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P-Semi Hollow-Lifting Modules

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Abstract

Let R be a ring with identity and Q be a unitary left Module over R. In this paper, we introduced the concept of p-semi hollow-lifting Module as generalization of semi hollow-lifting Module. Also, give a comprehensive study of basic properties of p-semi hollow-lifting Modules and some related concepts.

Keywords: Hollow module, lifting module, semi hollow module, semi hollow-lifting module.

المقاسات الرفع شبه المجوفة من النوع P

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الخلاصة

لتكن R حلقة ذات عنصر محايد ولتكن Q مقاس ايسر معرف على R. قدمت في هذا البحث مقاسات الرفع شبه المجوفة من النوع P كتعميم لمقاسات الرفع شبه المجوفة واعطيت بعض الخواص الاساسية لمقاسات الرفع شبه المجوفة من النوع P مع بعض المفاهيم المرتبطة.

الكلمات المفتاحية: المقاسات المجوفة، مقاسات الرفع، المقاسات شبه المجوفة، المقاسات الرفع شبه المجوفة.

Diyala Journal for Pure Science
International Conference for Pure Sciences (ICPS-2021)

Introduction

A submodule E of an R-Module Q is small submodule of Q ($E \ll Q$) if for any submodule U of Q s.t $Q = E + U$, then $U = Q$[5]. A submodule E of Q is semismall in Q ($E \ll_s Q$) if E = 0 or for each nonzero submodule U of E, E/U is small in Q/U [1]. Let E, U be submodules of an R-Module Q s.t $E \subset U \subset Q$. E is semicoessential submodule of U in Q ($E \subseteq_{sce} U$ in Q) if $\frac{U}{E} \ll_s \frac{Q}{E}$[6]. A nontrivial R-Module Q is semihollow if each proper submodule of Q is semismall in Q [1]. An R-Module Q is semihollow-lifting if for each submodule N of Q s.t $\frac{Q}{N}$ semihollow, ∃ a submodule W of Q s.t $Q = W \oplus W^*$ and $W \subseteq_{sce} N$ in Q [10]. An R-Module Q is semilifting Module if for each submodule W of Q, ∃ a direct summand E of Q s.t $E \subseteq_{sce} W$ in Q [1].

P-Semi hollow-Lifting Modules

We introduced the concept of p-semihollow-lifting Module and some properties of p-semihollow-lifting Modules.

An R-Module Q is p-semihollow, if for each proper cyclic submodule is semismall in Q.

Every Semihollow Module is P-Semihollow Module.

An R-Module Q is p-semihollow-lifting if for each cyclic submodule N of Q s.t $\frac{Q}{N}$ p-semihollow, ∃ a direct summand W of Q s.t $W \subseteq_{sce} N$ in Q.

Clearly, Z_4 as Z-Module is p-semihollow-lifting.

Every semihollow-lifting Module is p-semihollow-lifting Module

An R-Module Q have p-semihollow factor Module if ∃ a cyclic submodule E of Q s.t $\frac{Q}{E}$ is p-semihollow Module.

Every Module which has not any p-semihollow factor Module is p-semihollow-lifting.

Diyala Journal for Pure Science
International Conference for Pure Sciences (ICPS-2021)

Z as Z-Module is not p-semihollow-lifting, assume Z is p-semihollow-lifting. Let 4Z be cyclic submodule. Since $\frac{Z}{4Z}$ is p-semihollow, thus ∃ a direct summand W of Z s.t W $\subseteq_{sce}$ 4Z in Z. But Z is indecomposable, then W = 0 implies 4Z $\ll_s$ Z , a contradiction.

An R-Module Q is p-semilifting Module if for each cyclic submodule W of Q, ∃ a direct summand E of Q s.t E$\subseteq_{sce}$W in Q.
Every semilifting Module is p-semilifting Module

Every p-semilifting Module is p-semihollow-lifting, in particular every semisimple or p-semihollow Module is p-semihollow-lifting. For example, Z_{p^∞} as Z-Module, where p is a prime number. The converse is not true. For example, let Q be an indecomposable R-Module which has not any p-semihollow factor Module. Clearly Q is p-semihollow-lifting. Claim Q is not p-semilifting, suppose Q is p-semilifting and W be a proper cyclic submodule of Q. Since Q is p-semilifting, thus ∃ a submodule E of Q s.t E $\subseteq_{sce}$ W in Q and Q = E ⊕ E_1 for some E_1 ⊆ Q. But Q is indecomposable Module, thus E = 0 implies W$\ll_s$ Q. Therefore, Q is p-semihollow, which is a contradiction.

