Solving Fractional Initial Value Problems by Using Hybrid Series

Saad Naji Al-Azzawi [1], Ronak Bagelany[2] and Ahmed Murshed [3]

[1] Department of Mathematics - College of Science for Women - Baghdad University
[2] Presidency of Kirkuk University – Iraq
[3] Department of Mathematics - College of Science - Diyala University

[2]saeed.runak@yahoo.com

Received: 7 June 2017 Accepted: 10 October 2017

Abstract

In this paper, we suggest a new method for solving fractional initial value problems of different fractional orders. We call it hybrid Series Solution (Fractional with power, Fractional with fractional, Fractional with power and fractional). The main difference between Caputo and Riemann – Liouville formulas for the fractional derivatives as mentioned. The paper focuses on finding the exact solution of Bagley-Torvik equation and other nonhomogeneous fractional differential equations, illustrated by some Theorems and examples.

Key words: Hybrid Series, Bagley-Torvik equation, Fractional Calculus, Caputo and Riemann - Liouville derivatives.

حل مسائل القيم الابتدائية الكسرية باستخدام متسلسلات هجينة

سعد ناجي العزاوي[1]، روناك محمد سعيد باجلاني[2] و احمد مرشد كريم[3]

[1] قسم الرياضيات - كلية العلوم للبنات - جامعة بغداد
[2] رئاسة جامعة كركوك
[3] قسم الرياضيات - كلية العلوم - جامعة ديالى

الخلاصة

في هذا البحث تم اقتراح طريقة جديدة لحل مسائل القيم الابتدائية الكسرية ذات الرتب المختلفة. سميت هذه الطريقة بطريقة المتسلسلات الهجينة للحل (كسرية مع قوى، كسرية مع كسرية ، كسرية مع قوى مع كسرية). وكذلك قدمنا الاختلاف الرئيسي بين خصائص المشتقات الكسرية حسب تعريف ريمان- لوفيل وحسب تعريف كابوتو. نركز في هذا البحث على ايجاد الحل التام لمعادلة Bagley-Torvik وغيرها من المعادلات التفاضلية الكسرية غير المتجانسة وهذا يتضح من بعض النظريات والامثلة.

الكلمات المفتاحية : المتسلسلات الهجينة، معادلة Bagley-Torvik، التفاضل والتكامل الكسري، مشتقات Caputo وRiemann-Liouville .

Introduction

The fractional calculus deals with integrals and derivatives of real or even complex order [1]. The history of fractional calculus started at the same time when classical calculus was established. It was first mention in Leibniz's letter to l' Hospital in 1695, where the idea of semi derivative was suggested [2, 3]. During time, fractional calculus built on formal foundations by many famous mathematicians e.g. Liouville, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel etc. The fractional calculus finds an application in different fields of science, including theory of fractional, engineering, physics, numerical analysis, biology and economics [4]. Bagley-Torvik equation, which is ordinary fractional differential equation, firstly appeared in Bagley and Torvik seminal work. They proposed to model viscoelastic behavior of geological strata, metals and glasses by using fractional calculus and they have proved that this approach is effective in describing structures containing elastic and viscoelastic components, so, it plays important role in engineering and applied science [5,6] . In particular, the equation with derivative of 1/2 order or 3/2 order can describe the motion of real physical systems or the motion of a rigid plate immersed in a Newtonian fluid and a gas in a fluid respectively [7, 8]. There are several works to solve Bagley-Torvik equation, such as, numerical procedures for a reformulated Bagley-Torvik equation as a system of fractional differential equation of order ½ and a numerical way for solving this equation, a generalization of Taylor's and Bessel's collocation method. The aim of this work is to find the exact solution of Bagley-Torvik equation and other nonhomogeneous fractional differential equations by using hybrid series. The paper is organized as follows: In section 1 we introduced some necessary definitions and mathematical preliminaries of fractional calculus, section 2 is devoted to present some theorems and lemmas related to the fractional power series [9 - 15].

In section 3 our new method to solve nonhomogeneous fractional differential equation is used. In section 4 the Bagley-Torvik Equation is presented and also applied our new method to extract the exact solution by using three special cases of Bagley-Torvik Equation [16, 17]. Finally, a conclusion is given in section 5.

