Diyala Journal for Pure Science



# Semi Compactness Space in Bornological Space

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**Received: 25 April 2019** **Accepted: 1 August 2019**

**DOI:** https://dx.doi.org/10.24237/djps.15.04.505A

## Abstract

The idea of bounded subset of a topological vector space was introduced by von Neumann. As a result of playing an important role in functional analysis that motivated the concept of more general and abstract classes of bounded sets, it is called bornology. That means, it is applied to solve the questions of boundedness for any space or set $X$ in general way not only by usual definition of bounded set. However, we take a collection β of subset of $X$, such that, satisfying three conditions, $\beta$ cover X, and $\beta$ stable under hereditary, i.e., if $B \in \beta, A \subseteq B,$ then $A \in \beta$ also finite union. Basically, a bornological space is a type of spaces which possesses the minimum amount of the structure needed to address the question of boundedness of sets and functions. In addition, since bornology has shown to be a very useful tool in various aspects of functional analysis, it has been considered by several researchers in different areas. In this paper, we consider the concept of semi-compactness in bornological space and study some of its properties.

**Keywords**: Bornological set, Semi bounded set, Unbounded set, Unbounded linear map.

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**شبه الفضاء المرصوص ضمن الفضاء البرنولوجي**

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**الخلاصة**

أن فكرة المجوعة الجزئية المقيدة من الفضاء المتجه التبولوجي قد تم تقديمها من قبل فون نيومن قد سميت برنولوجي ونتيجة لأهمية الدور التي تلعبه في التحليل الدالي الذي حفز مفهوم المجموعات المقيده بشكل أكثر عمومية من الفئات المجردة. هذا يعني أنه يمكن تطبيقها لحل مسألة الحدودية لأي فضاء أو مجموعة $X$ بشكل عام ليس فقط بالتعريف الاعتيادي للمجموعة المقيدة. على أي حال نحن نأخذ مجموعة β هي عائله من المجاميع الجزئية من X, بحيث تحقق ثلاث شروط وهي: β غطاء$X$, β مستقرة وراثيا وكذلك عملية الإتحاد المنتهية متحققة. بصورة أساسية الفضاء البرنولوجي هو الفضاء الذي يمتلك الحد الأدنى للهيكلية اللازمة لمعالجة مسالة محدودية المجموعات والدوال. بالإضافة إلى ذلك، بما أنه قد ظهرت أهمية البرنولوجي على أنه أداة مفيدة في مختلف جوانب التحليل الدالي, لذلك تم أخذه بنظر الاعتبار من قبل العديد من الباحثين في مختلف المجالات. في هذه الدراسة، نحن ندرس مفهوم أشباه الفضاء المرصوص في الفضاء البرنولوجي، كذلك سندرس بعض خواصه.

**الكلمات المفتاحية:** الفضاء البرنو لجي، أشباه المجاميع المقيدة، دوال غير مقيدة، تطبيق خطي غير مقيد.

## Introduction

The bornological construction is studied for the first time by Hogbe-Nlend [1] in 1977. A bornology β on a set $X$ is defined as a family of subsets of $X$, such that, β is covering $X$, stable under finite unions and hereditary under inclusion, i.e., if $B \in \beta, A \subseteq B,$ then $A \in \beta$. The elements of β are called bounded set see [1]. Abdul Husein (2002) studied bornological properties of the space of entire function, and Anwar. N. Imran (2005) studied bornological construction of the space of entire functions and also included a new bornological construction. see [2 and 3].

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In (2009) the concept of bornology on a semi-bounded set, semi-bounded linear map, semi convergence in Bornological space was introduced by [4]. In (2011) Anwar. N. Imran [5] introduced the concept of semi-unbounded linear map in bornological space and product space. Semi bounded sets were first introduced and investigated in [4]. Since this time semi bounded sets have been used to investigate many new bornological proprieties. S-converge, S-bounded linear map, semi-unbounded map in bornological space and product space where defined in before, through this paper we assume $(X;\ \beta)$ to be bornological space, semi bounded set (writer S-bounded). The aim of this paper is to introduce the concept of semi compactness space in bornological space and study some of their basic properties.

## Preliminaries

In this section we recall several basic notions from the theory of bornological space. We state the definitions and the properties of semi-bounded set in the bornological space.

Consider $B$ is a bounded subset of a bornological set $X$, and let the union of the set of all upper and lower bounds of $B$ and $B$ is denoted by $\bar{B}$. That is,

$\bar{B}$ =$\{all\ upper\ and\ lower\ bounds\ of\ B\} \cup\ B.$

**Example 1 [4]** For $X\ =\ \mathbb{R}$

**i)** $B = \cup\,\{connected\ subsets\ of\ \mathbb{R}\} \Longrightarrow \bar{B}\ = \mathbb{R}.$

**ii)** $B = \cup\,\{disconnected\ subsets\ of\ \mathbb{R}\} \Longrightarrow \bar{B}\ \neq\ \mathbb{R}$, such that, if $B\ =\ (a;\ b) \cup (c;\ d)$ and $b\ <\ c$, then

$$\bar{B}= (-\infty, b)\ \cup\ (c,+\infty).$$

Now we will introduce below semi bounded set in bornological space.

**Definition 1 [4]** Let $(X;\ \beta)$ be a bornological set, then $A \subseteq X$ is called a semi-bounded (written as s-bounded) if $\exists\ B \subseteq X$, such that, $B$ is bounded and $B\ \subseteq\ A\ \subseteq\ \bar{B}$.

