تساعدنا على معرفة بنية المساحات الطوبولوجية. أحد المساحات الخاصة هو مساحة إيلينبرغ – ماكلين التي تلعب فيها دوراً مهماً في الطوبولوجيا. ثم يمكن العثور على مجموعات cohomology من هذا الفضاء مفيد لتصنيف بنية هذا الفضاء كفضاء طوبولوجي.

**الكلمات المفتاحية:** مجموعة كوهومولوجي K(Z, 4)، متواليات الطيفية K(Z, 4)، الفضاءايلينبرغ ماكلين، بناء K(Z,4)، مجموعة هومولوجي K(Z, 4).

## Introduction

Algebraic topology is one of the important branches of mathematics that examines the structure and properties of topological space. One of the tools that playing important role is serre spectral sequences. One of the significant and computational tool is serre spectral method in Algebraic Topology in case of computing homology and cohomology groups of topological spaces.

We know that cohomology groups of $K(Z,3)$ for $i < 14$ computed see [2]. In this paper first we introduce the notion of a spectral sequence, next we use it to compute the cohomology groups of $K(Z,4)$ with coefficient group Z.

### 1.Preliminaries

Definitions and theorems maybe will be uses in this paper as follows:

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**Definition 1.1:** [3] The group structure of loops in topological space is called fundamental group which denoted by $\pi(x, x)$.

**Definition 1.2:** [7] A space X is called an Eilenberg- Maclane space $K(G, n)$ if $\pi_i(X, x_0) = 0, \forall i \neq n$.

**Definition 1.3:** [6] A topological space is called simply- connected if it is path- connected and has trivial fundamental group.

**Definition 1.4:** [ 4] A topological space with basepoint $s_0$ is said to be n- connected if

$$\boldsymbol{\pi_i(S, s_0) = 0 \quad \forall i < n}.$$

**Theorem 1.1:** [8] (Hurewicz) If a space A is $(n - 1)$- connected for $n \geq 2$, then $H_i(A) = 0$ for $i < n$ and $\pi_n(A) \simeq H_n(A)$.

**Theorem 1.2:** [1] (Universal Coefficients for Cohomology)

If a chain complex C of free abelian groups has homology groups $H_i(C)$, then the cohomology groups $H^i(C, G)$ of cochain complex $Hom(C_i, G)$ are determined by split exact sequences

$$\mathbf{0 \to Ext((H_{n-1}(C), G) \to H^n(C, G) \to Hom(H_n(C), G) \to 0} \qquad \textbf{(1.1)}$$

**Theorem 1.3 :** [3] (Homology with Coefficient) If the homology groups $H_n$ and $H_{n-1}$ of a chain complex D of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$ ,then $H^n(D, Z) \simeq \left(\frac{H_n}{T_n}\right) \oplus T_{n-1}$ .

Which $T_n$ is a non- free part of abelian group.

**Theorem 1.4:** [5] If D is a chain complex of free abelian groups, then there are natural short exact sequences

$$\mathbf{0 \to H_n(D) \otimes G \to H_n(D, G) \to Tor(H_{n-1}(D), G) \to 0} \qquad \textbf{(1.2)}$$

## 2. Spectral Sequences

A spectral sequence is a tool to compute the cohomology of a chain complex. It arises from a filtration of the dual chain complex and it provides an alternative way to determine the cohomology of the dual chain complex. A spectral sequence consists of a sequence of intermediate dual chain complexes called pages $E_0, E_1, E_2, E_3, \ldots$ with differentials denoted by $d_0, d_1, d_2, d_3, \ldots$ such that $E_{r+1}$ is the cohomology of $E_r$. The various pages have accessible cohomology groups, which form a finer, and finer approximation of the cohomology H we wish to find out. This limit process may converge, in which case the limit page is denoted by $E_\infty$. Even if there is convergence to $E_\infty$, reconstruction is still needed to obtain H from $E_\infty$ although the differentials $d_r$'s cannot always be all computed, the existence of the spectral sequence often reveals deep facts about the dual chain complex. The spectral sequence and its internal mechanisms can still lead to very useful and deep applications [4].

**Theorem 2.1**: (Spectral Sequences for Cohomology)

For a fibration $F \to E \to B$ with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F, G)$, there is a spectral

sequences $\{E_r^{p,q}, d_r\}$ with:

**1.** $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ and $E_{r+1}^{p,q} = \frac{\mathrm{Ker}\, d_r}{\mathrm{Img}\, d_r}$ at $E_r^{p,q}$.