1. Basic definitions

Through this section we explain some mathematical definitions of the fractional calculus which are used in our work.

Definition 1.1: The Gamma function, denoted by $\Gamma(z)$,is a generalization of the factorial function n! and defined as.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad Re\, z > 0. \tag{1}$$

Below we show some basic properties of Γ function, namely:

$\Gamma(1) = \Gamma(2) = 1,$

$\Gamma(z+1) = z\Gamma(z)$

$\Gamma(z) = \frac{\Gamma(z+1)}{z},$ for negative value of z .

$\Gamma(n) = (n-1)!\ , \qquad n \in N_0,$

$\Gamma(n+1) = n!\,, \qquad n \in N_0.$

Whereas N_0 is the set of the non-negative integers. From the above we can get:

a) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

b) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,.\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

c) $\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{\frac{-3}{2}} = \frac{\Gamma\left(\frac{-1}{2}\right)}{\frac{-3}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-3}{2}\cdot\frac{-1}{2}} = \frac{4}{3}\sqrt{\pi}$

Definition 1.2: Suppose that $\sigma > \mathrm{o}, \mathrm{t} > \mathrm{a}, \sigma, \mathrm{a}, \mathrm{t} \in \mathrm{R}$ .Then we have

$$D^{\sigma} f(t) = \begin{cases} \frac{1}{\Gamma(n-\sigma)} \frac{d^n}{d\,t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\sigma+1-n}}\, d\tau, & n-1 < \sigma < n \in N, \\ \frac{d^n}{d\,t^n} f(t), & \sigma = n \in N. \end{cases} \tag{2}$$

This definition is called the Riemann-Liouville fractional derivative of order σ.

Definition 1.3: Suppose that $\sigma > 0,\ t > a,\ \sigma, c, t \in R$. The fractional Caputo operator has the form:

$$D_*^{\sigma} f(t) = \begin{cases} \frac{1}{\Gamma(n-\sigma)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\sigma+1-n}}\, d\tau, n-1 < \sigma < n \in N, \\ \frac{d^n}{d\,t^n} f(t), & \sigma = n \in N. \end{cases} \tag{3}$$

Definition 1.4: The Caputo fractional derivative of the power function is denoted by

$$D_*^{\sigma} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\sigma+1)} t^{\beta-\sigma}, & n-1 < \sigma < n,\ \beta > n-1, \beta \in R, \\ 0, & n-1 < \sigma < n,\ \beta \le n-1, \beta \in \mathbb{N} \end{cases} \tag{4}$$

Remark: The main difference between Caputo and Riemann – Liouville formulas for the fractional derivatives are:

a) Caputo fractional derivative of a constant equals zero while (Riemann – Liouville) fractional derivative of a constant does not equal zero.

b) The non-commutation, in Caputo fractional derivative we have:

$$D_*^{\alpha} D^m f(t) = D_*^{\alpha+m} f(t) \neq D^m D_*^{\alpha} f(t) \quad (5)$$

Where $\alpha \in (n-1, n), n \in N, m = 1,2,3, \ldots$

In general, the Riemann - Liouville derivative is also non-commutation as:

$$D^m D^{\alpha} f(t) = D^{\alpha+m} f(t) \neq D^{\alpha} D^m f(t) \quad (6)$$

Whereas $\alpha \in (n-1, n), n \in N, m = 1,2,3, \ldots$

Note that the formulas in (5) and (6) become equalities under the following additional conditions:

$f^{(s)}(0) = 0, \qquad s = n, n+1, \ldots, m.$ for D_*^{α} and

$f^{(s)}(0) = 0, \qquad s = 0,1,2, \ldots, m,$ for D^{α}

Definition 1.5: The power series is denoted by

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\sigma} = c_0 + c_1 (t-t_0)^{\sigma} + c_2 (t-t_0)^{2\sigma} + \cdots, \quad (7)$$

Where $0 \leq n-1 < \sigma \leq n$, $t \geq t_0$ is called a fractional power series about t_0 , where t is a variable and c_n's are constants called the coefficients of the series, particularly, if $t_0 = 0$, the expansion $\sum_{n=o}^{\infty} c_n t^{n\sigma}$ is called a Fractional Maclaurin Series.

