On the Approximation in the Weighted Spaces ($L_{p,\beta}(X)$) via Spline Polynomials

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Abstract

In this paper, we introduce a new norm and modulus in weighted spaces ($\boldsymbol{L_{p,\beta}(X)}$) of order $\boldsymbol{k}$. Via these modulus, we prove the direct and inverse spline approximation inequalities of unbounded functions in weighted spaces ($\boldsymbol{L_{p,\beta}(X)}$) **;** $0 < p \leq 1$ which is the main results of our paper.

Keyword: Direct and inverse inequalities, Unbounded functions, Spline polynomials, Weighted spaces.

حول التقريب في فضاءات الوزن ($L_{p,\beta}(X)$) بواسطة متعددات الحدود النقطية

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الخلاصة

في هذا البحث نقدم معياراً ومقاييساً جديدة في فضاءات الوزن من الرتبة k وباستعمال هذه المقاييس نقوم ببرهنة عدداً من المتراجحات المباشرة والمعكوسة للدوال غير المقيدة في فضاءات الوزن ($L_{p,\beta}(X)$) بواسطة تقريب متعددات الحدود

الكلمات مفتاحية: المتراجحات المباشرة والمعكوسة، متعددات الحدود النقطية ، ا لدوال غير المقيدة، فضاءات الوزن .

Introduction

Direct and inverse theorems which establish a relationship between the degree of best approximation of unbounded functions in weighted space with respect to spline polynomials and modulus of continuity of order $\boldsymbol{k}$. In 1998 Radzievakii and Zeng studied direct and inverse theorems [1, 2], using the notation of a $\boldsymbol{k}$-functional has two-sided estimates with regard to the modulus of continuity at least for bounded C_0 $-$groups. The purpose of this article is developing a theory of direct and converse theorems for spline approximation in weighted space ($L_{p,\beta}(X)$), $0 < p \leq 1$. This result were proved by Gorbachuk and Grushk [3, 4] and extended by Kochurov and Zoha [5, 6]. However the present article deals with a rather different setting which somehow related to the direct and inverse inequalities see [7]. Let $X = [0,1]$ be the periodic unit interval and $0 < p \leq 1$ and W be the set of all weight functions. Then as usual, the weighted space $L_{p,\beta}(X)$ is the quasi-norm of all unbounded functions which the following norm:

$$\|f\|_{p,\beta} = \left(\int_X |f(x)\beta(x)|^p dx\right)^{\frac{1}{p}} < \infty \qquad \dots (1)$$

Where $\beta: X \to \mathbb{R}^{+}$ the weight function on X such that, $|f(x)\beta(x)| < M$; M is positive real number.

Let us recall some definitions of modulus of order k which are used through this article. The kth symmetric difference of f is given by:

$$\Delta_h^k f(x) = \begin{Bmatrix} \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f(x+ih) & , if \ x, x+ih \in X \\ 0 & otherwise \end{Bmatrix} \quad \ldots (2)$$

Where $k = 1,2,3, \ldots$.

Then the $\boldsymbol{kth}$ modulus of continuity of $f \in L_{p,\beta}(X)$ is defined by:

$$\omega_k(f,\eta)_{p,\beta} = \underbrace{sup}_{0 \le h < \eta} \left\| \Delta_h^k f(.) \right\|_{p,\beta}, \qquad 0 < \eta < 1 \qquad \ldots (3)$$

Also, the kth modulus of continuity of, $g \in L_{p,\beta}(X)$, $0 < p \le 1$ which satisfies the following properties:

$$\omega_k(f+g,\eta)_{p,\beta}^p \le \omega_k(f,\eta)_{p,\beta}^p + \omega_k(g,\eta)_{p,\beta}^p \qquad \ldots (4)$$

$$\omega_k(f,\xi)_{p,\beta} \le C(\frac{\xi}{\eta})^{\frac{k+1}{p}} \omega_k(f,\eta)_{p,\beta} \ \eta < \xi \ \& \ C > 0 \qquad \ldots (5)$$

$$\omega_k(f,\eta)_{p,\beta} \le C \ \omega_{k-1}(f,\eta)_{p,\beta} \le C\|f\|_{p,\beta} \qquad \ldots (6)$$