**Example 2 [6]**

Let X=$R^2$ be the Euclidean plan with bornology define on $R^2$ which it is called usual bornology. Let $B = \{(x, y): 0 \leq x \leq a, 0 \leq y \leq b\}$ be bounded set belong to the usual bornology $\beta$ on $R^2$. Then $\bar{B} = B^U \cup B^L \cup B,$ such that:

All upper bounds of $B, B^U = \{(x, y): x > a \text{ or } x = a \text{ and } y > b\}$.

All lower bounds of $B, B^L = \{(x, y): x < 0 \text{ or } x = 0 \text{ and } y < 0\}$.

Then, the s-bounded set A is either $B$ or $\bar{B}$ , or any subset of $\bar{B}$ which contains B.

The following practical facts will be used in the next section.

**Remark 1 [4]**

i) Every bounded set is a semi-bounded; however, the opposite is not true.
Now we provide a counterexample which is semi-bounded but it is not bounded and this is the motivation for semi bounded concept.

**Example 3 [6]**

Let R be the real line with the canonical bornology, and A=[a,∞). Then A is an example of s-bounded set but not bounded set in R.

There are many anthers, let X={1,2,3}, $\beta = \{\emptyset\}$ then every subset of X is semi bounded set and in this example the class of all semi bounded set is also bornology on X which is discrete bornology.

The union of any two semi bounded sets is semi-bounded sets see [6].

**Definition 2 [5]** A set is semi-unbounded if its complement is semi bounded.

**Definition 3 [1]** Let $X$ and $Y$ be two bornological spaces, and let $U: X \longrightarrow Y$ be a map, then we say that $U$ is a bounded linear map if the image of every bounded subset of $X$ is bounded in $Y$. That is, $U(A) \in B(y) \; \forall \; A \in B(X)$.

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## Semi-compactness

In this section we state the definitions of semi compact space in bornological space, and we provide some functions that depend on the concepts of S-compact. Also, we prove some propositions and give some remarks and examples. Recall that the space $X$ is said to be compact space in bornological spaces iff for each bounded cover of $X$; there exists a bounded finite sub cover of $X$ see [1].

**Remark 2 [1]** In any bornological space,

**i)** An unbounded subset of a compact space is compact.
**ii)** A finite union of compact sets is compact.
**Proposition 1 [1]** Let $X$ be a bornological space and $K$ be a compact set in $X$, then $K \cap A$ is compact set in $X$ if $A$ is unbounded subset of $X$.

**Definition 4** A bornological space is said to be a semi-compact space (written as S-compact) if and only if for each semi bounded cover of $X$, there exists a finite sub cover of $X$.

In other words. A bornological space $(X; \beta)$ is S-compact space in bornological space if and only if every cover of $X$ by semi bounded set has a finite sub cover.

**Definition 5** A subset of a bornological space $X$ is relatively compact in $X$ if it is contained in a compact subset of $X$.

**Example 4** Let $X$ be an infinite set with the cofinite bornological space, then to show that

$X$ is S-compact let $\{B\ \alpha\}\ \alpha \in \gamma$ be an S-bounded cover of X. That is,

$$X \subseteq \bigcup_{\propto \in \gamma} \beta_{\propto}$$

Fix $\alpha_k \in \gamma$ $there$ $B_{\alpha k}$ to be bounded. Hence ${\beta_\alpha}^c$ is finite set in $X$ .

$X = \beta_{\alpha k} \cup {\beta_{\alpha k}}^{c}$

Where

$B_{\alpha k}^{c} = \{ x_1, x_2, \dots \dots, x_n \} \subseteq B_{\alpha 1} \cup B_{\alpha 2} \cup B_{\alpha 3} \cup \dots \dots . . \cup B_{\alpha n}.$

Hence

$B_{\alpha k} \cup {B_{\alpha k}}^{c} \subseteq B_{\alpha k} \cup B_{\alpha 1} \cup B_{\alpha 2} \cup B_{\alpha 3} \cup \dots \dots . . \cup B_{\alpha n}.$

So $\{B_{\alpha k}, B_{\alpha 1} \cup B_{\alpha 2} \cup B_{\alpha 3} \cup \dots \dots . . \cup B_{\alpha n}\}$. is a finite sub cover of the cover

$\{B_\alpha\}\alpha\epsilon\gamma$

Which cover $X$. Hence $X$ is S-compact.

**Theorem 1** The union of any two $S$-compact sets is $S$-compact.

Proof. Let $A$ and $B$ be $S$-compact sets, to prove that A ∪ B is $S$-compact.

Let $\{ B_\alpha\}_{\alpha\epsilon\gamma}$ be a S-bounded cover of $A$ and a $S$-bounded cover of B.

Since $A$ is $S$-compact, than $\{ B_\alpha\}_{\alpha\epsilon\gamma}$ has a finite sub cover of $A$.

Also, since B is S-compact, than $\{ B_\alpha\}_{\alpha\epsilon\gamma}$ has a finite sub cover of $B$.

The union of these two finite sub covers is a finite sub cover of the cover $\{ B_\alpha\}_{\alpha\epsilon\gamma}$ which covers $A \cup B$.

Hence $A \cup B$ is S-compact.

**Definition 6** Let $X$ be a bornological space and $F$ be a sub set of $X$; then $F$ is said to be a compactly unbounded set if and only if $F \cap K$ is compact for each compact $K$ in $X$.

**Example 5** Let $X = [a, b]$ and $\beta_x = [\{a\}; \emptyset ; X]$, then the collection of unbounded subset of $X$ is $[\{b\}; \emptyset ; X]$. Thus $\{a\}$ is compactly unbounded, but it is not unbounded.

## Conclusion

We introduce and study the concept of semi compactness in bornological space to restrict the condition of boundedness of the compact bornological space. Furthermore, investigate their properties, and establish their differences from compact space in bornological space. Four this purpose we employ semi bounded set instead of bounded set in the case of bornological space.

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