2. Fractional Power Series and Analytical Manipulations

Through this section, we review some theorems and lemmas related to our work.

Theorem 2.1[8]: Consider the fractional power series $\sum_{n=o}^{\infty} c_n t^{n\sigma}$, $t \geq 0$, there are two possible cases:

1- If the FPS $\sum_{n=o}^{\infty} c_n t^{n\sigma}$ converges when $t = d > 0$, then it converges whenever $0 \leq t \leq d$.

2- If the FPS $\sum_{n=o}^{\infty} c_n t^{n\sigma}$ diverges when $t = g > 0$, then it diverges whenever $t > g$.

Theorem 2.2[8]: Let the FPS $\sum_{n=0}^{\infty} c_n t^{n\sigma}$, and $t \geq 0$, there are only three possibilities.

1- The series converges only when t= 0,

2 -The series converges for each t ≥ 0,

3- There is a positive real number R such that the series converges whenever $0 \leq t < R$ and diverges whenever $t > R$

Theorem 2.3[8]: Suppose that the FPS $\sum_{n=0}^{\infty} c_n\, t^{n\sigma}$ has radius of convergence $R > 0$. If $f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n\, t^{n\sigma}$ on $0 \le t < R$, then for $0 \le n-1 < \sigma \le n$ and $0 \le t < R$,we have

$$D_0^{\sigma} f(t) = \sum_{n=1}^{\infty} c_n \frac{\Gamma(n\sigma+1)}{\Gamma((n-1)\sigma+1)} t^{(n-1)\sigma} \qquad (8)$$

Lemma 2.1 [16]: Suppose $y \in C^s[0, a]$whereas,$a > 0$ and $s \in \mathbb{N}$, let $\alpha \notin \mathbb{N}$ such that 0< $\alpha < s$. Then

$D_*^{\alpha} y(0) = 0$.

Lemma 2.2[16]: Suppose $y \in C^2[0, a]$ where $a > 0$, then:

1- $D_*^{1/2} D_*^{1/2} y = y'$, (9)

2- $D_*^{1/2} y' = D_*^{3/2} y$, (10)

3-$D_*^{1/2} D_*^{3/2}\, y = y''$. (11)

3. Double Fractional Series Solution

It known that one of the general methods of solution of differential equations is the series solution:

$$y(t) = \sum_{i=0}^{\infty} c_i t^i, \qquad (12)$$

While in a neighborhood of a regular singular point x_0 , the series solution has the form:

$$y(t) = \sum_{i=0}^{\infty} c_i t^{i+\sigma} , \qquad (13)$$

as mentioned in frobenius method, clearly the sum of convergent series is convergent. for homogeneous fractional differential equation of order σ , the fractional series solution has the form:

$$y(t) = \sum_{i=0}^{\infty} c_i t^{i\sigma}, \qquad (14)$$

but this form may not applicable for nonhomogeneous equations. Our suggested method is to write the solution as a sum of power series and fractional series. The efficiency of this suggested method is clear through an illustrated example, this form is a generalization of Taylor series for the solution. In this section, we apply a double fractional series for two examples:

Example 3.1: Consider the equation

$$y^{\frac{5}{2}}(t) + y^{\frac{3}{2}}(t) + y(t) = \Gamma\left(\frac{7}{2}\right) + \Gamma\left(\frac{5}{2}\right) + \Gamma\left(\frac{7}{2}\right)t + t^{\frac{5}{2}} + t^{\frac{3}{2}} + t + 1, \qquad (15)$$

With the initial conditions, $y(0) = y'(0) = 1$.

