Some New Properties of Convex and Concave Soft Sets

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Abstract

The main propose of this paper is to show the assertions of the conclusion proposed in [1]. Here we introduce new notions in soft set theory such as, strictly convex soft, strongly convex soft set, strictly concave soft set and strongly convex soft set, also introducing their connections among them.

Keywords: Soft sets, convex soft sets, concave soft sets, strictly convex soft sets, strongly convex soft set.

بعض الخصائص الجديدة للمجموعات المحدبة والمجوعات المقعدة الناعمة

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الخلاصة

الاقتراح الرئيسي لهذه الورقة هو إظهار وتوضيح تأكيدات الاستنتاج المقترح في [1]. هنا سوف نقدم مفاهيم جديدة مثل، ناعمة محدبة بدقة، مجموعة صارمة محدبة ناعمة، مجموعة ناعمة مقعرة بصرامة ومجموعة ناعمة محدبة بشدة، وايضا تقديم الاتصالات بينهم.

الكلمات المفتاحية: مجموعات ناعمة، مجموعات ناعمة محدبة، مجموعات ناعمة مقعرة، مجموعات ناعمة محدبة بدقة، مجموعة ناعمة محدبة بشدة.

Introduction

One of the most well-known and significant sources are the notion of soft set developed by Molodtsov [2]. It can be used as a general mathematical tool for dealing with uncertainty. It is now the main topic of many research areas in mathematics. In view of Molodtsov's work, scholars extensively studied soft sets such as [3, 4, 5, 6]. An important definition of convex and concave soft sets is introduced by Deli [1] and the author gave some properties of them. Here, we point out that assertions of conclusion in [1] are correct in general, we introduce reasonable dentitions to show and prove the results, and study some desired properties. In this section, we demonstrate the basic definitions of soft set theory and one can find some interesting result about this in [7, 8, 9, 10]. In section 2, we define strictly and strongly convex soft sets and then we show some properties of each notion. The last section contains conclusions related to our main results.

1. Preliminaries

Using the idea that there is a connection between soft sets and fuzzy sets [11], where fuzzy set may be considered as a soft set. We recall the definition of fuzzy set by Zadeh [12], which is defined by its membership function, whose values are defined on the closed interval $[0,1] \coloneqq I^{\circ}$. Let $f_{\tilde{\mu}_1}(x), f_{\tilde{\mu}_2}(x)$ be fuzzy sets on X. Then $\tilde{\mu}_1$ is a superset of $\tilde{\mu}_2$ is defined by $f_{\tilde{\mu}_1}(x) \geq f_{\tilde{\mu}_2}(x)$, for every $x \in X$. A fuzzy set $f_{\tilde{\mu}}(x)$ on $\mathcal{R}$ is convex [13, 14] if and only if $f_{\tilde{\mu}}(x) \geq f_{\tilde{\mu}}(x_1) \wedge f_{\tilde{\mu}}(x_2)$, where $x = \lambda x_1 + (1-\lambda)x_2$, $x_1, x_2 \in \mathcal{R}$ and $\lambda \in I^{\circ}$. The standard fuzzy intersection of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, $\tilde{\mu}_1 \cap \tilde{\mu}_2$, is defined by $f_{\tilde{\mu}_1 \cap \tilde{\mu}_2}(x) = \min_x f_{\tilde{\mu}_i}(x)$, $i = 1,2$; the standard fuzzy union of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, $\tilde{\mu}_1 \cap \tilde{\mu}_2$, is defined by $f_{\tilde{\mu}_1 \cup \tilde{\mu}_2}(x) = \max_x f_{\tilde{\mu}_i}(x)$, $i = 1,2$; and the complement of $\tilde{\mu}_1$, $X \backslash \tilde{\mu}_1$, is defined by $f_{X \backslash \tilde{\mu}_1}(x) = 1 - f_{\tilde{\mu}_1}(x)$, for all $x \in X$. The union of a fuzzy set $\tilde{\mu}_1$ and $X \backslash \tilde{\mu}_1$ should not necessarily give the whole X. Also, the intersection between $\tilde{\mu}_1$ and its complement $X \backslash \tilde{\mu}_1$ is not necessarily give the empty set.

