Diyala Journal for Pure Science

On Some Applications of Lidskii's Theorem

Duaa Taheir Bader* **and Fatima Mohammad Aboud**

Mathematics Department - College of Science - University of Diyala – Iraq

duaataheir@gmail.com

Received: 25 May 2021 **Accepted: 13 July 2021**

DOI: https://dx.doi.org/10.24237/djps.17.04.562B

Abstract

Lidskii's theory is considered one of the most important and recent theories for calculating the following categories and the relationship between them and the eigenvalues. This theory provides an easier way to prove the existence of the eigenvalues, and hence to prove the existence of solutions for some kind of problems.

This thesis article to prove that there are solutions to some problems for which the computation of eigenvalues is very complex and to prove that the existence of eigenvalues is also complex, in our work we try to take advantage of the fact that calculating the trace is much easier than calculating eigenvalues. Lidskii's theorem gives the relationship between Trace and eigenvalues and gives us a way to prove the existence of eigenvalues.

Keywords: Nonlinear eigenvalue problems, spectra, trace, quasi-homogenous operators

Diyala Journal for Pure Science

On Some Applications of Lidskii's Theorem

Duaa Taheir Bader and Fatima Mohammad Aboud

في بعض تطبيقات نظرية ليدسكي

دعاء طاهر بدر و فاطمه محمد عبود

قسم الرياضيات - كلية العلوم - جامعة ديالى

الخلاصة

في هذا البحث سندرس ايجاد القيم الذاتية التي يصعب مقارنة حسابها للعثور على الاثروقمنا باعطاء تطبيقات على مبرهنه ليدسكي باستخدام برنامج السيلاب وهو مكافئ لبرنامج الماتلاب

الكلمات المفتاحية: القيم الذاتية الالخطية، الطيف، الاثر، المؤثرات شبه المتجانس

Introduction

Lidskii's theorem gives the relationship between Trace and eigenvalue and gives us a way to prove the existence of eigenvalue. In fact, computing an eigenvalue is difficult compared to find the formula of the trace.

In 2002, S. Chanillo, B. Helffer, and A. Laptev show that the eigenvalue of a special family of compact non selfadjoint operators is not zero. They demonstrate how to apply old results obtained using ordinary differential equations techniques to the higher dimensional case. There results in a new class of hypoelliptic operators that are not analytic hypoelliptic operators [1].

In 2009, T. Phelan explained using a self-container derivation of the popular Lidskii trace theorem for Hilbert space operators.[2]

In 2009, F. Aboud, presents a technique to prove the existence of non-trivial eigenvalues for some classes of non-linear eigenvalue problems for any dimension d ≥1 and she consider specialy for the problems with odd dimentional spaces [3].

In 2014, F. Aboud and D. Robert considered a family of operators with a quadratic dependency on a complex parameter and generalized eigenvalue problems. They present a method for demonstrating the existence of nontrivial solutions to the equation L(λ)u = 0, which is true in all dimensions d ≥1 [4].

In 2018, Fatima Aboud examine the case of odd dimension d>1 for the family of quasi-homogenous and provide explanations for the cases where the space dimension is either d=3 or d=5. She investigates the conditions under which it is possible to demonstrate the existence of a nontrivial solution in each case [5].

In 2019, Fatima Aboud study the conditions for which the existence of non-trivial solution is guaranteed for the case of odd dimension d>1 for the family of quasi-homogeneous and quasi-elliptic operators and give some examples for the case for which d=3 [6].

In 2020. Fatima Aboud et al., give a review of theoretical results for quadratic operator spectra, especially for Schrodinger pencils, also they present the computational methods developed to compute the spectra, including spectral methods and finite difference discretization in infinite and bounded domains [7].

In this article we present a verification for the Lidskii theorem for some applications for a non-linear eigenvalue problems by using some numerical method. We try to present the proof in a simple way, we present applications on the trace calculation using the programmation by Scilab software. The advantage of this software is that it is a free application comparing with Matlab and it is similar and equivalent to Matlab.

