On Semi - Pre irresolute Topological Vector Space

Radhi I. M. Ali[1], Jalal H. Hussein[2] and Suhad K. Hameed[3]

Department of Mathematics - College of Science for Woman - Baghdad University

[3]a197915197915@gmail.com

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Abstract

In this work some properties of semi-pre irresolute topological vector space was introduced , also several characterizations of semi-pre Hausdorff are given. Moreover, we show that the extreme point of convex subset of semi-pre irresolute topological space X lies on the boundary.

Keywords: Topological Vector space, Semi-preirresolute Topological Vector Space , Semi-preHausdorff Space , and Extremely points .

ألفضاءات التبولوجيه المتجهه الشبه مترددة

راضي ابراهيم محمد علي، جلال حاتم حسين البياتي و سهاد كريم حميد

قسم الرياضيات - كليه العلوم للبنات - جامعه بغداد

ألخلاصة

في هذا البحث تم تعريف الفضاءات التبولوجيه المتجهه الشبه مترددة و دراسة خصائصها , وأعطاء تمثيل للفضاءات الشبه هاوزدورف (semi-pre Hausdorff) . أضافة الى ذلك تم اثبات ان القيم القصوى للمجموعات الجزئيه المحدبه تكون حدوديه بالنسبه للفضاءات التبولوجيه المتجهه الشبه مترددة .

الكلمات المفتاحيه: الفضاءات التبولوجيه المتجهه، الفضاءات التبولوجيه المتجهه الشبه مترددة، الفضاءات شبه هاوزدورف، القيم القصوى .

Introduction and Preliminaries

This study discussed the notion of semi-pre irresolute topological of vector space that is one of generalization of a topological of vector space. Many researches have been done in this field see [1, 2, 3]. We considered it in terms of semi-preopen set, on the sense of Andrijevi´c [4]. A subset A of a topological space X is called semi-preopen (sp - open) if $A \subset Cl\ (Int\ (Cl\ (A)))$ [4], while a complement for sp-open set is named by semi-preclosed (sp - closed). The semi-preclosure of A that subset of X is represented by $Cl_{sp}(A)$ which is intersection for all sp-closed subsets on X containing A [5]. Recall that the subset $U \subset X$ is called semi-preopen neighborhood for x if there is sp-open set A with *x belong to the set* $A \subset U$ [5]. The point x for a subset A is said to be semi-preinterior point on A which denoted by $Int_{sp}(A)$ [5] if there is sp-open subset U, $x \in U$, and $U \subseteq A$, and we used sp-irresolute continuous mapping. A mapping $f{:}X \to Y$ is named semi-pre irresolute continuous if an inverse image for each sp-open subset in Y is sp-open on X where X and Y is topological spaces [6]. Also, a mapping $f{:}X \to Y$ is called sp-irresolute continuous at point x on X if for all sp-open subset V on Y contained $f(x)$, there is sp-open subset U with $x \in U$ satisfy $f(U)$ is subset of V [6]. A function f from a topological space X to Y is named pre sp-open function if the image for each sp-open subset of X is sp-open set in Y. Moreover the notion of sp- homeomorphism [6] that is a mapping f from X to Y is called sp-homeomorphism if it's bijective, both f and f^{-1} are sp-irresolute.

1. Semi-pre irresolute Topological Vector Space (SPITVS).

Definition 1.1: A topology τ with a vector space X over a field F is called SPITVS whenever these conditions are satisfied :

(a) The vector addition map $S{:}X \times X \to X$

(b) The multiplication by scalar map $M{:}F \times X \to X$

are both sp-irresolute . Observe that for all x belong to X , the translation mapping $\boldsymbol{T}_x :X \to X$ defined by $\boldsymbol{T}_x(y) = y + x$ also the multiplication mapping $M_\lambda : X \to X$ defined by $M_\lambda(x) = \lambda.x$.

Remark: We denoted to the collection of sp-neighborhoods for $x \in X$, by N_x . As well as the set of all sp-neighborhoods of zero vector space 0 of X denoted by N_0 .