**2.** Stable terms $E_\infty^{p,n-p}$ is isomorphic to the successive quotients $\frac{F_p^n}{F_{p+1}^n}$ in a filtration

$$0 \subset F_n^n \subset F_{n-1}^n \subset \cdots \subset F_0^n = H^n(E, G)$$

**3.** $E_2^{p,q} \approx H^p(B, H^q(F, G))$ [3].

## 3. Cohomology Groups of K(Z,4) Via Fibrations

In algebraic topology, for each path connected space $(Y, y_0)$, there is a path fibration

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$$\Omega Y \to PY \to Y$$

where Y is the base space, PY is the contractible and ΩY is fiber over the base space $(Y, y_0)$, which called loop space. Now consider base space Y = K (Z, 4), so there is a path space fibration

$$\Omega K(Z,4) \to PK(Z,4) \to K(Z,4)$$

We know that topological space $K(Z,4)$ is (3 - connected), so by defintion (1.4) and theorem (1.1), we have

$$H_1(K(Z,4),Z) \simeq H_2(K(Z,4),Z) \simeq H_3(K(Z,4),Z) \simeq 0$$

Now with the using of theorem (1.2), it is easy to show that

$$H^1(K(Z,4),Z) \simeq H^2(K(Z,4),Z) \simeq H^3(K(Z,4),Z) \simeq 0$$

On the other hand, by Hurewicz theorem we see that:

$$\pi_4(Y) \simeq H_4(Y,Z) \simeq Z$$

Then from universal coefficient theorem we have

$$\mathbf{H^4(Y,Z) \simeq Ext(H_3(Y,Z)) \oplus Hom(H_4(Y,Z)) \simeq Z} \quad \textbf{(3.1)}$$

Now by setting

$$E_2^{p,q} := H^p(\ K(Z,4)\ ; H^q(K(Z,3))$$

By utilizing theorems (2.1), (2.2) and cohomology groups of topological space K (Z, 3) see [1], There are following results:

$$E_2^{3,0} = H^3(K(Z,4),Z) \simeq 0$$

$$E_2^{3,1} = H^3(K(Z,4),0) \simeq 0$$

$$E_2^{3,2} = H^3(K(Z,4),0) \simeq 0$$

$$E_2^{3,3} = H^3(K(Z,4),Z) \simeq 0$$

$$E_2^{3,4} = H^3(K(Z,4),0) \simeq 0$$

$$E_2^{3,5} = H^3(K(Z,4),0) \simeq 0$$

$$E_2^{3,6} = H^3(K(Z,4),Z_2) \simeq 0$$

$$E_2^{3,7} = H^3(K(Z,4),0) \simeq 0$$

$$E_2^{3,8} = H^3(K(Z,4),Z_3) \simeq 0$$

$$E_2^{3,9} = H^3(K(Z,4),Z_2) \simeq 0$$

$$E_2^{4,0} = H^4(K(Z,4),Z) \simeq Z$$

$$E_2^{4,1} = H^4(K(Z,4),0) \simeq 0$$

$$E_2^{4,2} = H^4(K(Z,4),0) \simeq 0$$

$$E_2^{4,3} = H^4(K(Z,4),Z) \simeq Z$$

$$E_2^{4,4} = H^4(K(Z,4),0) \simeq 0$$

$$E_2^{4,5} = H^4(K(Z,4),0) \simeq 0$$

$$E_2^{4,6} = H^4(K(Z,4),Z_2) \simeq Z_2$$

$$E_2^{4,7} = H^4(K(Z,4),0) \simeq 0$$

$$E_2^{4,8} = H^4(K(Z,4),Z_3) \simeq Z_3$$

$$E_2^{4,9} = H^4(K(Z,4),Z_2) \simeq Z_2$$

Now with the above results, we can compute page 2 as follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $Z_2$ | 0 | 0 | 0 | … | … | … | … | … | … | |
| $Z_2$ | 0 | 0 | 0 | $Z_2$ | 0 | 0 | $E_2^{7,9}$ | $E_2^{8,9}$ | $E_2^{9,9}$ | |
| $Z_3$ | 0 | 0 | 0 | $Z_3$ | 0 | 0 | $E_2^{7,8}$ | $E_2^{8,8}$ | $E_2^{9,8}$ | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $Z_2$ | 0 | 0 | 0 | $Z_2$ | 0 | 0 | $E_2^{7,6}$ | $E_2^{8,6}$ | $E_2^{9,6}$ | $E_2^{10,6}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Z | 0 | 0 | 0 | Z | $E_2^{5,0}$ | $E_2^{6,0}$ | $E_2^{7,3}$ | $E_2^{8,3}$ | $E_2^{9,3}$ | $E_2^{10,3}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Z | 0 | 0 | 0 | Z | $E_2^{5,0}$ | $E_2^{6,0}$ | $E_2^{7,0}$ | $E_2^{8,0}$ | $E_2^{9,0}$ | $E_2^{10,0}$ |