Molodtsov [2] developed the concept of soft sets from fuzzy sets. He defined soft set over the universe set X as a pair (f_s, E) such that E is a set of parameters that are describing the elements of X and f_s maps whole members in E to the power set of X, $P(X)$. It is noting that the soft set (f_s, E) is a parameterized family of subsets of the set X, and therefore it can be written as a set of ordered pairs $(f_s, E) = \{(x, f_s(x)): x \in E$ and $f_s(x) \subseteq X\}$. The empty soft set denoted by S_ϕ, and defined when $f_S(x) = \phi$ for all $x \in E$. For the given two soft sets such as S and T, S is a subset of T, when $f_s(x) \subseteq f_T(x)$ for all $x \in E$. The complement for the soft set S is denoted by $X \backslash S$, and defined by $f_{X\backslash S}(x) = X \backslash f_S(x)$, for all $x \in E$. The approximation function $f_{S\cup T}$ (resp. $f_{S\cap T}$) is defined by $f_{S\cup T}(x) = f_s(x) \cup f_T(x)$ (resp. $f_{S\cap T}(x) = f_s(x) \cap f_T(x)$, for all $x \in E$). In this paper, the set of all soft sets over X will be denoted by $SS(X)$. Convex and concave soft sets are defined as the set $(f_s, E) \in SS(X)$ satisfy $f_S(\lambda x_1 + (1-\lambda)x_2) \supseteq f_S(x_1) \cap f_S(x_2)$ and $f_S(\lambda x_1 + (1-\lambda)x_2) \subseteq f_S(x_1) \cap f_S(x_2)$, respectively, for all $x_1, x_2 \in E$ and $\lambda \in I^\circ$.

2. Soft Set Convexity

In this section, we define strictly convex soft, strongly convex soft set, strictly concave soft set and strongly convex soft set, and then give some properties of them.

Definition 2.1. The soft set $(f_s, E) \in SS(X)$ is called a strongly convex soft set if

$$f_S(\lambda x_1 + (1-\lambda)x_2) \supset f_S(x_1) \cap f_S(x_2)$$

for every $x_1, x_2 \in E$, $x_1 \neq x_2$ and $\lambda \in I^\circ := (0,1)$.

Definition 2.2. The soft set $(f_s, E) \in SS(X)$ is called a strongly concave soft set if

$$f_S(\lambda x_1 + (1-\lambda)x_2) \subset f_S(x_1) \cap f_S(x_2)$$

for every $x_1, x_2 \in E$, $x_1 \neq x_2$ and $\lambda \in I^\circ := (0,1)$.

Definition 2.3. The soft set $(f_s, E) \in SS(X)$ is called a strictly convex soft set if

$$f_S(\lambda x_1 + (1-\lambda)x_2) \supset f_S(x_1) \cap f_S(x_2)$$

for all $x_1, x_2 \in E$, $f_S(x_1) \neq f_S(x_2)$ and $\lambda \in I^\circ$.

Definition 2.4. The soft set $(f_s, E) \in SS(X)$ is called a strictly concave soft set if

$$f_S(\lambda x_1 + (1-\lambda)x_2) \subset f_S(x_1) \cap f_S(x_2)$$

for all $x_1, x_2 \in E$, $f_S(x_1) \neq f_S(x_2)$ and $\lambda \in I^\circ$.

Theorem 2.5. Let $(f_s, E) \in SS(X)$ be a strictly convex soft set. If there exists $\lambda \in I^\circ$, for all $x_1, x_2 \in E$ such that

$$f_S(\lambda x_1 + (1-\lambda)x_2) \supseteq f_S(x_1) \cap f_S(x_2),$$

then (f_s, E) is a convex soft set.

Proof: Suppose that $f_S(x_1) \subseteq f_S(x_2)$ and there exist $x_1, x_2 \in E$, $\mu \in I^\circ$ such that

$$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supseteq X \setminus \{f_S(x_1) \cap f_S(x_2)\} \quad \textbf{(1)}$$

If $f_S(x_1) \subset f_S(x_2)$, then (1) contradicting (f_s, E) is a strictly convex soft set.