Lemma 1.2[6]: Let A is a sp-open subset of a topological space X , B is any open subset of X , then the set $A \cap B$ is sp-open set .

Lemma 1.3[6]: Let $f:X \to Y$ (X and Y are topological spaces) is sp-irresolute mapping , so for sp-neighborhood V for f(x), there sp-neighborhood U of x satisfied $f(U) \subseteq V$.

Theorem 1.4 : Let X is a SPITVS , then the following hold.

a. If $U \in N_x$, and V is a neighborhood for x in X , then $U \cap V \in N_x$.
b. If $U \in N_0$, then $\lambda U \in N_0$ for a non-zero element $\lambda \in R$.
c. If $U \in N_0$, then $x+ U \in N_x$

Proof: a) If U is sp-neighborhood of x ,and V is a neighborhood for x, then there is a sp-open subset A and an open set B with $x \in A \subset U$ and x belong to $B \subset V$. Then x *belong to* $A \cap B \subseteq U \cap V$ and by Lemma 1.2, $A \cap B$ is sp-open. So $A \cap B$ is a sp-neighborhood for x. To prove (b) and(c) suppose U is sp-neighborhood for *zero* since $S:(x, y)$ is sp-irresolute , we can define the map T_x: $X \to X$ by $T_x(y)= y+ x$. Therefore $T_x(y)=S_x(x, y)$, $T_x(y)$ is sp-irresolute also $T^{-1}_x (y)= S_x(x, -y)$ is also sp-irresolute(since the addition is sp-irresolute) ,therefore T_x is sp-homeomorphism ,by lemma 1.3 for a sp-neighborhood U for *zero*, there sp-neighborhood $U+ x$ *for a point* x.

Definition 1.5[7]: The subset A on the vector spaces X named balanced if $\lambda A \subseteq A$ for $|\lambda| \leq 1$and absorbing if every x belong to X , there is $\varepsilon > 0$ such that $\lambda x \in A$ for $|\lambda| \leq \varepsilon$. It is named absolutely convex if the subset both balanced and convex.

Theorem 1.6 : Let X be an SPITVS ,then

(a) Every sp-neighborhood U of 0 is absorbing.

(b) For sp-neighborhood U for *zero* there balanced $V \in N_0$ satisfy that $V \subseteq U$.

Proof: (a) Suppose U sp-neighborhood for *zero* , therefore there is sp-open subset $U_1 \in N_0(X)$ with $U_1 \subseteq U$, we have the scalar multiplication map M_λ is sp-irresolute , so there exist sp-neighborhood of 0 V_1 , V_2 satisfying $M_\lambda (V_1 \times V_2) \subseteq U_1$ .The set V_1 contains an open interval of the form (-ε, ε) , therefore $tx \in U_1$ for all $t \in (-\varepsilon, \varepsilon)$ and for all $x \in V_2$. That implies U_1 is absorbing.

(b). Since the multiplication map $M_\lambda : R \times X \to X$ is sp-irresolute therefore for every sp-neighborhood U for zero in X , there is sp-neighborhood for 0 with $M_\lambda (V) \subseteq U$, so there exists ε> 0 with $V= V_1 \times V_2$, $(-\varepsilon,\varepsilon) \subseteq V_1$, V_1 is a sp-neighborhood for 0 on R , V_2 is sp-neighborhood for 0 in X . Define $W= \bigcup_{|t|<\varepsilon} t\, V_2$ and tV_2 is sp-neighborhood for 0 from theorem 1.4 , for $t \neq 0$ and $tV_2 \subseteq U$ for $t < \varepsilon$. Now we have to show that W is balanced. If $r < 1$, then $rW= \bigcup_{|t|<\varepsilon} (rt) V_2$ and $|r\,t| < \varepsilon$ $|r| < \varepsilon$, it follows that $rW = \bigcup_{|s|<\varepsilon} sV_2 \subseteq W$, where $s= r\,t$, hence W is balanced.

