# Properties of a Subclass of Meromorphic Univalent Functions By Using Multiplier transformation

**Thamer Khalil[1] and Lieth A. Majed[2]**

[1] College of Education - University of Al-Mustansirya
[2]College of Science - University of Diyala

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## Abstract

In this present paper, we introduced and defined properties of a subclass of meromorphic univalent Functions defined by multiplier transformation in the puncture unit disk $\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. we obtain some properties, like, theorem of coefficient inequality, linear combination, extreme points and convex set.

**Key words:** coefficient inequality, linear combination, extreme points and convex set.

## خواص الاصناف الجزئية للدوال الميرومورفية احادية التكافوء باستخدام التحويلات المضاعفة

**ثامر خليل محمد صالح[1] و ليث عبد الطيف مجيد [2]**

[1]الجامعة المستنصرية كلية التربية قسم الرياضيات
[2]جامعة ديالة كلية العلوم قسم الرياضيات

## الخلاصة

في البحث نحن قدمنا وعرفنا خواص الاصناف الجزئية للدوال الميرومورفية احادية التكافوء المعرفة بواسطة التحويلات المضاعفة في قرص الوحدة المثقوب

$\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$ حصلنا على بعض الخواص مثل متراجحة المعاملات التركيب الخطي النقاط الحرجة والمجموعة المحدبة

**الكلمات المفتاحية :** متراجحة المعاملات، التركيب الخطي، النقاط الحرجة ،والمجموعة المحدبة .

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## Introduction

Let *MF* denote the class of meromorphic functions $f$ of the form

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \ , \tag{1}$$

defined on the punctured unit disk $\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Also, denote by $\Omega$ the subclass of *MF* consisting of functions of the from

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \ , \quad (a_m \geq 0\,). \tag{2}$$

Now, we define on $\Omega$ multiplier transformation, we define the operator $L_1(r,\gamma)$ by the following infinite series when

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

then

$$L_1(r,\gamma)h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{m+\gamma}{1+\gamma}\right)^r a_m z^m \ (\gamma \geq 0\,). \tag{3}$$

The operator $L_1(r,\gamma)$ was considered by Cho and Srivastava [3] and Cho and Kim [2].

**Definition (1):** The function $k \in \Omega$ be of the form (2) is said to be in the new class $L_1(\tau,\alpha,\mu,r,\gamma)$ if it satisfies the following condition:

$$\left| \frac{\dfrac{\frac{z^2\tau}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))} - \tau}{\alpha - \dfrac{\frac{z^2\alpha\mu}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}} \right| < 1, \tag{4}$$

for $0 < \mu \leq \frac{1}{2},\ 0 < \tau < 1, 0 < \alpha < 1 and\ r \in z = \{\dots, -1,0,1,\dots\}$.

The following interesting geometric properties of this function subclass were studied by several authors for another classes, like, Darus [2], Atshan [1].

Now**,** we obtain the necessary and sufficient condition for a function $h$ to be in the class $L_1(\tau,\alpha,\mu,r,\gamma)$.

**Theorem (1):** Let $h \in \Omega$. Then $h \in L_1(\tau,\alpha,\mu,r,\gamma)$ if and only if

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)a_m \leq \alpha(1-\mu) \qquad (5)$$

where $0<\mu\leq\frac{1}{2}, 0<\tau<1, and\ 0<\alpha<1.$

The result is sharp for the fumctiom

$$h(z)=\frac{1}{z}+\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}m(m-1)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}z^m, m\in\mathbb{N}.$$