Let $n = 1,2, \ldots \ldots$ the partitions of X is y setting

$\Pi_n = \{0 = j_{0,n} < j_{1,n} < j_{2,n} < \cdots < j_{n-1,n} < 1\}$, are defining by:

$$s_{i,n} = \begin{Bmatrix} 2^{-l-1}.i & ; i = 0,1, \ldots, 2j \\ 1 - 2^{-l}.(n-i); & i = 2j, \ldots, n-1 \end{Bmatrix}$$

For $n = 2^{-l} + j > 1$ where $j = 1,2, \ldots, 2^{l}$ and $l = 0,1,2, \ldots$ are a unique determined by n.

Furthermore, we set:

$$\mathcal{s}_{ln+i,n} = \mathcal{s}_{i,n} \;,\; \mathcal{s}'_{ln+i,n} = \mathcal{s}_{i,n} + l \;,\; l = 0, \mp 1, \dots \;.$$

For $k = 0,1,2, \dots$ we denoted by $\mathcal{S}_n^{(k)}(X)$ the n-dimensional subspace of $C^{(k-1)}(\text{X})$ (of $L_\infty(\text{X})$ if $k = 0$) consisting of all periodic spline functions of order k with respect to Π_n .

The corresponding B-splines

$S_{i,n}^{(k)}(x) = \sum_{l=-\infty}^{\infty} S_{ln+i,n}^{(k)}(x)$, where, $S_{l,n}^{(k)}(x) = \left(\mathcal{s}'_{l+k+1,n} - \mathcal{s}'_{l,n}\right)[\mathcal{s}'_{l,n}, \dots, \mathcal{s}'_{l+k+1,n}, (\mathcal{s} - x)^k]$, $x \in (-\infty, \infty)$, have the following properties

$$supp S_{i,n}^{(k)}(x) = [\mathcal{s}_{i,n}, \mathcal{s}_{i,n+k+1}] \quad \dots (7)$$

$$\sum_{i=0}^{n-1} S_{i,n}^{(k)}(x) = 1 \,,\; S_{i,n}^{(k)}(x) \geq 0 \;\;, \forall\, x \in X \quad \dots (8)$$

$$\left\{S_{i,n}^{(k)}(x)\right\}_{i=0}^{n-1} \text{ Form an algebraic basic in } S_n^{(k)}(x) \quad \dots (9)$$

$$\frac{d}{dx} S_{i,n}^{(k)}(x) = k \left\{\frac{S_{i,n}^{(k)}(x)}{\mathcal{s}'_{i+k,n} - \mathcal{s}'_{i,n}} - \frac{S_{i,n}^{(k)}(x)}{\mathcal{s}'_{i+k+1,n} - \mathcal{s}'_{i,n}}\right\} \quad \dots (10)$$

If $\psi(x) \in \mathcal{S}_n^{(k)}(X)$ & $(x) = \sum_{i=0}^{n-1} \alpha_i S_{i,n}^{(k)}(x)$, $\quad \dots (11)$

Then $1 \leq p < \infty$,$C(p)\{\sum_{i=0}^{n-1} n^{-1} |\alpha_i|^p\}^{\frac{1}{p}} \leq \|\psi\|_{p,\beta} \leq C^*(\text{p})\{\sum_{i=0}^{n-1} n^{-1} |\alpha_i|^p\}^{\frac{1}{p}}$

Where $C(p)$ and $C^*(\text{p})$ are constant independent of $\psi(x)$ & n.