1. Preliminaries

In the following section we recall some basic concepts that we need in our work.

Theorem 1-1 (Lidskii's Theorem): If A is a trace class operator and λn are the eigenvalues of A, then for any orthonormal basis ηm, we have [2]

Diyala Journal for Pure Science

$$\sum_{n} \lambda n(A) = Tr(A) := \sum_{m} (\eta m\,,\, A\eta m).$$

Definition 1-1 (Positive operator): Let H be the space of a Hilbert. Linear operator A is known as positive operator A. if for all $x \in H, (Ax, x) \geq 0.$ one writes $B \geq 0$ if B is positive and $A \geq B$ if $A - B \geq 0$.

Definition 1.2 (The Trace): Let H be a space of Hilbert and {φi}be an orthonormal basis for H. there we define the trace of A for any positive operator A ∈ L(H) to be

$$\text{tr (A)} = \sum_{i=1}^{\infty} (\varphi_{i\,,}\, A\varphi_i\,)$$

Where (,) is the inner product in space H.

In the following we give some properties of the trace:

The quantity tr A is independent of the collection {φi} of

Orthonormal foundation

tr (A+B) = tr (A) + tr (B).

tr (λA) = λ tr (A) for all λ ≥ 0.

Definition 1.3 (The family of all trace operators): An operator A ∈ L (H) is referred to as the trace class if and only if

tr |A| < ∞. The family of all trace class operators is denoted by T1 and the map are denoted by the families of trace class operator

$$tr\, : T_1 \to ₵ \text{ Define by } A \mapsto \sum_{i=1}^{\infty} (\varphi_{i\,,}\, A\varphi_i\,)$$

Called operator trace A, where ℂis the field of the complex number.

Definition 1.4 (Non-linear eigenvalue): In the case of a linear eigenvalue problem, a complex number λ is referred to be an eigenvalue of T if $Tx = \lambda x$ for a nonnull vector x.

In the case of a non-linear eigenvalue problem, λ is said to be a nonlinear eigenvalue of T if the problem $T(\lambda)x = 0$ has a non-trivial solution x [8].

2. Theoretical part

As stated Lidskii's theorem was done a relation between the trace and the eigenvalues of the trace class operators. To indicate that the problem has a non-trivial eigenvalue, we will use the following proposition which is a direct result of Lidskii's theorem.

Proposition 2.1: Suppose that A is a trace class operator. If the spectrum $\sigma(A)$ satisfies $\sigma(A) = \{0\}$, then $Tr(A^k) = 0$, $\forall k \in N^*$.

So we will use the converse of this proposition, then $Tr(C^k) \neq 0$, then $\sigma(C) \neq \{0\}$ [6].

This means that in case that the trace of any power of the operator is different from zero, then the spectrum of this operator is also different from zero.

We proceeded as following:

- We check if the trace of the operator or any power of it is different from zero, in this case, we obtain directly that the spectrum of this operator is different from zero.
- We use this technique in two applications that we give in the next section. These two examples have already been done in [4-8] and [1], in these references the calculations were done for the eigenvalues but in our work, we will calculate the trace.

3. Applications

In this section give a numerical application by considering some family of non-linear eigenvalue problem with either periodic boundary conditions or homogenous Dirichlet conditions (for a matter of simplicity). To find the trace and the eigenvalues numerically we discretized the linearized problem by using the finite difference method and we find the ordinary eigenvalues and trace for the corresponding linearized discrete matrix.