Proposition1 Let Q = $Q_1 \oplus Q_2$ be a Module where Q_1 and Q_2 be p-semihollow Modules. Then Q is p-semihollow lifting Module iff Q is p-semilifting Module.

Proof: ⇒) Let U be a cyclic submodule of Q and $\pi_1 : Q \to Q_1$ and $\pi_2 : Q \to Q_2$ be two natural projections maps. First case, if $\pi_1(U) \neq Q_1$ and $\pi_2(U) \neq Q_2$. Thus $\pi_1(U) \ll_s Q_1$ and $\pi_2(U) \ll_s Q_2$. So, by [1], $\pi_1(U) \oplus \pi_2(U) \ll_s Q_1 \oplus Q_2$. Claim U ⊆ $\pi_1(U) \oplus \pi_2(U)$, let u ∈ U then u ∈ Q = $Q_1 \oplus Q_2$ and hence u = (q_1, q_2), where $q_1 \in Q_1$, $q_2 \in Q_2$. Second case, assume $\pi_1(u) = \pi_1((q_1,q_2)) = q_1$ and $\pi_2(u) = \pi_2((q_1,q_2)) = q_2$. Thus u = $(\pi_1(u), \pi_2(u))$ and get U ⊆ $\pi_1(U) \oplus \pi_2(U)$ and hence U$\ll_s$ Q. Then Q is p-semilifting Module. Now, assume that $\pi_1(U) = Q_1$, then $\pi_1(U) = \pi_1(Q)$. Thus Q = U + H_2. By second isomorphism theorem, $\frac{U+Q_2}{U} \cong \frac{Q_2}{U \cap Q_2}$. Since H_2 is p-semihollow Module, then $\frac{Q_2}{U \cap Q_2}$ is p-semihollow and hence

Diyala Journal for Pure Science
International Conference for Pure Sciences (ICPS-2021)

$\frac{Q}{U}$ is p-semihollow. But Q is p-semihollow-lifting, therefore ∃ a semicoessential submodule of U in Q which is a direct summand of Q. Then Q is p-semilifting.

⇐) Clear.

One can use the previous proposition to give the following examples:

1. Consider the Module $Q = Z_2 \oplus Z_4$, clearly, Z_2 and Z_4 as Z-Module are p-semihollow Modules. Since for each cyclic submodule W of Q, ∃ a direct summand E of Q s.t $E \subseteq_{sce} W$ in Q, thus $Q = Z_2 \oplus Z_4$ is p-semilifting and hence p-semihollow-lifting.
2. Consider the Module $Q = Z_2 \oplus Z_8$, clearly, Z_2 and Z_8 as Z-Module are p-semihollow Modules. One can easily to see that $Q = Z_2 \oplus Z_8$ is not p-semilifting. Thus, Q is not p-semihollow-lifting.
3. Let p be any prime integer. Since the Module $Z/p^2Z \oplus Z/p^3Z$ is p-semilifting [11, prop.A.7], then it is p-semihollow-lifting. But $Z/pZ \oplus Z/p^3Z$ is not p-semihollow-lifting because it is not p-semilifting [11, prop. A.7].

Proposition2 Every p-semihollow Module is indecomposable.

Proof: Clear.

Proposition3 Let Q be a Module, if Q is a p-semihollow Module, thus $\frac{Q}{W}$ is a p-semihollow Module, for every proper cyclic submodule W of Q.

Proof: Let H/W cyclic submodule of Q/W. Since Q is p-semihollow, then $H \ll_s Q$ and hence $H/W \ll_s Q/W$. Thus $\frac{Q}{W}$ is p-semihollow.

Proposition4 An R-Module Q is a p-semihollow Module iff for some proper cyclic submodule D of Q, $\frac{Q}{D}$ is p-semihollow and $D \ll_s Q$.

Proof: ⇒) Suppose that Q is a p-semihollow Module and D is any proper cyclic submodule of Q, then D $\ll_s$ Q and by prop.3, $\frac{Q}{D}$ is p-semihollow.