If $n = 2^j$, then:

$$S_{i,n}^{(k)}(x) = S_{i,n}^{(k)}(x - i.2^j) \quad \dots (12)$$

Our interesting is about approximation of unbounded functions in weighted space $L_{p,\beta}(X)$ by spline polynomial belong to, $\mathcal{S}_n^{(k)}(X)$. For $f \in L_{p,\beta}(X)$, let

$$\mathcal{E}_n^{(k)}(f)_{p,\beta} = inf\left\{\|f - S_n\|_{p,\beta} \,, S_n \in \mathcal{S}_n^{(k)}\right\} \quad \dots (13)$$

be the degree of best spline approximation of unbounded functions in $L_{p,\beta}(X)$ have the following properties:

$$\mathcal{E}_n^{(k)}(f+g)_{p,\beta} \leq \mathcal{E}_n^{(k)}(f)_{p,\beta} + \mathcal{E}_n^{(k)}(g)_{p,\beta} \quad \dots (14)$$

$$\mathcal{E}_n^{(k)}(\xi f)_{p,\beta} \leq |\xi|\, \mathcal{E}_n^{(k)}(f)_{p,\beta} \quad \dots (15)$$

$$\mathcal{E}_n^{(k)}(f)_{p,\beta} \leq \mathcal{E}_{n-1}^{(k)}(f)_{p,\beta} \quad \dots (16)$$

$$\mathcal{E}_n^{(k)}(f + S_n)_{p,\beta} \leq \mathcal{E}_n^{(k)}(f)_{p,\beta} \quad \dots (17)$$

For, $, g \in L_{p,\beta}(X)$, $S_n \in \mathcal{S}_n^{(k)}$ and ξ scalor.

Auxiliary lemmas

Lemma 1 [8]

Let $k = 1,2,3, \dots..$, $s = 0,1,2, \dots..$ and $n = 2^j > s + k + 1$ be given. Then for every spline $\psi(x) = \sum_{i=1}^{n-1} \alpha_i S_n^{(k)} \in \mathcal{S}_n^{(k)}(X)$, there exist a step function

$\not{f}(x) \in \mathcal{S}_n^{(0)}(X)$, such that

$$\psi(x) - \not{f}(x) = \sum_{s=0}^{n-1} A_s(x) \quad \dots (18)$$

Where the function $A_s(x)$, $s = 0,1, \dots, n-1$ satisfy

$$supp\, A_s(x) \subset [s_{n,s}, s_{n,s+s+1}) = I_{n,s} \quad \dots (19)$$

$$\|A_s\|_\infty \leq C \sum_{i=s-k}^{s-1} |\Delta\alpha_i| \quad \dots (20)$$

Where $\alpha_{s+i,n} = \alpha_i$ for $i = 0,1, \dots, n-1$, $s = 0, \mp 1, \mp \cdots$

and

$$\int_{I_{0,s}} A_{sc}(x + s_{n,s})\, x^q dx = 0,\ q = 0,1, \dots, s \quad \dots (21)$$

The positive constant C in eq. **(20)** is independent of n & $\psi(x)$.

Lemma 2

If $k^{\circ} = 0,1,2, \dots , k > k^{\circ}, 0 < p \leq 1$ and $n = 2^j > k + 1$, then for any $\psi(x) \in \mathcal{S}_n^{(k)}(X)$, there exist a spline $\psi(x) \in \mathcal{S}_n^{(k^{\circ})}(X)$ satisfying

$$\|\psi - \psi^{\circ}\|_{p,\beta} \leq C\omega_{k^{\circ}}(\psi, \tfrac{1}{n})_{p,\beta} \qquad \dots (22)$$

Proof:

Let $\psi(x) = \sum_{i=0}^{n-1} \alpha_i S_i(x)$ be the B-spline with respect to, $\psi(x)$.

From eq. **(10)** we have, $\psi^{(k^{\circ})}(x) = \frac{d^{k^{\circ}}}{dx^{k^{\circ}}}\psi(x) = n^{k^{\circ}} \sum_{i=0}^{n-1} \Delta^{k^{\circ}} \alpha_{i-k} S^{(k-k^{\circ})}(x)$, where $S^{(k-k^{\circ})} \in \mathcal{S}_n^{(k-k^{\circ})}(X)$.