3.1. Non-Linear eigenvalue with periodic boundary conditions

We consider the quadratic family:

$$l(\lambda) = -\Delta + (p(x) - \lambda)^2$$

Where p(x) is a polynomial. To find the trace and the eigenvalues for the following problem with the periodic conditions:

$$-\Delta + (p(x)^2 - 2\lambda p(x) + \lambda^2)u = 0$$

By the linearization we obtain:

$$A = \begin{pmatrix} 0 & 1 \\ -H_0 & H_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -(-\Delta + p(x)^2) & -(-2p(x)) \end{pmatrix}$$

We need to approximant the system by using the finite difference method so the internal will be $= 1, 2, \dots, N$. [7]

We approximant the 2^w derivative by

$$\frac{d^2}{dx^2} = \delta\delta^*$$

Where

$$Su(n) = u(n+1) - u(n),$$

$$S^{*}u(n) = u(n) - u(n-1),$$

$$\frac{d^2u(n)}{dx^2} = u(n-1) - 2u(n) + u(n+1)$$

So

$$lu(n) \approx -(\delta\delta^{*})\, u(m) - (n^k - \lambda)u$$

I.e. we approximate:

$$x \; by \; n$$

$$u(x) \;\; by \;\; u(n)$$

$$Du(x) \;\; by \;\; \delta\delta^{*}u(n)$$

For the problem:

$$-(\delta\delta^{*})u(n) + (n^k - \lambda)^2 u(n) = 0$$

$$n = 1,2,\dots,N$$

$$u(j) = u(j+N) \;, \quad j = 0,1$$

$$-(\delta\delta^{*})u(n) + n^{2k}u(n) - 2n^k\lambda u(n) + \lambda^2 u(n) = 0$$

$$\Longrightarrow H_0 + \lambda H_1 + \lambda^2 \mathbb{I} = 0$$

Where $H_0 = -(\delta\delta^{*})u(n) + n^{2k}u(n), H_1 = -2n^k$. To find the material formula of $H_0 and\ H_1$.

Diyala Journal for Pure Science

For the formula of H_1 we need to find $-2n^k\lambda\mathbb{I}$:

$$n = 1, \qquad 1^k u(1) + 0 + \cdots + 0$$

$$n = 2, \qquad 2^k u(2) + 0 + \cdots + 0$$

$$\vdots$$

$$n = s, \qquad 0 + \cdots + j^k\, u(j) + 0 + \cdots + 0$$

$$\vdots$$

$$n = N - 1 \qquad 0 + \cdots + 0 + (N-1)^k u(N-1) + 0 + \cdots + 0$$

$$n = N\,, \qquad 0 + \cdots + 0 + N^k u(n)$$

So we get

$$H_1 = -2\begin{pmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ 0 & 2^k & 0 & \cdots & & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & & \cdots & 0 & (N-1)^k & 0 \\ 0 & \cdots & & & \cdots & 0 & N^k \end{pmatrix},$$

For H_0 we need $-(\delta\delta^*)u(n) + n^{2k}u(n)$

$$\delta\delta^* \;:\; u(n-1) - 2u(n) + u(n+1)$$

$$n = 1: \qquad u(0) - 2u(1) + u(2) = 0$$

$$n = 2: \qquad u(1) - 2u(2) + u(3) = 0$$

$$n = j: \qquad u(j-1) - 2u(j) + u(j+1) = 0$$

$$\vdots$$

$$n = N-1: \quad u(N-2) - 2u(N-1) + u(N) = 0$$

$$n = N: \quad u(N-1) - 2u(N) + u(N+1) = 0$$

So $\delta\delta^*$ is given by the following matrix

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & \cdots & & & 0 & 1 \\
1 & -2 & 1 & 0 & 0 & \cdots & & & 0 \\
0 & 1 & -2 & 1 & 0 & & & & 0 \\
 & & \vdots & & \vdots & \vdots & & & \\
0 & \cdots & 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
 & & \vdots & & \vdots & \vdots & & & \\
0 & \cdots & & & \cdots & 0 & 1 & -2 & 1 \\
1 & 0 & \cdots & \cdots & & \cdots & 0 & 1 & -2
\end{pmatrix}$$