⇐) Suppose that $D \ll_s Q$ and $\frac{Q}{D}$ is p-semihollow. Let Y be a proper cyclic submodule of Q. Then Y+D ≠ Q, so $\frac{Y+D}{D} \ll_s \frac{Q}{D}$. Let $Q = Y + V$, where $V \subseteq Q$, then $\frac{Q}{D} = \frac{Y+V}{D} = \frac{Y+D}{D} + \frac{V+D}{D}$. But $\frac{Y+D}{D} \ll_s \frac{Q}{D}$ therefore, Q = V+D. Since D$\ll_s$Q, then Q = V. Thus, Q is a p-semihollow Module.

Proposition5 Let Q be an R-Module. If Q is a p-semihollow Module then each non-zero factor Module of Q is indecomposable.

Proof: Assume Q is p-semihollow Module and $\frac{Q}{D}$ non-zero factor Module of Q. Then by prop.3, $\frac{Q}{D}$ is p-semihollow. Hence, by prop.2, $\frac{Q}{D}$ is indecomposable.

Proposition6 An indecomposable Module Q is a p-semihollow-lifting Module iff Q is p-semihollow or Q has not any p-semihollow factor Modules.

Proof: ⇒) Assume Q has p-semihollow factor Module. Thus ∃ a proper cyclic submodule W of Q s.t $\frac{Q}{W}$ p-semihollow. Since Q is p-semihollow-lifting, ∃ a direct summand U of Q s.t $U \subseteq_{sce} W$ in Q. But Q is indecomposable Module, therefore $U = 0$ hence $W \ll_s Q$.Thus by prop.4, Q is p-semihollow.

⇐) Clear.

Proposition7 Let $Q_1 , \ldots , Q_n$ be Modules having not any p-semihollow factor Modules. Thus Q = $Q_1 \oplus \cdots \oplus Q_n$ is p-semihollow-lifting.

Diyala Journal for Pure Science
International Conference for Pure Sciences (ICPS-2021)

Proof: Assume Q has a cyclic submodule D s.t Q/D is p-semihollow. Since Q_1+D/D+⋯ +Q_n+D/D = Q/D, ∃ i∈{1,. . .,n}s.t Q_i+D/D = Q/D is p-semihollow. So Q_i has a p-semihollow factor Module, a contradiction.Thus Q is p-semihollow-lifting.

proposition6 gives an idea to find an example of a p semihollow lifting Module that is not p-semilifting Module. In fact, every indecomposable Module Q which has not any p-semihollow factor Module is p-semihollow-lifting, but it is not a p-semilifting Module, let E be any indecomposable Module having not any p-semihollow factor Module and X be a semisimple Module. If Y cyclic submodule of Q = E⊕X s.t Q/Y is p-semihollow, then E + Y = Q or X + Y = Q. Since E has not any p-semihollow factor Modules and E + Y/Y ≅E/E∩Y, X + Y = Q. Since X is semisimple. ∃ a submodule D of X s.t X = D⊕(X∩Y). Therefore D⊕Y = Q. Thus, Y is a direct summand of Q. Consequently, Q is p-semihollow-lifting. Clearly Q is not p-semilifting (E is not semihollow).

Proposition8 Let Q be an indecomposable p-semihollow-lifting Module, If Q has a maximal cyclic submodule, then it is unique.

Proof: Assume W be a maximal cyclic submodule of Q. Suppose Q has another maximal cyclic submodule k which is different from W, thus Q = W + K. By [9], $\frac{Q}{W}$ is a simple Module and hence p-semihollow. But Q is p-semihollow-lifting Module, thus ∃ a direct summand A of Q s.t $A \subseteq_{sce} W$ in Q. But Q is indecomposable Module thus A = 0, hence $W \ll_s Q$ implies Q = K, a contradiction, then Q has a unique maximal cyclic submodule.

Proposition9 Let W be a submodule of p-semihollow-lifting Module Q and Y be a cyclic submodule of Q such that $\frac{Q}{Y}$ p-semihollow and Q = W+Y, then there exists a direct summand D of Q and $D \subseteq_{sce} Y$ in Q.