**Proof:** For $|z|=1$, we have

$$\left|\frac{z^2\tau}{2}\left(L_1(r,\gamma)(h)(z)\right)''-\tau\left(L_1(r,\gamma)(h)(z)\right)\right| - \left|\alpha\left(L_1(r,\gamma)(h)(z)\right)-\frac{z^2\alpha\mu}{2}\left(L_1(r,\gamma)(h)(z)\right)''\right|$$

$$=\left|\sum_{m=1}^{\infty}\left(\frac{\tau}{2}(m^2-m)-\tau\right)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$-\left|\alpha\left(z^{-1}+\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right)-\frac{z^2\alpha\mu}{2}\left(2z^{-3}+\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_n m(m-1)z^{m-2}\right)\right|$$

$$=\left|\sum_{m=1}^{\infty}\left(\frac{\tau}{2}m(n-1)-\tau\right)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$-\left|\alpha z^{-1}+\alpha\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m-\alpha\mu z^{-1}-\sum_{m=1}^{\infty}\frac{\alpha}{2}\mu m\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$=\left|\sum_{m=1}^{\infty}(\frac{\tau}{2}m(m-1)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$-\left|(\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$\leq\sum_{m=1}^{\infty}(\frac{\tau}{2}m(m-1)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m-(\alpha-\alpha\gamma)+\sum_{m=1}^{\infty}(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m$$

$$=\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}(\frac{\tau}{2}m(m-1)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1))a_m-\alpha(1-\mu)\leq 0.$$

by hypothesis. Hence, $h\in L_1(\tau,\alpha,\mu,r,\gamma)$.

Conversely, assume that $h\in L_1(\tau,\alpha,\mu,r,\gamma)$, then from (4), we have

$$\left|\frac{\dfrac{\frac{z^2\tau}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}-\tau}{\alpha-\dfrac{\frac{z^2\alpha\mu}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}}\right|$$

$$=\left|\frac{\sum_{m=1}^{\infty}\quad(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}{(\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}\quad(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}\right|<1.$$

Since Re (z) ≤ $|z|$ ∀ z (z ∈ $\Delta^*$), we get

$$\text{Re}\left\{\frac{\sum_{m=1}^{\infty}\quad(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}{(\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}\quad(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}\right\}\leq 1. \quad (6)$$

We can choose value of z on the real axis $\frac{z^2(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}$ ∈ Re.

$$\sum_{m=1}^{\infty}(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^r a_m z^m$$
$$\leq (\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}(-\alpha+\frac{\alpha\mu m}{2}(-1+m))\left(\frac{m+\gamma}{1+\gamma}\right)^r a_m z^m$$

Let Re z $\rightarrow 1^-$

$$\sum_{m=1}^{\infty}(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^r a_m$$
$$\leq -\sum_{m=1}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(-1+m))a_m+\alpha(1-\mu).$$

we can write (6) as

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(-1+m))a_m \leq \alpha(1-\mu).$$

Finally,

$$h_m(z) = z^{-1} + \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(n-1))} z^n, m=1,2,\ldots \quad (7)$$

**Corollary (1):** Let $h \in L_1(\tau,\alpha,\mu,r,\gamma)$ Then

$$a_{\mathrm{m}} \leq \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1))}, m=1,2,\ldots \quad . \quad (8)$$

In the following theorem, we will show the class $L_1(\tau,\alpha,\mu,r,\gamma)$ is linear combination

**Theorem (2):** Let

$$h_i(z) = z^{-1} + \sum_{m=1}^{\infty} a_{m,i}\, z^m \in L_1(\tau,\alpha,\mu,r,\gamma) \;\; i \in \{1,2,\ldots,\ell\} \text{ and}$$

$0 < c_i < 1,$

Such that

$$\sum_{i=1}^{\ell} c_i = 1.$$

Then

$$H = \sum_{i=1}^{\ell} c_i h_i(z)$$

is also in the class $L_1(\tau, \alpha, \mu, r, \gamma)$.

**Proof:** By Theorem (1) for every $i \in \{1,2, \dots, \ell\}$ we have

$$\sum_{m=1}^{\infty} \frac{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(-1+m)\right)}{\alpha(1-\mu)} a_{m,i} \le 1.$$

Since

$$H(z) = \sum_{i=1}^{\ell} c_i h_i(z) \quad = \sum_{i=1}^{\ell} c_i \left( \mathrm{z}^{-1} + \sum_{\mathrm{m}=1}^{\infty} a_{\mathrm{m,i}} \, \mathrm{z}^{\mathrm{m}} \right)$$

$$= \frac{1}{z} + \sum_{\mathrm{m}=1}^{\infty} \left( \sum_{i=1}^{\ell} c_i a_{m,i} \right) \mathrm{z}^{\mathrm{m}}.$$