Theorem 1.7 : Let X be an SPITVS. If $A \subseteq X$, so $Cl_{sp}(A)= \cap(A+U)$. In particular $Cl_{sp}(A) \subseteq A+U$ *for all* U belongs to N_0.

Proof: Assume $x \in Cl_{sp}(A)$, and let U is a sp-neighborhood of 0 then by theorem 1.6(b) there balanced neighborhood V for zero with $V \subseteq U$, so $x+ V$ sp-neighborhood for x and $x \in Cl_{sp}(A)$, there $(x+ V) \cap A \neq \emptyset$, that implies $x \in A\text{-}V$. Since V is balanced, $A\text{-}V$ equal to $A+V$, then $x \in A+ V$ *subset of* $A+ U$, hence $Cl_{sp}(A) \subseteq \cap(A+U)$. Conversely if $x \notin Cl_{sp}(A)$,so there balanced neighborhood U for zero satisfy that $(x+U) \cap A= \emptyset$, so that $x \notin A\text{-}U = A+ U$.

Theorem1.8: Let X be an SPITVS. Then

(a) Every U belong to N_0 , there V *belong to* N_0 satisfy $V + V \subseteq U$.

(b) For every $U \in N_0$, there is a sp-closed balanced $V \in N_0$ satisfy that $V \subseteq U$.

Proof: (a) Assume $U \in N_0(X)$, since the addition map $S: X \times X \to X$ is sp-irresolute , then there are sp-neighborhood for 0 V_1 and V_2 satisfy $S(V_1,V_2) \subseteq U$, that mean $V_1+V_2 \subseteq U$, set $V= V_1 \cap V_2$ that show $V+V \subseteq V_1+V_2 \subseteq U$.

(b) Let U sp-neighborhood of zero on X , by part (a) there is sp-neighborhood V for zero with $V+V \subseteq U$. By Part (b) of Theorem 1.6, there neighborhood W for 0 which is balanced with

$W \subseteq V$ .By Theorem 1.7 $Cl_{sp}(W) \subseteq W+V$ and $Cl_{sp}(W) \subseteq W+V \subseteq V+V \subseteq U$. This shows that U contains the sp-closed neighborhood.

Definition 1.9: A topological space X called sp- Hausdorff , if each two distinct points x and y in X , there exist disjoint sp-open sets U,V such that $x \in U$, and $y \in V$.

Now we give some properties of sp-Hausdorff space.

Theorem 1.10 : Let X be an SPITVS. The statements are equivalently.

(a) X sp- Hausdorff.

(b) Let x belong to X therefore U_0 satisfy $x \notin U_0$.

Proof: (a) $\rightarrow$ (b) Assume x be a non-zero vector belong to X . Then there are disjoint sp-neighborhoods U for 0, and V for x , where U *belong to* N_0, V *belong to* N_x and $x \notin U$.

(b) $\rightarrow$(a) let x, y belong to X with x not equal to y. Therefore there exists a sp-neighborhood U_0 for 0 with x-$y \notin U_0$. By part (a) of Theorem 1.8, there exists sp-neighborhood W of 0 satisfy $W+W \subseteq .U_0$ In part (b) of Theorem 1.6 W is balanced, let $V_1 = x+ W$ and $V_2 = y +W$, therefore V_1 , V_2 is an sp-neighborhood of x, y respectively . To prove that $V_1 \cap V_2 = \emptyset$, let $s \in V_1 \cap V_2$ then $-(s -x) \in W$, since W is balanced and $s - y \in W$. Which implies that $x- y = (s - y)+(-(s - x)) \in W+W \subseteq U_0$ which is a contradiction. So we have $V_1 \cap V_2 = \phi$. Finally, the space X is sp- Hausdorff.

Corollary 1.11: Let X be an SPITVS. Then these statements are equivalently

(a) Let X sp- Hausdorff.

(b) $\cap\{U : U \in N_0\} = \{0\}$.