Where the exact solution is $y(t) = t^{\frac{5}{2}} + t^{\frac{3}{2}} + t + 1$,

Since t is in the nonhomogeneous part, so, to solve (15) , suppose that ,

$$y(t) = \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} b_n t^{\frac{5}{2}n} + \sum_{n=1}^{\infty} c_n t^{\frac{3}{2}n} \qquad (16)$$

Getting the derivatives $y^{\frac{5}{2}}(t)$and $y^{\frac{3}{2}}(t)$, respectively, and putting the outputs in (15), (by neglecting all the equations when $t < 0$) : satisfying

$$\sum_{n=3}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma\left(\left(n-\frac{5}{2}\right)+1\right)} t^{n-\frac{5}{2}} + \sum_{n=1}^{\infty} b_n \frac{\Gamma(\frac{5n}{2}+1)}{\Gamma\left(\left(\frac{5n}{2}-\frac{5}{2}\right)+1\right)} t^{\frac{5n}{2}-\frac{5}{2}} + \sum_{n=2}^{\infty} c_n \frac{\Gamma(\frac{3n}{2}+1)}{\Gamma\left(\left(\frac{3n}{2}-\frac{5}{2}\right)+1\right)} t^{\frac{3n}{2}-\frac{5}{2}} +$$

$$\sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma\left(\left(n-\frac{3}{2}\right)+1\right)} t^{n-\frac{3}{2}} + \sum_{n=1}^{\infty} b_n \frac{\Gamma(\frac{5n}{2}+1)}{\Gamma\left(\left(\frac{5n}{2}-\frac{3}{2}\right)+1\right)} t^{\frac{5n}{2}-\frac{3}{2}} + \sum_{n=1}^{\infty} c_n \frac{\Gamma(\frac{3n}{2}+1)}{\Gamma\left(\left(\frac{3n}{2}-\frac{3}{2}\right)+1\right)} t^{\frac{3n}{2}-\frac{3}{2}} +$$

$$\sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} b_n t^{\frac{5}{2}n} + \sum_{n=1}^{\infty} c_n t^{\frac{3}{2}n} = \Gamma\left(\frac{7}{2}\right) + \Gamma\left(\frac{5}{2}\right) + \Gamma(\tfrac{7}{2})\, t + t^{\frac{5}{2}} + t^{\frac{3}{2}} + t + 1,$$

By equalizing the coefficients in equations (17-25) below:

$$\Gamma\left(\frac{7}{2}\right) b_1 + \Gamma\left(\frac{5}{2}\right) c_1 + a_0 = \Gamma\left(\frac{7}{2}\right) + \Gamma\left(\frac{5}{2}\right) + 1, \qquad (17)$$

$$a_3 \frac{6}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} + c_2 \frac{6}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} + a_2 \frac{2}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} = 0 , \qquad (18)$$

$$b_1 \Gamma(\tfrac{7}{2}) t + a_1 t = \Gamma(\tfrac{7}{2}) t + t, \qquad (19)$$

$$a_4 \frac{24}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + a_3 \frac{6}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_2 \frac{6}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_1 t^{\frac{3}{2}} = t^{\frac{3}{2}} \qquad (20)$$

$$c_3 \frac{\Gamma(\frac{11}{2})}{2} t^2 + a_2 t^2 = 0\,, \qquad (21)$$

$$a_5 \frac{120}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} + b_2 \frac{120}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} + a_4 \frac{24}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} + b_1 t^{\frac{5}{2}} = t^{\frac{5}{2}}, \qquad (22)$$

$$c_3 \frac{\Gamma(\frac{11}{2})}{6} t^3 + a_3 t^3 + c_2 t^3 = 0, \qquad (23)$$

$$a_6 \frac{6!}{\Gamma(\frac{9}{2})} t^{\frac{7}{2}} + c_4 \frac{6!}{\Gamma(\frac{9}{2})} t^{\frac{7}{2}} + a_5 \frac{5!}{\Gamma(\frac{9}{2})} t^{\frac{7}{2}} + b_2 \frac{5!}{\Gamma(\frac{9}{2})} t^{\frac{7}{2}} = 0\,, \qquad (24)$$

$$a_4 t^4 = 0, \qquad (25)$$

And so on , by solving this system when $y(0)$ =y'(0)=1, we get : $a_0 = a_1 = 1$,