By lemma **(1)** there is $\psi_1^{(k^{\circ})}(x)$ such that, $\psi^{(k^{\circ})}(x) - \psi_1^{(k^{\circ})}(x) = \sum_{s=0}^{n-1} A_s(x)$, where $A_s(x)$ satisfies eq. **(19), (20)** and eq. **(21),** we have

$$\|A_s\|_{\infty} \leq Cn^k \sum_{i=s-k}^{s-k^{\circ}-1} \left|\Delta^{k^{\circ}+1}\alpha_i\right| \qquad \dots (23)$$

We consider the functions

$$G_s(x) = \int_{j_s}^{1} \dots \int_{j_s}^{y_{k^{\circ}-1}} A_s\left(y_{k^{\circ}}\right) dy_{k^{\circ}} \dots dy_1 \qquad \dots (24)$$

By using eq. **(19), (20), (23)** and eq. **(24),** we have

$$supp\ G_s(x) \subset I_s \qquad \dots (25)$$

$$\|G_s\|_{\infty} \leq C \left\{\sum_{i=s-k}^{s-k^{\circ}-1} \left|\Delta^{k^{\circ}+1}\alpha_i\right| n^p\right\} |I_s|^p \qquad \dots (26)$$

$$\int_{I_{0,s}} G_s(x + j_s)\, x^q dx = 0 \,,\ q = 0, \dots , [\tfrac{1}{p-1}] \qquad \dots (27)$$

Thus the functions $G_s(x)$ are multiples of $(p, \infty, \left[\frac{1}{p-1}\right])$ and by eq. **(11)** and eq. **(12)** we get

$$\psi(x) - \psi^{\circ}(x) = \sum_{s=0}^{n-1} G_s(x)$$

$$\|\sum_{s=0}^{n-1} G_s\|_{p.\beta} = (\int_X |\sum_{s=0}^{n-1} G_s(x)\ .\beta(x)|^p dx)^{\frac{1}{p}}$$

$$\leq C(\int_X \left(\sum_{s=0}^{n-1}\left(\sum_{i=s-k}^{s-k^{\circ}-1}\left|\Delta^{k^{\circ}+1}\alpha_i(x)\ \beta(x)\right|^p\right)\right) dx)^{\frac{1}{p}}$$

$$\leq\ C(\int_X \left(\sum_{s=0}^{n-1}\left|\Delta^{k^{\circ}+1}\alpha_i(x)\ \beta(x)\right|^p\right) dx)^{\frac{1}{p}}$$

$$\leq\ C(\int_X \left(\sum_{s=0}^{n-1}\left|\Delta^{k^{\circ}+1}\alpha_i(x) S_i(x)\ \beta(x)\right|^p\right) dx)^{\frac{1}{p}}$$

$$\leq C \left\|\Delta_{n^{-1}}{}^{k^{\circ}+1}\ \psi\right\|_{p,\beta}$$

$$\leq C\omega_{k^{\circ}}(\psi, n^{-1})_{p,\beta}\ .\blacksquare$$

Lemma 3

If $(x) \in \mathcal{S}_n^{(k)}(X)$, $k = 0,1,2, \dots , n = 2,3, \dots$ and $0 < p \leq 1$, then:

$$\left\|\Delta_h^k \psi\right\|_{p,\beta} \leq C(hn)^k \|\psi\|_{p,\beta} \qquad \dots (28)$$

Where $0 \leq h \leq \frac{C}{n}$, $C > 0$ independent of n, h.

Proof:

Let k, n and $\psi(x)$ be as above, $l = 1,2, \dots$ and $\frac{1}{l} < p < 1$. Then

$$\left\|\Delta_h^l \psi\right\|_{p,\beta} \leq C(hn)^l \|\psi\|_{p,\beta} \qquad \dots (29)$$

We will to prove eq. **(29).** Let $\Delta_h^l \psi(x) = \sum_{i=0}^{n-1} \alpha_i \Delta_h^l S_i(x)$ ... (30)

be the formula for the difference of order l to the B-spline of $\psi(x)$.

It's clear for $n \geq n(k,l)$ and $0 \leq h \leq C(k,l)\frac{1}{n}$.