(c) $\cap\{U : U \in N_x\} = \{x\}$

I. Theorem 1.12**:A SITVS *X* is sp-Hausdorff if iff one-point subset is sp- closed in *X* .**

Proof: Let $x \in X$ and y belong to $X/\{x\}$, then x not equal to y that mean x-$y \neq 0$,therefore there an sp-neighborhood U for zero satisfy $y-x \notin U$. Then there is sp-closed , balanced W of 0 with $W \subseteq U$ by theorem 1.8,b . Which implies that $y-x \notin W$ then $y-x \in X-W$. Therefore y *belong to* $(X-W)+\{x\}$, also $(X$-$W)+\{x\}$ is sp-open , W is sp-closed, $(X-W)+\{x\}$ contained on $(X-\{x\})$. That shows $X/\{x\}$ is sp-open, thus $\{x\}$ is sp-closed. Converse , let x *belong to* X and assume the singleton $\{x\}$ is sp-closed. Therefore by Theorem 1.7 $\{x\}=Cl_{sp}\{x\}=\cap\{U+\{x\}$: U is sp-neighborhood for zero$\}$= $\{W$: W is sp-neighborhood of $x\}$ where $W = U+\{x\}$. Thus by Corollary 1.11, X is sp- Hausdorff.

Definition 1.13:A topological space X is called sp-compact if for any cover for X by sp-open subsets have a finites subcover.

Theorem 1.14 : Let C, K be subsets in a SPITVS X , *and* $C \cap K=\emptyset$, with C sp-closed, K sp-compact. Therefore there sp-neighborhood U for 0 satisfy $(K+U)\cap(C+U)=\phi$.

Proof: If $K=\phi$, then the proof is trivial . Otherwise, let $x \in K$, and $x=0$. Then $X-C$ is an sp-open set of 0 . Since the addition mapping is sp-irresolute and sp-continuous, and $0=0+0+0$, therefore there an sp- neighborhood U for zero satisfy $3U=U+U+U \subset X-C$. Define $\tilde{U}= U\cap(-U)$ which is sp-open , symmetric and $3\tilde{U}= \tilde{U}+ \tilde{U}+ \tilde{U} \subset X$-$C$. Which is implies that $\emptyset=\{x+x+x, x\in \tilde{U}\}\cap C=\{x+x, x \in \tilde{U}\}$ intersected$\{y$- x , $x\in \tilde{U}$, $y\in C\}$ and $\tilde{U}\cap\{C+ \tilde{U}\} \subset \{2x, x \in \tilde{U}\}\cap\{y$- x , $x\in \tilde{U}$, $y \in C\}$.This is for one point. Now since K is sp-compact, then by the above argument for every $x \in K$, we have a symmetric sp-neighborhood V_x such that $(x+2V_x)\cap(C+V_x)=\phi$. The sets $\{V_x : x \in K\}$ are a sp-open that covers K, and since K is sp-compact subset therefore for finitely number for points $x_i \in K, i=1,\dots,n,$ we have $K \subset \bigcup_{i=1}^{n}(x_i+V_{x_i})$. Define the sp-neighborhood V of 0 by $V=\cap_{i=1}^{n} Vx_i$ Therefore $(K+V)$ *intersected* $(C+V) \subset \cup_{i=1} n$ $(x_i+ Vx_i+V)\cap (C+V) \subset \cup_{i=1} n$ $(x_i+ 2\, Vx_i)\cap(C+ Vx_i) = \emptyset$. That is complete the proof .

Lemma 1.15: Let U be sp-open subset of a SPITVS X, and A any subset such that $U \cap A = \phi$, then $U \cap Cl_{sp}(A) = \emptyset$.

Proof: Let $x \in U \cap Cl_{sp}(A)$. Thus $x \in Cl_{sp}(A)$, U sp-neighborhood for x, since U is sp-open subset, then $X - U$ sp-closed subset contain A, and so $Cl_{sp}(A) \subseteq X\text{-}U$ and $x \notin Cl_{sp}(A)$ which implies a contradiction, therefore $U \cap Cl_{sp}(A) = \emptyset$.