If $f_S(x_1) = f_S(x_2)$ and $\mu \in [0, \lambda]$, let $x_3 = \frac{\mu}{\lambda}x_1 + \left(1 - \frac{\mu}{\lambda}\right)x_2$ and $\gamma = \left(\frac{1}{\lambda} - 1\right)\left(\frac{1}{\mu} - 1\right)^{-1}$. Thus, by hypothesis

$$f_S(\mu x_1 + (1-\mu)x_2) = f_s\left(\lambda\left(\frac{\mu}{\lambda}x_1 + \left(1 - \frac{\mu}{\lambda}\right)x_2\right) + (1-\lambda)x_2\right)$$

$$= f_S(\lambda x_3 + (1-\lambda)x_2)$$

$$\supseteq f_S(x_2) \cap f_S(x_3) \quad \textbf{(2)}$$

and

$$f_S(x_3) = f_s\left(\frac{\mu}{\lambda}x_1 + \left(1 - \frac{\mu}{\lambda}\right)x_2\right)$$

$$= f_s\big(\gamma x_1 + (1-\gamma)(\mu x_1 + (1-\mu)x_2)\big) \quad \textbf{(3)}$$

From (1), (2) and $f_S(x_1) = f_S(x_2)$, it follows that

$$f_S(\mu x_1 + (1-\mu)x_2) \supseteq f_S(x_3) \quad \textbf{(4)}$$

From (1), (3), $f_S(x_1) = f_S(x_2)$ and strictly convex soft set condition, it follows that

$$f_S(x_3) \supset f_S(\mu x_1 + (1-\mu)x_2) \quad \textbf{(5)}$$

or

$$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supseteq X \setminus f_S(x_3) \quad \textbf{(6)}$$

Hence, (4) and (6) gives contradicts.

If $f_S(x_1) = f_S(x_2)$ and $\in [\lambda, 1]$, let $x_4 = \frac{\mu-\lambda}{1-\lambda}x_1 + \frac{1-\mu}{1-\lambda}x_2$. Thus, by hypothesis

$$f_S(\mu x_1 + (1-\mu)x_2) = f_S(\lambda x_1 + (1-\lambda)x_4)$$

$$\supseteq f_S(x_1) \cap f_S(x_4) \quad \textbf{(7)}$$

From (1), (7) and $f_S(x_1) = f_S(x_2)$, it follows that

$$f_S(\mu x_1 + (1-\mu)x_2) \supseteq f_S(x_4) \quad \textbf{(8)}$$

On the other hand, $\mu x_1 + (1-\mu)x_2 = \lambda x_1 + (1-\lambda)x_4$ gives

$$x_4 = \frac{1}{1-\lambda}(\mu x_1 + (1-\mu)x_2) - \frac{\lambda}{1-\lambda}x_1$$

$$= \frac{1}{1-\lambda}(\mu x_1 + (1-\mu)x_2) - \frac{\lambda}{1-\lambda}\left(\frac{1}{\mu}(\mu x_1 + (1-\mu)x_2) - \frac{1-\mu}{\mu}x_2\right)$$

$$= \frac{\mu-\lambda}{(1-\lambda)\mu}(\mu x_1 + (1-\mu)x_2) + \left(1 - \frac{\mu-\lambda}{(1-\lambda)\mu}\right)x_2 \quad \textbf{(9)}$$

From (1), (9), $f_S(x_1) = f_S(x_2)$ and strictly convex soft set condition, it follows that

$$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supseteq X \setminus f_S(x_4) \quad \textbf{(10)}$$

Hence, (8) and (10) gives a contradict.

Remark 2.6. Theorem 3.11 in [1] allows us to easily apply all results about convex soft set to concave soft set.

Theorem 2.7. Let $(f_s, E) \in SS(X)$ be a strictly concave soft set. If there exist $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$ such that

$f_S(\lambda x_1 + (1-\lambda)x_2) \subseteq f_S(x_1) \cap f_S(x_2)$,

then (f_s, E) is a concave soft set.

Proof: The proof of this theorem is the same steps of the previous theorem for convex soft set, by just taking the complement of the equations gives the result for strictly concave soft set.

Theorem 2.8. Let $(f_s, E) \in SS(X)$ be a convex soft set. If there exists $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$, $f_S(x_1) \neq f_S(x_2)$ implies

$f_S(\lambda x_1 + (1-\lambda)x_2) \supset f_S(x_1) \cap f_S(x_2)$,

then (f_s, E) is a strictly convex soft set.