From equation (19), we get $b_1 = 1$,

From equation (17), we get $c_1 = 1$,

From equation (25), we get $a_4 = 0$,

From equation (18) we get $a_2 = 0$,

From equation (21) we get $c_3 = 0$, and so on,

by putting these outputs in (16),we can obtain the exact solution :

$$y(t) = t^{\frac{5}{2}} + t^{\frac{3}{2}} + t + 1\,.$$

Example 3.2: Consider the equation

$$y^{\frac{3}{2}}(t) - \frac{3\sqrt{\pi}}{8} t^{\frac{1}{2}} y^{\frac{1}{2}}(t) + y = \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}} + 1, \qquad (26)$$

with the exact solution $y = t^2 + 1$, and our hypothesis is

$$y = \sum_{n=0}^{\infty} c_n t^{\frac{3n}{2}} + \sum_{n=0}^{\infty} b_n t^n \qquad (27)$$

To apply our method we must compute the derivatives $y^{\frac{3}{2}}$, $y^{\frac{1}{2}}$, so,

$$D_*^{\frac{3}{2}} y(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(\frac{3n}{2}+1)}{\Gamma(\frac{3n}{2}-\frac{3}{2}+1)} t^{\frac{3n}{2}-\frac{3}{2}} + \sum_{n=0}^{\infty} b_n \frac{\Gamma(n+1)}{\Gamma(n-\frac{3}{2}+1)} t^{n-\frac{3}{2}}, \qquad (28)$$

$$D_*^{\frac{1}{2}} y(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(\frac{3n}{2}+1)}{\Gamma(\frac{3n}{2}-\frac{1}{2}+1)} t^{\frac{3n}{2}-\frac{1}{2}} + \sum_{n=0}^{\infty} b_n \frac{\Gamma(n+1)}{\Gamma(n-\frac{1}{2}+1)} t^{n-\frac{1}{2}}, \qquad (29)$$

By substituting (27), (28) and (29) in (26) , satisfying

$$\sum_{n=0}^{\infty} c_n \frac{\Gamma(\frac{3n}{2}+1)}{\Gamma(\frac{3n}{2}-\frac{3}{2}+1)} t^{\frac{3n}{2}-\frac{3}{2}} + \sum_{n=0}^{\infty} b_n \frac{\Gamma(n+1)}{\Gamma(n-\frac{3}{2}+1)} t^{n-\frac{3}{2}} -$$

$$\frac{3\sqrt{\pi}}{8} t^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} c_n \frac{\Gamma(\frac{3n}{2}+1)}{\Gamma(\frac{3n}{2}-\frac{1}{2}+1)} t^{\frac{3n}{2}-\frac{1}{2}} + \sum_{n=0}^{\infty} b_n \frac{\Gamma(n+1)}{\Gamma(n-\frac{1}{2}+1)} t^{n-\frac{1}{2}} \right) +$$

$$\sum_{n=0}^{\infty} c_n t^{\frac{3n}{2}} + \sum_{n=0}^{\infty} b_n t^n = \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}} + 1 ,$$

For $n = 0,1,2,3, \ldots,$ with $t \geq 0$, we obtain :

$$c_1 \Gamma\left(\frac{5}{2}\right) + c_2 \frac{\Gamma(4)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_3 \frac{\Gamma(\frac{11}{2})}{\Gamma(4)} t^3 + 0 + b_2 \frac{2}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} + b_3 \frac{\Gamma(4)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} - \frac{3\sqrt{\pi}}{8} t^{\frac{1}{2}} \left[\left(c_1 \Gamma\left(\frac{5}{2}\right) t + \right.\right.$$

$$\left.\left. c_2 \frac{\Gamma(4)}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} + c_3 \frac{\Gamma(\frac{11}{2})}{\Gamma(5)} t^4 \right) + \left(b_1 \frac{1}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} + b_2 \frac{2}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + b_3 \frac{\Gamma(4)}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} \right) \right] +$$

$$c_0 + c_1 t^{\frac{3}{2}} + c_2 t^3 + c_3 t^{\frac{9}{2}} + b_0 + b_1 t + b_2 t^2 + \mathrm{b}_3 \mathrm{t}^3 = \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}} + 1 ,$$

So, we can get the Coefficients:

$b_2 \frac{2}{\frac{\sqrt{\pi}}{2}} = \frac{4}{\sqrt{\pi}}$, $b_2 = 1$, and

$c_1 \Gamma\left(\frac{7}{2}\right) + c_1 = 0$, $c_1 = 0$, led to $c_2 = c_3 = b_1 = b_3 = 0$

$c_1 \Gamma\left(\frac{5}{2}\right) + \mathrm{c}_0 + \mathrm{b}_0 = 1$, led to $c_1 = 0$, $c_0 = b_0 = \frac{1}{2}$,

Finally, putting these values into the equation (27), we reach to the exact solution: $y = t^2 + 1$.

4. Bagley-Torvik Equation

4.1. The Origin of Bagley-Torvik Equation

Bagley and Torvik (1984), found that the fractional calculus can be identified in the solution to a classic problem in the motion of viscous fluids, and they showed that the resulting shear stress at any point in the fluid can be expressed by fractional order time derivative of the fluid velocity profile. Thus, the fractional derivative is found to apply in the differential equation that describes the motion of some physical systems defined by localized motion in a viscous fluid, when they applied their work they arrived at the surprising result. Starting with the diffusion equation,

$$\alpha \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial z^2}$$

Where α is the fluid density, μ is the viscosity of time t, z is the distance from the (wetted plate), after that, they found that the differential equation to describe the displacement X on the plate is:

$m\ddot{X} = F_x = -GX - 2A\,\delta(t,0),$

(They considered a rigid plate of mass m immersed in a Newtonian fluid of infinite extent and connected by a massless spring of stiffness G to a fixed point)

Finally,

$m\frac{d^2x}{dt^2} + 2A\sqrt{\mu\alpha}D_t^{3/2}X + GX = 0,$

Where,

$D^{3/2}X = D^{1/2}\frac{dx}{dt} = \frac{d}{dt}D^{1/2}X$. (more details can be found in [10]).

4.2. Solution of Certain Forms of Bagley –Torvik Equation

In this section, we use our method to solve three special cases of Bagley – Torvik equation and extract the exact solution. The general form of nonhomogeneous Bagley-Torvik equation is [9]:

$AD_*^2\, y(t) + BD_*^{\frac{3}{2}}y(t) + Cy(t) = f(t)\,, \quad (t \geq 0)$ (30)

With the initial conditions $y(0) = 0$, $y'(0) = 1$, and $A = B = C = 1.$

Example 4.1:

Consider the nonhomogeneous Bagley-Torvik equation:

$D_*^2\, y(t) + D_*^{\frac{3}{2}}y(t) + y(t) = t^2 + 4\sqrt{\frac{t}{\pi}} + 2\,, \ (t \geq 0)$ (31)

With the initial conditions,$y(0) = 0$, $y(5)$ =25.

Where the exact solution is $y(t) = t^2$,

Since the series form $\sum_{n=0}^{\infty} c_n\, t^{\frac{3n}{2}}$, is not applicable because we cannot get a form $t^{\frac{1}{2}}$, so we suppose the series mixed form or in the form:

$y = \sum_{n=0}^{\infty} b_n t^n$, (32) In order to apply our method, we must compute the functions $y', y'', y^{\frac{3}{2}}$,to complete the fractional power series. So,

$$y' = \sum_{n=0}^{\infty} b_n n\, t^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} b_n n(n-1)\, t^{n-2} \quad (33)$$

$$y^{\frac{3}{2}} = \sum_{n=2}^{\infty} b_n \frac{\Gamma(n+1)}{\Gamma(n-\frac{3}{2}+1)} t^{n-\frac{3}{2}}. \quad (34)$$

To get the solution of equations (31), substitute the expansion formulas of equations (32),(33) and (34) into (31), getting :

$$\sum_{n=0}^{\infty} b_n n(n-1) t^{n-2} + \sum_{n=2}^{\infty} b_n \frac{\Gamma(n+1)}{\Gamma(n-\frac{3}{2}+1)} t^{n-\frac{3}{2}} + \sum_{n=0}^{\infty} b_n t^n = t^2 + 4\sqrt{\frac{t}{\pi}} + 2. \quad (35)$$