**Figure 3.1:** $E_2$ page

Since the topological space PK(Z, 4) is contractible, then for any p and q, $E_\infty^{p,q}$ converges to $E^{p+q}(PK(Z,4),Z)$. In the other words, we have

$$E_\infty^{p,q} \Rightarrow H^{p+q}\,(PK(Z,4),Z) \;=\; 0$$

**Lemma 3.1:** $E_2^{5,0} \;=\; H^5(K(Z,4),Z) \;=\; 0$

Proof. Since every differential map in page 2 and other pages, comes in and out $E_2^{5,0}$ is zero, then $E_2^{5,0}$ will be remained to page infinity and converges to $E_\infty^{5,0} \;=\; 0$, then

$$E_2^{5,0} \;=\; H^5(K(Z,4),Z) \;=\; 0$$

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**Lemma 3.2**: $E_2^{6,0} = H^6(K(Z,4),Z) = 0$

Proof. Similarly, we use the same method, we have $E_2^{6,0} = H^6(K(Z,4),Z) = 0$.

**Lemma 3.3:** $E_2^{7,0} = H_7(K(Z,4),Z) = Z_2$.

Proof. $E_2^{7,0} = H^7(K(Z,4),Z)$ because of converging to $E_\infty^{7,0} = 0$ it is equal to zero. The only differential map is:

$$Z_2 \simeq E_2^{0,6} \to H^{7,0}(K(Z,4),Z)$$

We know, the generator of $Z_2$ has finite order then there are two linear maps Identity and Zero from $Z_2$ to $E_2^{7,0}$.

If the map would be Zero, thus $E_2^{7,0}$ remaind as subgroup of $E_\infty^{7,0} = 0$ so $E_2^{7,0} = 0$.

Otherwise $Z_2$ also remained as subgroup of $H^0(PK(Z,4), H^6(K(Z,3)) = 0$, therefore

$Z_2$ should be zero, that is not accurate, and then the map must be identity.

**Lemma 3.4:** $E_2^{8,0} = H^8(K(Z,4),Z) \simeq Z$.

Proof. A spectral sequence $\{E_r^{p,q}, d_r\}$ is a collection of vector space or groups $E_r^{p,q}$, equipped with differential map $d_r$ (i.e. $d_r \circ d_r = d_r^2 = 0$).

$$E_{r+1}^{p,q} = H(E_r^{p,q}, d_r).$$

Now consider following sequence, such that $d_r^2 = 0$

$$\mathbf{0 \to Z_2 \simeq E_2^{0,6} \to E_2^{4,3} \simeq Z \to E_2^{8,0} \to 0} \qquad \textbf{(3.2)}$$

We know, the generator of $Z_2$ has finite order but there is no finite order element in Z then from Hom algebric properties, there is no linear map from $Z_2$ to $Z$. So, the left differential map in above exact sequence should be zero, but because of exactness and converging properties, the right map should be isomorphism.

Consider that right differential map not be an isomorphism, so we will have non

trivial subgroup of $E_2^{8,0}$such that converges to $E_\infty^{8,0} = 0$, which will not be true.

So, we have

$$E_2^{8,0} = H^8(K(Z,4),Z) \simeq Z.$$

**Lemma 3.5:** $E_2^{9,0} = H^9(K(Z,4),Z) = Z_3$

Proof. Actually $E_2^{9,0} = H^9(K(Z,4),Z)$ because of converging to $E_\infty^{9,0} = 0$

Similarly, the map should be

$$Z_3 \simeq E_2^{0,8} \rightarrow H^{9,0}(K(Z,4),Z),$$

and so, it must be isomorphism.

**Lemma 3.6:** $E_2^{10,0} \simeq 0$ or $Z_2$.

there are two maps within started at $E_2^{0,9}$ and ended at $E_2^{4,6} = Z_2$ or $E_2^{10,0}$.

Proof. Finally, for the disappearing $E_2^{0,9} = Z_2$ in total space, there are two maps within started at $E_2^{0,9}$ and ended a$E_2^{4,6} = Z_2$ t or $E_2^{10,0}$.

Firstly, if the map will be ended at $E_2^{4,6} = Z_2$ then should be identity so because of contractible of total space, the value of $E_2^{10,0}$should be zero.

Otherwise if possibly the map ended at $E_2^{4,6} = Z_2$ be zero, then because of contractibility we should have $E_2^{10,0} = Z_2$.

Totally, we will have $E_2^{10,0} \simeq 0$ or $Z_2$.

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## Conclusion

Overall, Eilenberg maclane space has lots of application in algebraic topology, such as classifying cohomology and the homology groups of spaces. Hatcher and other have found cohomology and homology groups of these space for K (Z, n), which n = 1,2 and 3. We calculated cohomology and respectively homology groups of these spaces for n=4. Totally cohomology groups of these space are algebra and exterior algebra rings, but locally, because of hardness of finding the differential maps, it should not be easy to find.

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