Proof: Suppose that there exist $x_1, x_2 \in E$, $\mu \in I^{\circ}$ such that

$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supset X \setminus \{f_S(x_1) \cap f_S(x_2)\}$ **(11)**

If $f_S(x_1) \supset f_S(x_2)$, then (11) gives

$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supset X \setminus f_S(x_1)$ **(12)**

On the other hand, from the convex soft set condition we have that

$f_S(\mu x_1 + (1-\mu)x_2) \supseteq f_S(x_1) \cap f_S(x_2)$ **(13)**

From (11) and (13), it follows that

$f_S(\mu x_1 + (1-\mu)x_2) = f_S(x_1) \cap f_S(x_2)$ **(14)**

Which together with $f_S(x_1) \supset f_S(x_2)$ getting that

$f_S(\mu x_1 + (1-\mu)x_2) = f_S(x_2)$ **(15)**

or

$$f_S(\mu x_1 + (1-\mu)x_2) \subset f_S(x_1) \quad \textbf{(16)}$$

Thus, from (16) and the hypothesis

$$f_S\big(\lambda x_1 + (1-\lambda)(\mu x_1 + (1-\mu)x_2)\big) \supset f_S\big((\mu x_1 + (1-\mu)x_2)\big) \quad \textbf{(17)}$$

More generally, for $n \in \{1,2,3,\dots\}$ can easily shown that

$$f_S(\lambda^n x_1 + (1-\lambda^n)(\mu x_1 + (1-\mu)x_2)) \supset f_S(\mu x_1 + (1-\mu)x_2) \quad \textbf{(18)}$$

Let $x_3 = \gamma x_1 + (1-\gamma)x_2$ where $\gamma = \mu - \lambda^n\mu + \lambda^n \in I^{\circ}$ for some n. Then from (18) we see that

$$f_S(x_3) = f_S(\gamma x_1 + (1-\gamma)x_2)$$

$$= f_S\big(\lambda^n x_1 + (1-\lambda^n)(\mu x_1 + (1-\mu)x_2)\big)$$

$$\supset f_S(\mu x_1 + (1-\mu)x_2) \quad \textbf{(19)}$$

Also, let $x_4 = \beta x_1 + (1-\beta)x_2$ where $\beta = \mu - \lambda^n + \frac{1}{1-\lambda} + \frac{\lambda^n\mu}{1-\lambda} \in I^{\circ}$ for some n. Then,

$$f_S(\mu x_1 + (1-\mu)x_2) = f_S(\lambda x_3 + (1-\lambda)x_4) \quad \textbf{(20)}$$

Now, if $f_S(x_3) \subseteq f_S(x_4)$, then (20) and (f_s, E) is a convex soft set implies that

$$X \setminus f_S(x_3) \supset X \setminus f_S(\mu x_1 + (1-\mu)x_2)$$

this contradicts (19).

If $X \setminus f_S(x_3) \subseteq X \setminus f_S(x_4)$, then (20) and the hypothesis of the theorem implies that

$$f_S(\mu x_1 + (1-\mu)x_2) \supset f_S(x_3) \cap f_S(x_4)$$

$$\supseteq \big(f_S(x_1) \cap f_S(x_2)\big) \cap \big(f_S(x_1) \cap f_S(x_2)\big) = f_S(x_1)$$

This contradicts (16).

Corollary 2.9. Let $(f_s, E) \in SS(X)$ be a concave soft set. If there exists $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$, $f_S(x_1) \neq f_S(x_2)$ implies

$f_S(\lambda x_1 + (1-\lambda)x_2) \subset f_S(x_1) \cap f_S(x_2)$,

then (f_s, E) is a strictly concave soft set.

Proof: The proof follows from Remark 2.6 and Theorem 2.8.

Theorem 2.10. Let $(f_s, E) \in SS(X)$ be a strongly convex soft set. If there exists $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$ such that

$$\boldsymbol{f_S(\lambda x_1 + (1-\lambda)x_2) \supseteq f_S(x_1) \cap f_S(x_2)} \quad \textbf{(21)}$$

then (f_s, E) is a convex soft set.

Proof: Suppose that there exist $x_1, x_2 \in E$ and $\mu \in I^{\circ}$ such that

$$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supseteq X \setminus \{f_S(x_1) \cap f_S(x_2)\} \quad \textbf{(22)}$$

If $x_1 \neq x_2$, then (22) contradicting that (f_s, E) is a strongly convex soft set.

If $x_1 = x_2$, then choose $\mu \neq \mu_1 \in I^{\circ}$ such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_1$.