For $n = 0,1,2,3,\ldots$, getting the values of the constants:

$2b_2 + b_0 = 2$, (Scalar value)

$b_2 \frac{\Gamma(3)}{\Gamma(\frac{3}{2})} = \frac{4}{\sqrt{\pi}}$, (Coefficient of $t^{\frac{1}{2}}$)

$12b_4 + b_2 = 1$, (Coefficient of t^2)

So, $b_2 = 1$, and the remaining Coefficient are zeros , by substituting all the outputs in (32), getting:

$$y = \sum_{n=0}^{\infty} b_n t^n = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4, \ldots,$$

We can easily obtain the exact solution $y(t) = t^2$. If we see the same example in [2] and make a comparison between the results, we will obtain the same results.

Example 4.2:

Now if we solve the nonhomogeneous Bagley-Torvik equation:

$$D_*^2\, y(t) + D_*^{\frac{3}{2}} y(t) + y(t) = 7t + \frac{8}{\sqrt{\pi}} t^3 + 1, (t \geq 0) \quad (36)$$

With the initial conditions,$y(0) = 1$, $y'(0) = 1$.

Where the exact solution is $y(t) = t^3 + t + 1$,

If we use the same steps in example (5.1) to solve equation (36), and use the same hypothesis $y = \sum_{n=0}^{\infty} c_n t^{\frac{3n}{2}} + \sum_{n=0}^{\infty} b_n t^n$, we will get the following results :

$2c_2 t + 6b_3 t + b_1 t = 1$,

$c_2 \frac{6}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + b_3 \frac{6}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_1 t^{\frac{3}{2}} = \frac{8}{\sqrt{\pi}} t^{\frac{3}{2}}$,

$c_3 \frac{\Gamma(\frac{11}{6})}{6} t^3 + c_2 t^3 + b_3 t^3 = t^3$,

$2b_2 + c_1 \Gamma\left(\frac{5}{2}\right) + c_0 + b_0 = 1$,

$\frac{63}{4} c_3 t^{\frac{5}{2}} + b_4 \frac{24}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} = 0$,

$12\, b_4 t^2 + b_2 t^2 = 0$,

$b_4 t^4 = 0$,

$c_3 t^{\frac{9}{2}} = 0$.

By identifying the coefficients, we can obtain the following:

$c_1 = c_3 = b_2 = 0$ and $b_1 = 1$, whereas $c_0 = b_0 = c_2 = b_3 = \frac{1}{2}$.

Finally, we obtain the solution of y(t) .

$y(t) = c_0 + c_1 t^{\frac{3}{2}} + c_2 t^3 + b_0 + b_1 t + b_2 t^2 + b_3 t^3$,

$= \frac{1}{2} + 0 + \frac{1}{2} t^3 + \frac{1}{2} + t + 0 + \frac{1}{2} t^3$,

Consequently, $y(t) = 1 + t^3 + t$.

Return to reference [15], this equation has solved by fractional iteration method (VIM), for comparing the results in our example and the same example in [15]. We obtained the same results**.**

Example 4.3: Consider the nonhomogeneous Bagley-Torvik equation :

$$D_*^2\, y(t) + D_*^{\frac{3}{2}} y(t) + y(t) = t + 1 \ (t \geq 0), \qquad (37)$$

With the initial condition $y(0) = 1$, $y'(0)=1$,

Whereas the exact solution $y(t) = t + 1$.

To solve equation (37) by our method, we can use the hypotheses $y = \sum_{n=0}^{\infty} c_n\, t^{\frac{3n}{2}} + \sum_{n=0}^{\infty} b_n t^n$,

to get the same results. If we follow the same steps as the previous example, we can obtain the solution $y(t) = t + 1$.

Returning to example 2 in reference [9], we can see that they solved this example by (Applying generalized differential transform) and obtained the same results.

In general, we can prove that our method can applied on most of these kinds of equations, as in the applicable examples above.