The functions $\Delta_h^l S_i(x)$ have the following properties

$$supp\Delta_h^l S_i(x) \subset I_i = [s_{i-1}, s_{i+k+1}) \quad \text{... (31)}$$

$$\int_0^{|I_i|} \Delta_h^l S_i(x + s_{i-1})x^q dx = (-1)^l \int_0^{|I_i|} S_i(x + s_{i-1})\Delta_h^l x^q dx = 0 \quad \text{... (32)}$$

$q = 0,1, \dots, l-1$, and from eq. **(7),** we have

$$\|\Delta_h^l S_i\|_{p,\beta} \leq C(hn)^{\min(l,k)}\frac{1}{n} \leq C\left\{(hn)^{\min(l,k)}\frac{1}{n^{\frac{1}{q}}}\right\}|I_i|^{1-\frac{1}{p}}.$$

There for by using eq. **(11)**

$$\|\Delta_h^l \psi\|_{p,\beta} \leq C\left\{\sum_{i=0}^{n-1}|\alpha_i|^p\,(hn)^{\min(l,k)}\frac{1}{n}\right\}^{\frac{1}{p}} \leq C(hn)^{\min(l,k)}\|\psi\|_{p,\beta} \quad \text{... (33)}$$

This yield eq. **(29)** and eq. **(31)** in the case $0 < p \leq 1$, we still have the approximate of eq. **(33)** and the obvious the relation $\|\Delta_h^l S_i\|_{p,\beta} \leq C\|\Delta_h^l S_i\|_1$. Thus eq. **(31)** remains valid.■

Results

In this section, we give certain results, which are necessary to prove it. The direct results of Jackson type are presented in the following:

Theorem 1

Let $\in L_{p,\beta}(X)$, $0 < p \leq 1$ and $k = 0,1,2, \dots$. Then the following hold for $n = 1,2, \dots$

$$\mathcal{E}_n^{(k)}(f)_{p,\beta} \leq C(p)\omega_k(f,\tfrac{1}{n})_{p,\beta} \quad \dots (34)$$

Where $C(p) > 0$ depending on p.

Proof:

Let $k^\circ = 0,1,2,\dots,\ 0 < p \leq 1$ and for any integer $k > k^\circ$ such that eq.(34) holds, this meaning:

If $\mathcal{E}_n^{(k)}(f)_{p,\beta} = inf\left\{\|f-\psi\|_{p,\beta}, \psi \in \ \mathcal{S}_n^{(k)}\right\} \leq C(p)\omega_k(f,\frac{1}{n})_{p,\beta}$

$$\leq C(p)\omega_{k^\circ}(f,\tfrac{1}{n})_{p,\beta} \quad \ldots (35)$$

Where $n = 1,2, \ldots$, on the other hand $\delta = 2^j > k$ by lemma **(2)** there exists a corresponding spline $\psi^\circ \in \mathcal{S}_n^{(k^\circ)}$ with

$$\left\|\psi - \psi^\circ\right\|_{p,\beta} \leq C(p)\omega_k(f,\tfrac{1}{n})_{p,\beta} \leq C(p)\omega_{k^\circ}(f,\tfrac{1}{n})_{p,\beta} \quad \ldots (36)$$

Now the wanted results can be attained by typical consideration from eq. **(6), (16), (35)** and eq. **(36)** If $2^l \leq n \leq 2^{l+1}$, then

$$\mathcal{E}_n^{(k^\circ)}(f)_{p,\beta} \leq \mathcal{E}_{2^l}^{(k^\circ)}(f)_{p,\beta} \leq \left\|f - \psi^\circ_{2^l}\right\|_{p,\beta}$$

$$\leq C\left\{(\textstyle\int_X\left|[f(x)-\psi_{2^l}(x)]\beta(x)\right|^p dx)^{\frac{1}{p}} + (\int_X\left|[\psi_{2^l}(x)-\psi^\circ_{2^l}(x)]\beta(x)\right|^p dx)^{\frac{1}{p}}\right\}$$

$$\leq C\left\{(\textstyle\int_X\left|[f(x)-\psi_{2^l}(x)]\beta(x)\right|^p dx)^{\frac{1}{p}} + \omega_{k^\circ}(f,\frac{1}{2^l})_{p,\beta}\right\}$$

$$\leq C(p)\omega_{k^\circ}(f,\frac{1}{2^l})_{p,\beta} \leq C(p)\omega_{k^\circ}(f,\frac{1}{n})_{p,\beta}$$