Let $x_1 = \mu_1 x_1 + (1-\mu_1)x_2$, $x_2 = \mu_1 x_1 + (1-\mu_1)x_2$. Then (22) implies that

$$X \setminus f_S(x_1) \supseteq X \setminus \{f_S(x_1) \cap f_S(x_2)\} \quad \textbf{(23)}$$

$$X \setminus f_S(x_2) \supseteq X \setminus \{f_S(x_1) \cap f_S(x_2)\} \quad \textbf{(24)}$$

According to (21), (23) and (24), we have

$f_S(\lambda x_1 + (1-\lambda)x_2) \supseteq f_S(x_1) \cap f_S(x_2)$

$\subset \{f_S(x_1) \cap f_S(x_2)\} \cap \{f_S(x_1) \cap f_S(x_2)\}$

$= f_S(x_1) \cap f_S(x_2)$

Which contradicts that (f_s, E) is a strongly convex soft set.

Corrollary 2.11. Let $(f_s, E) \in SS(X)$ be a strongly concave soft set. If there exists $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$ such that

$$f_S(\lambda x_1 + (1-\lambda)x_2) \subseteq f_S(x_1) \cap f_S(x_2),$$

then (f_s, E) is a concave soft set.

Proof: The proof follows from Remark 2.6 and Theorem 2.10.

Theorem 2.12. Let $(f_s, E) \in SS(X)$ be a convex soft set. If there exists $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$, $x_1 \neq x_2$ implies

$$f_S(\lambda x_1 + (1-\lambda)x_2) \supset f_S(x_1) \cap f_S(x_2),$$

then (f_s, E) is a strongly convex soft set.

Proof: Suppose that there exist $(x_1 \neq x_2)x_1, x_2 \in E$, $\mu \in I^{\circ}$ such that

$$X \setminus f_S(\mu x_1 + (1-\mu)x_2) \supset X \setminus \{f_S(x_1) \cap f_S(x_2)\} \quad \textbf{(25)}$$

Thus, from (25) and the convex soft set condition we get that

$$f_S(\mu x_1 + (1-\mu)x_2) = f_S(x_1) \cap f_S(x_2) \quad \textbf{(26)}$$

Furthermore, it can be easily seen that

$$\lambda x_1 + (1-\lambda)x_2 = \mu x_1 + (1-\mu)x_2 \quad \textbf{(27)}$$

where both x_1 and x_2 are of the form $x_1 = \mu x_1 + (1-\mu)x_2$ and $x_2 = \mu x_1 + (1-\mu)x_2$ for choosing $\mu \in I^{\circ}$.

On the other hand, from the convex soft set condition and our definition of x_1 and x_2, getting

$$f_S(x_1) \supseteq f_S(x_1) \cap f_S(x_2) \quad \textbf{(28)}$$

$f_S(x_2) \supseteq f_S(x_1) \cap f_S(x_2)$ **(29)**

Therefore, from (27), (28), (29) and the hypothesis of the theorem, we get that

$f_S(\mu x_1 + (1-\mu)x_2) = f_S(\lambda x_1 + (1-\lambda)x_2) \supset f_S(x_1) \cap f_S(x_2)$

$\supseteq \{f_S(x_1) \cap f_S(x_2)\} \cap \{f_S(x_1) \cap f_S(x_2)\} = f_S(x_1) \cap f_S(x_2)$

This contradicts (26).

Corrollary 2.13. Let $(f_s, E) \in SS(X)$ be a concave soft set. If there exists $\lambda \in I^{\circ}$, for all $x_1, x_2 \in E$, $x_1 \neq x_2$ implies

$f_S(\lambda x_1 + (1-\lambda)x_2) \subset f_S(x_1) \cap f_S(x_2)$,

then (f_s, E) is a strongly concave soft set.

Proof: The proof follows from Remark 2.6 and Theorem 2.12.

Conclusion

In this paper, we have defined some notions, and give a detailed theoretical study on convexity of soft sets. The theorems are relations of convex soft sets and concave soft sets with some new notions such as strictly convex soft, strongly convex soft set, strictly concave soft set and strongly convex soft set. We have demonstrated some conditions for which strictly convex soft sets (resp. strictly concave soft sets) become convex soft sets (resp. concave soft sets) and found simpler condition for vice versa. Moreover, we declared relations between strongly convex soft sets and convex soft sets, and some conditions for which convex soft sets become strongly convex soft sets are given. Finally, we presented relationship between these versions of soft convexity with the concave property.

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