Hence $H_0 = (-(\delta\delta^*) + n^{2k})\mathbb{I}$ is given by

$$\begin{pmatrix}
-2+1 & 1 & 0 & 0 & \cdots & & & 0 & 1 \\
1 & -2+2^{2k} & 1 & 0 & 0 & \cdots & & & 0 \\
0 & 1 & -2+3^{2k} & 1 & 0 & & & & 0 \\
 & & \vdots & & \vdots & \vdots & & & \\
0 & \cdots & 0 & 1 & -2+j^{2k} & 1 & 0 & \cdots & 0 \\
 & & \vdots & & \vdots & \vdots & & & \\
0 & \cdots & & & \cdots & 0 & 1 & -2+(N-1)^{2k} & 1 \\
1 & 0 & \cdots & \cdots & & \cdots & 0 & 1 & -2+N^{2k}
\end{pmatrix}$$

We find the trace and the eigenvalues of the linearized matrix A by using Scilab software for N=500, 1000, and k=2 and we obtain the following figures:

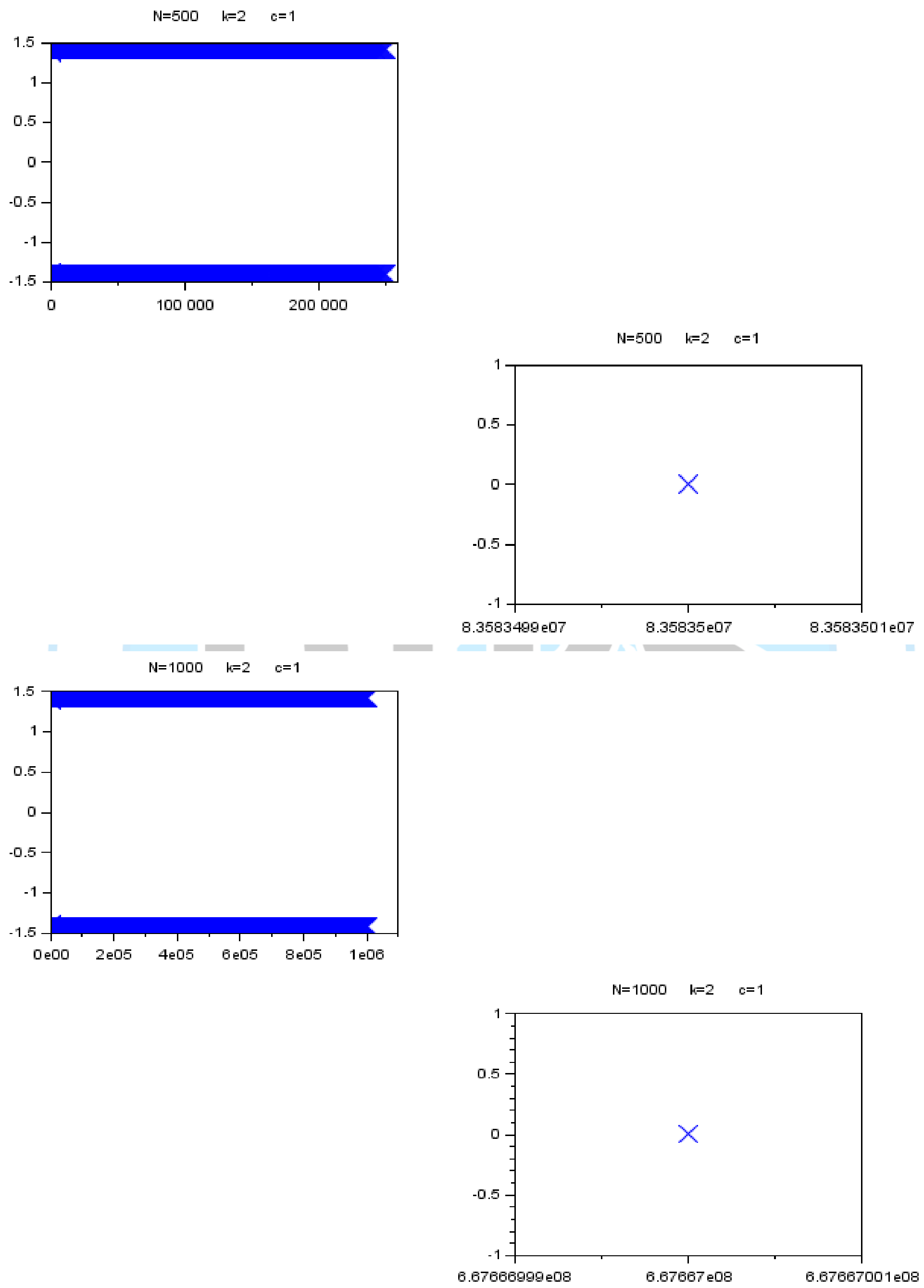

Figure 1: The figure on the right represents the trace, which does not equal zero, and the figure on the left represents the eigenvalues

3.2. Non-linear eigenvalue problem with homogeneous Dirichlet boundary conditions

Now we consider the following homogenous Dirichlet boundary conditions:

$$-(\delta\delta^*)u(n) + (n^k - \lambda)^2 u(n) = 0$$

$$u(0) = u(N+1) = 0$$

We obtain the following nonlinear eigenvalue problem

$$H_0 + \lambda H_1 + \lambda^2 \mathbb{I} = 0$$

With $H_0 = -(\delta\delta^*)u(n) + n^{2k}u(n), H_1 = -2n^k$, $\mathbb{I}$ is the NxN identity matrix and H_1, H_0 are given as follows:

$$H_1 = -2\begin{pmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ 0 & 2^k & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ 0 & \cdots & & \cdots\ 0 & (N-1)^k & 0 \\ 0 & \cdots & & \cdots & 0 & N^k \end{pmatrix},$$

$$H_{0,} = H_{0,d} + H_{0,+1} + H_{0,-1} ,$$

where

$$H_{0,d}=\begin{pmatrix} 2+1 & 0 & 0 & \cdots & & & & \cdots & 0 & 0 \\ 0 & 2+2^{2k} & 0 & 0 & \cdots & & & & \cdots & 0 \\ & & & \vdots & & & \vdots & & & \\ 0 & \cdots & \cdots & 0 & 0 & 2+j^{2k} & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 0 & 2+(N-1)^{2k} & 0 \\ 0 & 0 & \cdots & & & & \cdots & 0 & 0 & 2+N^{2k} \end{pmatrix},$$

$$H_{0,+1}=\begin{pmatrix} 0 & -1 & \cdots & & \cdots & 0 \\ 0 & 0 & -1 & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & -1 \\ 0 & \cdots & & \cdots & 0 & 0 \end{pmatrix}, A_{0,-1}=\begin{pmatrix} 0 & 0 & \cdots & & \cdots & 0 \\ -1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ 0 & \cdots & & \cdots & 0 & 0 & 0 \\ 0 & \cdots & & \cdots & -1 & 0 \end{pmatrix}.$$

We find the trace and the eigenvalues of the linearized matrix A by using Scilab software for N=500, 1000, and k=0 and we obtain the following figures:

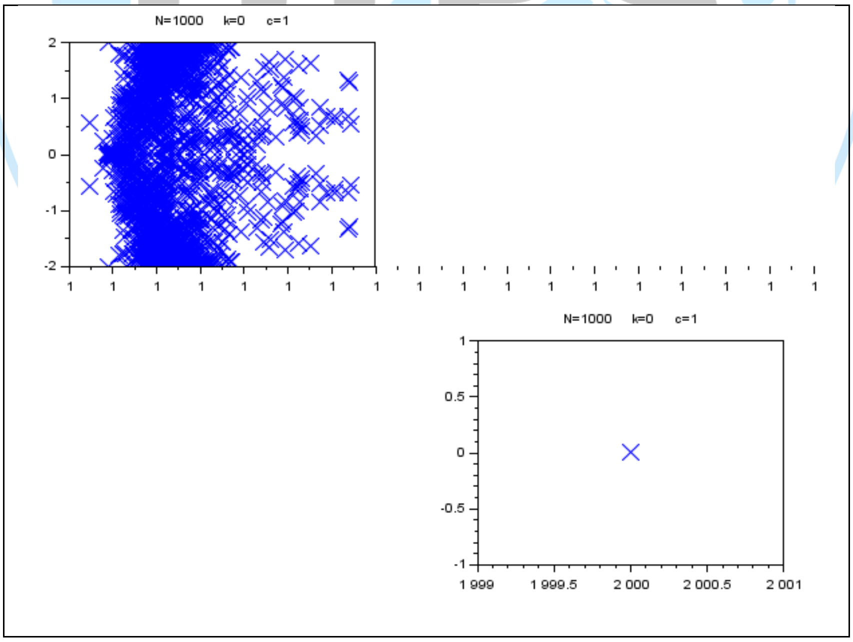

Figure 3: The figure represents the trace, which does not equal zero

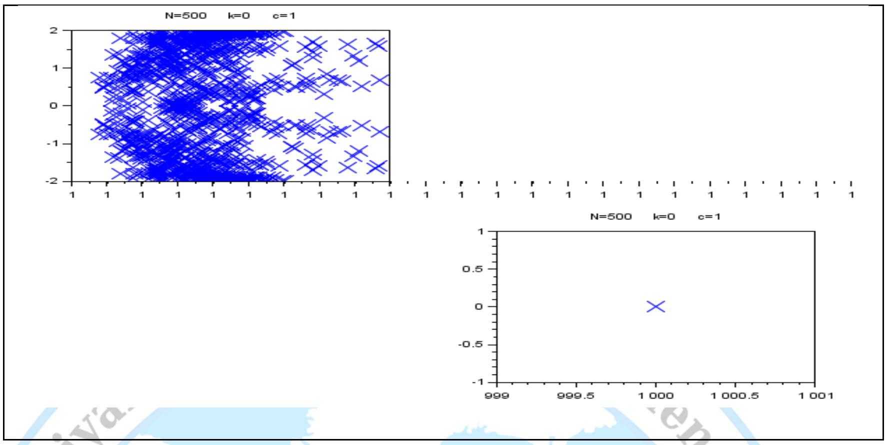

Figure 3: the figure represents the eigenvalues

To begin, we compute the eigenvalues for various N values and the operator $\mathcal{L}$. The eigenvalues of certain perturbations of the operator $\mathcal{L}$ are then computed, i.e. we consider the discrete operator:

$$\mathcal{L}u(n) = H_0 u(n) + \lambda H_1 u(n) + \lambda^2 \mathbb{I} u(n), n = 1, \cdots, N$$

And the same homogeneous Dirichlet boundary conditions as before. Instead of calling the non-linear problem, we consider the linearization system problem. As a result, we investigate the linear system's spectrum.[7]

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbb{I} \\ -H_0 & -H_1 \end{pmatrix},$$

Conclusion

We conclude that the numerical methods of computing trace confirm the theoretical results

Diyala Journal for Pure Science

On Some Applications of Lidskii's Theorem

Duaa Taheir Bader and Fatima Mohammad Aboud

References

1. S. Chanillo, B. Helffer, A. Laptev, Journal of Functional Analysis, 209 (2), 425–443 (2004).
2. T. Phelan, Trace Class Operators and Lidskii's Theorem, 1–11(2009).
3. F. Aboud, Problèmes aux valeurs propres non-linéaires Fatima Aboud To cite this version, Doctoral dissertation, Université de Nantes (2009).
4. F. Aboud, D. Robert, F. Aboud, D. Robert, Journal de mathématiques pures et appliquées, vol. 93(2), 149–162(2014).
5. F. Mohammed Aboud, Diyala Journal for Pure Science, 14(4), 88–104(2018)
6. F. M. ABOUD, Zanco Journal of Pure and Applied Sciences, 31(2),70–80(2019)
7. F. Aboud, F. Jauberteau, G. Moebs, D. Robert, J. Math. Study, 53, 12-44(2020)
8. B. Helffer, D. Robert, and X. Wang, St. Petersburg Mathematical Journal, 16(1),285-296 (2005)