Conclusion

A new method for solving fractional initial value problems of different fractional orders ($D^{\frac{3}{2}}, D^{\frac{5}{2}}$) has been applied, by using mixed power and fractional series solution. The usefulness of this method is to get the exact solution of Bagley-Torvik equation and other nonhomogeneous fractional differential equations. Consequently, we proved that the results we obtained were very accurate as the results in the references [2], [9]and [15], by applying our new method to the same examples.

References

1. Mohammadi F., Mohyud-DinS. T., (2016), "*A fractional-order Legendre collocation method for solving the Bagley-Torvik equations*", *Advances in Difference Equations* , Springer, ,pp. 3-10.
2. Mohammadi F., (2014), "*Numerical solution of Bagley-Torvik equation using Chebyshev wavelet operational matrix of fractional derivative*",Int. J. Adv. Appl. Math. ,vol. 2, no. 1,pp. 83-87.
3. Ishteva M. K., M.Sc. Thesis ,(2015), "Properties and Applications of the Caputo Fractional Operator", University of Karlsruhe (TH), (Bulgaria), pp. 9-20.
4. Gorenflo R., Mainardi F.,(1997), " *FRACTIONAL CALCULUS : Integral and differential Equations of Fractional Order*", International Centre for Mechanical Sciences, Springer Vienna,vol.378, pp. 230-240.
5. Mladenov V., Mastorakis N., (2014)," *Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling*", World Scientific and Engineering Academy and Society, p. 4-15.
6. Wei H. M., Zhong X. C., Huang Q. A., (2016), "*Uniqueness and approximation of solution for fractional Bagley–Torvik equations with variable coefficients*." International Journal of Computer Mathematics ,pp. 1-6.
7. Xu, F., Gao, Y., Yang, X., Zhang, H., (2016)," *Construction of fractional power series solutions to fractional Boussinesq equations using residual power series method*", *Mathematical Problems in Engineering*, Hindawi Publishing Corporation, *pp.*3-6.
8. El-Ajou, A., Arqub, O. A., Zhour, Z. A., Momani, S. , (2013)," *New results on fractional power series: theories and applications*". *Entropy*, vol. 15 no. 12, pp. 5305-5323.
9. Bansal M. K., Jain R.,(2016) "*Analytical solution of Bagley Torvik Equation by Generalize Differential Transform*", International Journal of Pure and Applied Mathematics ,vol.110, no.2,pp. 266-272.
10. Bagley R. L., Torvik P. J., (1984), "On the appearance of the fractional derivative in the behavior of real materials" , Journal of Applied Mechanics, vol.51, no.2, pp. 295-297.

11. Öğrekçi S.,(2015), "*Generalized Taylor series method for solving nonlinear fractional differential equations with modified Riemann-Liouville derivative*" *Advances in Mathematical Physics*, Hindawi Publishing Corporation, vol. 2015,pp.1-4 .
12. Labecca, W., Osvaldo G., José R., Piqueira C.,(2015), "*Analytical solution of general Bagley-Torvik equation*" *Mathematical Problems in Engineering*, Hindawi Publishing Corporation, vol. 2015, pp. 1-3.
13. Raja, M. A. Z., Junaid A. K., Ijaz M. Q.,(2011), "*Solution of fractional order system of Bagley-Torvik equation using evolutionary computational intelligence*" *Mathematical Problems in Engineering* , Hindawi Publishing Corporation,vol.2011 ,pp. 1-6.
14. Yan, T., Shuanghua L., (2016), "*Local polynomial smoother for solving Bagley-Torvik fractional differential equations*", School of Mathematics and Statistics, China, pp. 1-2.
15. Mekkaoui T., Hammouch Z., (2012),"*Approximate analytical solutions to the Bagley-Torvik equation by the fractional iteration method*" Annals of the University of Craiova-Mathematics and Computer Science Series ,vol.39,no.2, pp. 251-254.
16. Diethelm K., Ford J.,(2002), "*Numerical solution of the Bagley-Torvik equation*" *BIT Numerical Mathematics* ,vol.42,no.3,pp. 490-507.
17. Wei H. M., Zhong X. C., Huang Q. A.,(2016), "*Uniqueness and approximation of solution for fractional Bagley–Torvik equations with variable coefficients*", International Journal of Computer Mathematics ,pp. 1-15.