Proof: Let E be a submodule of Q and Y cyclic submodule of Q s.t $\frac{Q}{Y}$ p-semihollow. Since Q is p-semihollow-lifting Module, ∃ a direct summand D of Q s.t $D \subseteq_{sce} Y$ in Q. Now, since Q = E + Y, then $\frac{Q}{D} = \frac{E+Y}{D} = \frac{E+D}{D} + \frac{Y}{D}$. But $D \subseteq_{sce} Y$ in Q, therefore $\frac{Q}{D} = \frac{E+D}{D}$. Thus Q = E + D.

Let W be a submodule of an R-Module Q. A submodule U of Q is supplement of W in Q. If U is a minimal element in the set of submodule $V \subseteq Q$ with $W + V = Q$. Equivalently, $Q = W + U$ and $W \cap U << U$[9].

An R-Module Q is an amply supplemented Module, if for any two submodules W and D of Q with $+D = Q$, D contains a supplement of W in Q [2].

Let Q be an R-Module, and W be a submodule of Q. A submodule V of W is coclosure submodule of W in Q, if D is a coessential submodule of W in Q and coclosed of Q. That is, $\frac{W}{D} << \frac{Q}{D}$ and whenever $Y \subseteq V$ with $\frac{D}{Y} << \frac{Q}{Y}$ implies $Y = V$[3].

Proposition10[3] Let Q be an amply supplemented Module. Then each submodule of Q has a coclosure submodule.

Proposition11 Let Q be an R-Module and let W and V be submodules of Q such that W⊂ V⊂Q, if $W \subseteq_{sce} V$ in Q and $\frac{Q}{V}$ p-semihollow Module then $\frac{Q}{W}$ p-semihollow Module.

Proof: By third isomorphism theorem, $\frac{Q}{V} \cong \frac{\frac{Q}{W}}{\frac{V}{W}}$. Since $\frac{Q}{V}$ is p-semihollow and $W \subseteq_{sce} V$ in Q, then by prop.4, $\frac{Q}{W}$ is p-semihollow.

A submodule W of an R-Module Q is coclosed of Q ($W \subseteq_{cc} Q$), if $\frac{W}{U} << \frac{Q}{U}$ implies that $W = U$ for all $U \subseteq Q$ contained in W [7].

Proposition12 Let Q be a p-semihollow-lifting Module, then each coclosed cyclic submodule U of Q with $\frac{Q}{U}$ p-semihollow is a direct summand of Q. The converse is true if Q is amply supplemented.

Proof: Assume Q is a p-semihollow-lifting Module and U coclosed cyclic submodule in Q s.t $\frac{Q}{U}$ p-semihollow. Since Q is p-semihollow-lifting, then ∃ a direct summand N of Q s.t $N \subseteq_{sce} U$ in Q. But K is coclosed in Q, so $U = N$. Thus, K is a direct summand of Q.

Conversely, suppose Q is amply supplemented Module and each coclosed proper cyclic Submodule U of Q with $\frac{Q}{U}$ p-semihollow is a direct summand. To prove Q is p-semihollow-lifting, let W be a cyclic submodule of Q with $\frac{Q}{W}$ p-semihollow. So, by prop.10, W has a coclosure submodule U in Q. Thus $U \subseteq_{sce} W$ in Q and $U \subseteq_{cc} Q$. Since $\frac{Q}{W}$ is p-semihollow, then by prop.11, $\frac{Q}{U}$ is p-semihollow. Thus, by assumption, U is a direct summand, hence Q is p-semihollow-lifting.

An R-Module Q have (D3) if for each direct summands Y and V of Q with $Q = Y + V, Y \cap V$ is a direct summand of Q [4].

The submodules K and W are called mutual supplements in R-Module Q, if they are supplements of each other [9].

Proposition13 Let Q = W + U be a p-semihollow-lifting Module, where W and U are cyclic mutual supplements in Q with $\frac{Q}{W}$ and $\frac{Q}{U}$ are p-semihollow Modules. If Q has (D3), then Q = W⊕U

Proof: Let Y and D be two cyclic submodules of Q which are mutual supplements in Q, with $\frac{Q}{Y}$ and $\frac{Q}{D}$ are p-semihollow Modules. Then by [3,lemma1.1], Y and D are coclosed submodules of Q. But Q is p-semihollow-lifting, therefore by prop.12, Y and D are direct summands of Q. Since Q = Y + D and Q has (D3), thus Y∩D is a direct summand of Q so Q = (Y∩D) ⊕X, for

Diyala Journal for Pure Science
International Conference for Pure Sciences (ICPS-2021)

some X⊆Q. But D is a supplement of Y then Y∩D≪ D and hence Y∩D ≪ Q. So $Q = X$ and Y∩D = 0. Thus, Q = Y⊕D.