Therefore

$$\sum_{m=1}^{\infty} \frac{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(m-1)\right)}{\alpha(1-\mu)} \left(\sum_{i=1}^{\ell} c_i \, a_{m,i}\right)$$

$$= \sum_{i=1}^{\ell} c_i \left( \sum_{m=1}^{\infty} \frac{\left(\frac{n+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma n}{2}(-1+m))}{\alpha(1-\mu)} a_{m,i} \right)$$

$$\leq \sum_{i=1}^{\ell} c_i = 1.$$

Hence $H \in L_1(\tau, \alpha, \mu, r, \gamma)$ and the proof is complete.

In the following theorem, we obtain the extreme points of the class $L_1(\tau, \alpha, \mu, r, \gamma)$.

**Theorem (3):** Let $h_0(z) = \frac{1}{z}$ and

$$h_m(z) = z^{-1} + \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(-1+m))} z^m, (m \geq 1).$$

Then $h \in L_1(\tau, \alpha, \mu, r, \gamma)$, if and only if

$$h(z) = w_0\, h_0(z) + \sum_{m=1}^{\infty} w_n\, h_n\,(z), \quad (w_m \geq 0,\ w_0 + \sum_{m=1}^{\infty} w_m = 1).$$

**Proof:** Suppose that

$$h(z) = w_0 h_0(z) + \sum_{m=1}^{\infty} w_m\, h_m\,(z)$$

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r (\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(-1+m))} z^m$$

then

$$\sum_{m=1}^{\infty}\frac{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}{\alpha(1-\mu)}$$

$$\times w_{n}\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}$$

$$=\sum_{m=1}^{\infty}w_{m}=1-w_{0}\leq 1.$$

So by Theorem (1), $h\in L_{1}(\tau,\alpha,\mu,r,\gamma)$. Conversly, we suppose

$h\in L_{1}(\tau,\alpha,\mu,r,\gamma)$. By (8), we have

$$a_{\mathrm{m}}\leq\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)},m\geq 1.$$

Setting

$$w_{m}=\frac{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(n-1)\right)}{\alpha(1-\mu)}a_{m},\quad m\geq 1,$$

and

$$w_{0}=1-\sum_{m=1}^{\infty}w_{m}.$$

Then

$$h(z)=w_{0}h_{0}(z)+\sum_{m=1}^{\infty}w_{m}\,h_{m}\,(z)$$

Then

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\alpha(1-\mu)w_n}{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(-1+m)\right)} z^m$$

$$= \frac{1}{z} + \sum_{m=1}^{\infty} (h_m - z^{-1})w_m$$

$$= \frac{1}{z}\left(1 - \sum_{m=1}^{\infty} w_m\right) + \sum_{m=1}^{\infty} w_m h_m$$

$$= z^{-1}w_0 + \sum_{m=1}^{\infty} w_m h_m$$

$$h(z) = w_0 h_0(z) + \sum_{m=1}^{\infty} w_m\, h_m\,(z).$$

In the following theorem, we will prove the class $L_1(\tau,\alpha,\mu,r,\gamma)$, is a convex set.

**Theorem (4):** The class $L_1(\tau,\alpha,\mu,r,\gamma)$ is convex set.

**Proof:** Let $f_1$ and $f_2$ be the arbitrary elements of the class $L_1(\tau,\alpha,\mu,r,\gamma)$. Then for every k $(0 < k < 1)$, we will show that

$(1-Q)h_1 + Qh_2 \in L_1(\tau,\alpha,\mu,r,\gamma)$.

Thus, we have

$$(1-Q)h_1 + Qh_2 = \frac{1}{z} - \sum_{m=1}^{\infty}[(1-Q)a_m + Qb_m]z^m.$$

Hence,

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)\ [(1-Q)a_{m}+Qb_{m}]$$

$$=(1-Q)\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)a_{m}$$

$$+Q\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)b_{m}$$

$$\leq(1-Q)\alpha(1-\mu)+\alpha(1-\mu)Q=\alpha(1-\mu).$$

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