Corollary 1.16: Let C, K be disjoint sets in a SPITVS X with C sp-closed, K sp-compact. Therefore there sp-neighborhood U for zero satisfy $Cl_{sp}(K+U) \cap (C+U) = \emptyset$.

Proof: In theorem 1.14 we have for any disjoint sp-closed C set and sp-compact set K, so there sp-neighborhood U for 0 satisfy $(K+U) \cap (C+U) = \emptyset$. The set $C+U = \{y+U : y \in C\}$ is an sp-open set, then by lemma 1.15 $Cl_{sp}(K+U) \cap (C+U) = \emptyset$.

Definition 1.17: Let X be a vector space with field F, an algebra dual for X is the collection of linear functional which define in X and represented by X^*.

Theorem 1.18: Let X be an SPITVS and $0 \neq f \in X^*$, then $f(E)$ is sp-open in F whenever E is sp-open in X.

Proof: Let E be non-empty subset of X, and $0 \neq x_0 \in X$ with $f(x_0) = 1$. then for any point $a \in E$, we have to prove that $f(a) \in Int_{sp}(f(E))$. E is sp-open neighborhood for a then by Theorem 1.4 we have $E - a$ is sp-neighborhood of 0. By Theorem 1.6 $E - a$ is absorbing, then there exists $\epsilon > 0$, such that $\lambda x_0 \in E - a$ for $\lambda \in R$ with $|\lambda| \leq \in$. Therefore for any $\beta \in R$ with $|\beta - f(a)| \leq \in$ we have $(\beta - f(a))x_0 \in E - a$, therefore $f((\beta - f(a)x_0) \in f(E-a) \Longrightarrow (\beta - f(a))\, f(x_0) \in f(E-a) \Longrightarrow (\beta - f(a))\,(1) \in f(E-a) = f(E) - f(a)$ which implies $\beta \in f(E)$, $f(a) \in [\beta - \in, \beta + \in]$, so that $f(a) \in Int(f(E)) \subseteq Int_{sp}(f(E))$, therefore $f(E) = Int_{sp}(f(E))$.

Lemma 1.19: [7] Let X be a vector space, $\emptyset \neq A \subseteq X$. For $x \in A$

The following statements are equivalents.

1) x is extremely point on A

2) if $a, b \in A$ such that $x = \frac{1}{2}(a+b)$, then x *equal to* a *equal to* b.

3) let $a,b \in A$, with $a \neq b$, let $\lambda \in (0, 1)$, $x = \lambda a + (1-\lambda)b$. Then we have either $\lambda = 0$, or $\lambda = 1$

An extremely point for subset V which is convex in a vector X is the point $x \in V$ which is not interior point on a segment $U \subset V$.

Theorem 1.20: Let X be SPITVS, $K \subset X$ be convex. Therefore $Int_{sp}(K) \cap (\delta K) = \emptyset$.

Proof: If $Int_{sp}(K) = \emptyset$, the proof is trivial. Suppose that the $Int_{sp}(K) \neq \emptyset$ and let $x \in Int_{sp}(K)$. Therefore there is sp-neighborhood U for 0 satisfy $x+U \subset K$. As a mapping $\varphi : \Re \to X$ where $\varphi(\mu) = \mu x$ continuous at $\mu = 1$, for this the sp-neighborhood $x+U$, there is an $s>0$ such that $\mu x \in x+U$ whenever $|\mu-1| \leq s$. In particular, we have $(1+s)x \in x+U \subset K$ and $(1-s)x \in x+U \subset K$. Now consider $x = \lambda(1+s)x + (1-\lambda)(1-s)x$ and take $\lambda = \frac{1}{2}$. So that $x = \frac{1}{2}(1+s)x + (1-\frac{1}{2})(1-s)x$, which implies the point x is not extremely on K.

Conclusion

Throughout our work we give some main results. Which are the semi-pre topological vector space satisfies that any neighbourhood for 0 containing a neighborhood for 0 which is balanced, semi-pre topological vector space satisfies the separation axiom (semi-pre hausdorff) and in a semi-pre topological vector space the semi-preinterior of a convex set doesn't intersect with the boundary of the set .

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