The following proposition gives a condition under which a direct summand of a p-semihollow-lifting Module is p-semihollow-lifting.

Proposition14 Let Q be a p-semihollow-lifting Module having (D3). Then each direct summand of Q is p-semihollow-lifting.

Proof: Let W be a direct summand of Q. Thus Q = W ⊕ W^* for some submodule W^* of Q. Let Y be cyclic submodule in W s.t W/Y is p-semihollow. Now, Q/Y = (W⊕W^*)/Y = W/Y⊕(W^*⊕Y)/Y, By [5, corr.3,44], Q/Y/(W^*⊕Y)/Y ≅ W/Y, thus by third isomorphism theorem, Q/Y/(W^*⊕ Y)/Y ≅Q/(W^*⊕ Y). But W/Y p-semihollow, so Q/(W^*⊕Y) is p-semihollow. Since Q is p-semihollow-lifting, ∃ a direct summand V of Q s.t X$\subseteq_{sce}$ (W^*⊕ Y) in Q. Now, Q/X = (W ⊕ W^*)/X = (W+X)/X+(W^*+X)/X. Claim that Q ≠ W^*+X(if Q = W^*+X) this implies Q = W^* + Y which is contradiction). But by prop.11, Q/X is p-semihollow, so Q/X = (W+X)/X. Thus Q = W + X. Thus, by prop.4, W∩(W^*⊕Y)/(X∩W)$\ll_s$Q/(X∩W). Then $X \cap W \subseteq_{sce} Y$ in Q. But Q has (D3), thus X∩W is a direct summand of Q and X∩W is a direct summand of W. Since Y/(X∩W) ≤ W/(X∩W) and W/(X∩W) is a direct summand of Q/(X∩W), then by [1], X∩W$\subseteq_{sce}$Y in W. Thus, W is p-semihollow-lifting.

A submodule Y of an R-Module Q is called a fully invariant submodule if u(Y) ⊆Y, for each u ∈ $Hom(Q, Q)$[5].

An R-Module Q is duo-Module if each submodule of Q is fully invariant [8].

Lemma 15 [8] Let Q be an R-Module.If $Q = Q_1 \oplus Q_2$, then $\frac{Q}{Y} = \frac{Y+Q_1}{Y} \oplus \frac{Y+Q_2}{Y}$, for each fully invariant submodule Y of Q.

The following proposition gives a condition under which a factor of a p-semihollow-lifting Module is p-semihollow-lifting.

Proposition16 Let Q be an R-Module. If Q is a p-semihollow-lifting Module, then $\frac{Q}{U}$ is p-semihollow-lifting for each cyclic fully invariant submodule U of Q.

Proof: Let $\frac{W}{D}$ be a cyclic submodule of $\frac{Q}{D}$ such that $\frac{\frac{Q}{D}}{\frac{W}{D}}$ is p-semihollow. Then by third isomorphism theorem, $\frac{\frac{Q}{D}}{\frac{W}{D}} \cong \frac{Q}{W}$ is p-semihollow. Since Q is a p-semihollow-lifting Module, thus $\exists$ a submodule K of Q s.t $K \subseteq_{sce} W$ in Q and Q = $V \oplus V^*$, for some $V^* \subseteq Q$. Now, clearly $V + D \subset W$ and hence $\frac{V+D}{D} \subset \frac{W}{U}$. Let $f : \frac{Q}{V} \to \frac{Q}{V+D}$ be a map defined by $f(q + V) = q + (V + D)$, for all $q \in Q$. Clearly, f is an epimorphism. But $V \subseteq_{sce} W$ in Q, thus by [1], $f\left(\frac{W}{V}\right) \ll_s \frac{Q}{V+D}$ and hence V+D$\subseteq_{sce}$W in Q. Then by third isomorphism theorem, $\frac{V+D}{D} \subseteq_{sce} \frac{W}{D}$ in $\frac{Q}{D}$. Now, by lemma15, $\frac{Q}{D} = \frac{V \oplus V^*}{D} = \frac{V+D}{D} \oplus \frac{V^*+D}{D}$. Therefore $\frac{V+D}{D}$ is a direct summand of $\frac{Q}{D}$. Then $\frac{Q}{D}$ is p-semihollow-lifting.