So by eq. **(34)** with $\boldsymbol{k}^\circ$ instead of $\boldsymbol{k}$ is followed. ■

The converse results of Bernstein type are including:

Theorem 2

If $f \in L_{p,\beta}(X)$, $0 < p \leq 1$ and $k = 0,1,2, \ldots$, then the following inequalities holds

$$\omega_k(f,\tfrac{1}{n})_{p,\beta} \leq C(p)\tfrac{1}{n^k}\left\{\textstyle\sum_{j=1}^{n} l^{k(p-1)}\mathcal{E}_j^{(k)}(f)_{p,\beta}^p\right\}^{\frac{1}{p}} \quad \ldots (37)$$

For $n = 1,2, \ldots$ with $C > 0$ depending on p.

Proof:

Let $\psi_j(x)$ be the best spline approximation to $\in L_{p,\beta}(X)$, $0 < p \leq 1$ (i.e.)

$$\mathcal{E}_{2^j}^{(k)}(f)_{p,\beta} = inf\left\{\left\|f-\psi_j\right\|_{p,\beta}, \psi_j \in \ \mathcal{S}_n^{(k)}(X)\right\}.$$

From eq. **(4), (16)** and eq. **(28),** we have

$$\omega_k(f,\tfrac{1}{n})_{p,\beta} \leq C(p)\tfrac{1}{n^k}.2^{kj}\left\|\psi_j - \psi_{j-1}\right\|_{p,\beta}$$

$$\leq C(p)\frac{1}{n^k}.2^{kj}\{(\int_X|(\psi_j(x)-f(x))\beta(x)|^p dx)^{\frac{1}{p}}$$

$$+(\int_X|(f(x)-(\psi_{j-1}(x))\beta(x)|^p dx)^{\frac{1}{p}}\}$$

$$\leq C(p)\frac{1}{n}.2^j\left\{\mathcal{E}^{(k)}_{2^j}(f)_{p,\beta}+\mathcal{E}^{(k)}_{2^{j-1}}(f)_{p,\beta}\right\}$$

$$\leq C(p)\frac{1}{n}.2^j\left\{\mathcal{E}^{(k)}_{2^{j-1}}(f)_{p,\beta}\right\}, \text{ for } n>2^j \qquad \dots (38)$$

So, from eq. **(4),** eq. **(6)** and eq. **(38),** we obtain:

$$\omega_k(f,\tfrac{1}{n})^p_{p,\beta}\leq\omega_k(f-\psi_j,\tfrac{1}{n})^p_{p,\beta}+\sum_{j=1}^{n}\omega_k(\psi_j-\psi_{j-1},\tfrac{1}{n})^p_{p,\beta}$$

$$\leq C(p)\left\{\|f-\psi_j\|^p_{p,\beta}+\sum_{j=1}^{n}\frac{1}{n}.2^j\,\mathcal{E}^{(k)}_{2^{j-1}}(f)^p_{p,\beta}\right\}$$

$$\leq C(p)\frac{1}{n^{pk}}\left\{\sum_{j=0}^{n}.2^{kj}\,\mathcal{E}^{(k)}_{2^j}(f)^p_{p,\beta}\right\}$$

The inverse theorem follows.■

Conclusion

We mentioned obviously that the Direct and inverse theorems which state the relationship between the degrees of best approximation of unbounded functions in weighted space with respect to spline polynomials and modulus of continuity of order $\boldsymbol{k}$, in our paper developed the Direct and inverse theorems of unbounded functions in weighted spaces $(\boldsymbol{L_{p,\beta}(X)})$; $0<p\leq 1$ by spline approximation in terms the $\boldsymbol{kth}$ modulus of continuity.

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