If Q is a p-semihollow-lifting Module and A is not cyclic fully invariant submodule of Q, then $\frac{Q}{A}$ need not be p-semihollow-lifting. For example, consider the Z-Module $Q = \frac{Z}{4Z} \oplus \frac{Z}{8Z}$, clearly, Q is p-semihollow-lifting [8, Example 2.2]. Let $A = \frac{2Z}{4Z} \oplus 0$ be a submodule of Q, then $\frac{Q}{A}$ is not p-semihollow-lifting. To see that: $\frac{Q}{A} = \frac{\frac{Z}{4Z} \oplus \frac{Z}{8Z}}{\frac{2Z}{4Z} \oplus 0} \cong \frac{\frac{Z}{4Z}}{\frac{2Z}{4Z}} \oplus \frac{Z}{8Z}$, then $\frac{Q}{A} = \frac{Z}{2Z} \oplus \frac{Z}{8Z}$ is not p-semihollow-lifting.

The following corollary gives another condition under which a direct summand of a p-semihollow-lifting Module is p-semihollow-lifting.

Corollary17 Let Q be a duo p-semihollow-lifting Module. Then each direct summand of Q is a p-semihollow-lifting.

Theorem18 Let Q be a non-zero indecomposable Module over a commutative ring R. Then the following are equivalent:

1. Q is p-semihollow-lifting.
2. Q is p-semilifting.
3. Q is p-semihollow.

Proof: 2) ⇔ 3) By [1].

3) ⇒ 1) Clear.

1) ⇒ 3) Let W be a proper cyclic submodule of Q. Since Q is p-semihollow-lifting, ∃ a submodule U of Q s.t $Q = U \oplus U^*$ and $\frac{W}{U} \ll_s \frac{Q}{U}$. Since Q indecomposable Module, then U = 0 and hence W $\ll_s$ Q. Thus, Q is p-semihollow.

Lemma 19 [9, p.63] Let $f: Q \rightarrow V$ be an epimorphism of R-Modules and $Q = D + Y$ where D and Y are submodules of Q then:

1. $V = f(D) + f(Y)$.
2. If $kerf = D \cap Y$, then $V = f(D) \oplus f(Y)$.

Proposition20 Epimorphic image of p-semihollow Module is p-semihollow.

Proof: Let Q, Q' be R-Modules, Q be p-semihollow and f: Q → Q' be an R-epimorphism, Let W be a proper cyclic submodule of Q'. Thus $f^{-1}(W)$ is a proper cyclic submodule of Q. Since Q p-semihollow, $f^{-1}(W)$ is semismall in Q, $f(f^{-1}(W))$ is semismall in Q'. Thus, W is semismall in Q' and hence Q' is p-semihollow.

Proposition21 Let f: Q → U be an epimorphism of R-Modules, let W be submodules of Q and Y be a cyclic submodule of Q such that $Q = Y + W$ and $kerf = Y \cap W$. If U is a p-semihollow-lifting Module and W is p-semihollow, then $U = Q_1 \oplus Q_2$, where $Q_1 \subseteq_{sce} f(Y)$ in U and Q_2 is p-semihollow.

Proof: By lemma19, $U = f(Y) \oplus f(W)$. Since W is p-semihollow, then by prop.20, f(W) is p-semihollow.Thus, by second isomorphism theorem, $\frac{U}{f(Y)} \cong f(W)$. So $\frac{U}{f(Y)}$ is p-semihollow. But

U is a p-semihollow-lifting Module, thus ∃ a direct summand Q_1 of U s.t $Q_1 \subseteq_{sce} f(Y)$ in U. Thus $U = Q_1 \oplus Q_2$, where $Q_2 \subseteq U$. Now, $\frac{U}{Q_1} = \frac{f(Y) \oplus f(W)}{Q_1} = \frac{f(Y)}{Q_1} + \frac{f(W) \oplus Q_1}{Q_1}$. This implies $U = f(W) \oplus Q_1$. By second isomorphism theorem, $\frac{U}{Q_1} \cong f(W)$ and $\frac{U}{Q_1} \cong Q_2$ therfore $Q_2 \cong f(W)$. Then Q_2 is p-semihollow.

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