On Suitable Sets in a Top-Group

Majid Mohammed Abed[*] and Faisal Ghazi Al-Sharqi

Department of Mathematics - Faculty of Education for Pure Sciences - University of Anbar

[*]majidabed12@yahoo.com

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Abstract

Many researchers for their importance and their useful applications had studied the suitable sets. This study presents full description of these sets with a number of topological concepts. A suitable set is presented as a special type with the conditions for some of the topological concepts. In addition, a special subset of the topological G has been described in two different ways to achieve the same goal. The first one is based on several concepts, such as the closure of subset, discrete topology and closed of the set of the identity element, whereas, the second way is based on two concepts, such as the locally compact and the separation axioms. It is, however, important to note that some of the topological groups have suitable sets, while, others do not.

Keywords: Topological space, Topological group, Locally compact, Hausdorff group, Discrete topology, Suitable set.

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حول المجموعة المناسبة في الزمرة التبولوجية

ماجد محمد عبد و فيصل غازي الشرقي

قسم الرياضيات -كلية التربية للعلوم الصرفة - جامعة الانبار

ألخلاصة

المجموعات المناسبة درست من قبل عدد من الباحثين بسبب اهميتها والاستفادة من تطبيقاتها. هذه الدراسة تقدم وصف كامل لهذه المجموعات مع عدد من المفاهيم التبولوجية. المجموعة المناسبة قدمت هنا كنوع خاص مع شروط معينة لبعض المفاهيم التبولوجية. كذلك المجاميع المناسبة من الزمر التبولوجية درست بطريقين الاول اعتمد على عدة مفاهيم مثل الانغلاق للمجموعة والتبولوجي المتقطع اما الطريق الثاني اعتمد على مفاهيم مثل التراص المحلي وبديهيات الفصل في التبولوجيا .على اية حال من المهم ان نقول بانه يوجد نوعين من المجاميع على الزمر التبولوجية وهما المجموعة المناسبة والمجموعة الغير مناسبة.

الكلمات المفتاحية: زمرة هاوزدورف، التبولوجي المتقطع، الفضاء التبولوجي، الزمرة التبولوجية، التراص المحلي، المجموعة المناسبة.

Introduction

Comfort, Morris, Robbie, Svetlichny and Tkacenko [2] have studied the subject of suitable set in 1998. A suitable set D means it is a sub-set of G, Clo($\prec$D$\succ$)= G, D has Dis-Top and D$\cup$\{e\} is a closed, where e is the identity of G. In the paper [2], we found all locally compact groups and metrizable groups have suitable sets. Dikranjan, Tkacenko and Tkachuk [3] said neither Lindelof nor countably compact groups need have suitable sets. Hoffmann and Morris [5] investigated further properties in topological group. Moreover; a Top-space X is Hausdorff if $\forall$ x, y in X with x $\neq$y $\exists$ U $\supset$ x and V $\supset$ y $\ni$ U$\cap$V=$\emptyset$. Armacost [1] explained the locally compact in details, therefore we worked to create a strong relationship between locally compact and suitable set in details and he defined the L-compact over Top-gp. After three years Spivak [7] defined the locally compact and gave an equivalent between T_0-space and Hausdorff. Dikran Dikranjan [4] introduced new characterizing subgroups of any comm. group. Juhsz [6] studied closed discrete subspaces of product spaces. Further; if G is not compact, it has a closed suitable set. If (X, τ) is a Dis- Top-space, then it is a metrizable space and so has suitable set. Also, from Urysohn Metrization Principal, let (X, τ) be a regular space as topological group and X have a countable basis, then X is metrizable and has suitable set. Our purpose in this paper is to give some criteria for the different types of conditions in order to study suitable sets.

1. Suitable Set

In this section, we introduce and determine the necessary conditions in order to study suitable set. The set of the identity element e is closed iff any point is a closed-set (i.e. iff T_1-Top-gp). The set G is called a topological group (Top-gp) if:

- ■ G satisfy all the conditions of a group.
- ■ G satisfy all the conditions of a Top-space.
- ■ Satisfy the condition of inversion continuous.

In the next theorem, we state three conditions to get suitable set, such as T_1-space, Discrete topology and closure set.

Theorem 1.1. Let G be Top-gp and {e} set of the identity in G. If:

(1) G is a T_1-Top-gp,

(2) H in G is closed-Dis-Top,

(3) Clo(≺D≻) = G. Then H is called suitable set.

Proof. Let G be a T_1-Top-gp. Then {e} is closed-set with H is closed-Dis-Top in G, we get the union of H with the identity is closed in G. From hypothesis, G is a Top-gp. But from condition (3), Clo(≺D≻) = G, H is a suitable set.

Example 1.2. Let (X, C) be a f-complement space such that any set D in G is closed-Dis-Top and Clo(≺D≻) = G. Then the set X has suitable set. Because for any x, y in X $\ni x \neq y$, then $U_x = \{y\}^c$, $V_y = \{x\}^c$ are open-sets. Also, y not in $\{y\}^c = U_x$, y in $\{x\}^c = V_y$ and x not in $\{x\}^c = V_y$, x in $\{y\}^c$ = Ux. So (X, C) is a finite complement space is T_1-space, (see [1]). Moreover, assume that x, y in X $\ni x \neq y$ and (X, τ) is T_3-space. Hence {x} is a closed set and y not in x. But (X, τ) is regular. Therefore, any two sets U, V in $\tau \ni y$ in U, $\{x\}^c \subseteq V$, $U \cap V = \emptyset$. Hence (X, τ) is T_2. So is T_1. Now if (X, τ) satisfy all conditions in this example, then we obtain X has suitable set.

Now to prove that any subset H of infinite group is suitable, we need to use several conditions on G with K compact Hausdorff group over real numbers R.

Theorem 1.3. Let G be an infinite group and it is a homeomorphic to $R^n \times K \times D$ with H subset of G. If:

(1) G is a Top-space,

(2) (x, y) $\mapsto$xy: G $\to$G; the inversion are continuous,

(3) K is D-Top-space,

(4) The real numbers R is a Top-gp,

(5) K is a com-Hausdorff-gp. Then H is a suitable set.

Proof. Let G be infinite gp. From conditions (1) and (2), G is infinite Top-gp. But from condition (3), K is Dis-Top-space. We have R is a real number and Top-gp, so K is L-compact. But K is a compact-Hausdorff-group. Thus, H is a suitable set.

Proposition 1.4. Let G be a Top-gp. If the following are true:

(1) D $\subseteq$G is compact,

(2) D$\trianglelefteq$G,

(3) G/D is L-compact,

(4) The set {1} is open. Then D is a suitable set.

Proof. Let G be a Top-gp. Then G is a group. Now The natural homomorphism ∅H: G→G/D is a closed mapping. For every gd ∈ G/D, we have $\emptyset^{-1}$D(gD) = gD , and thus $\emptyset^{-1}$D (gD) is a compact subset of G. Hence, we can apply the preceding lemma to the mapping ∅D, and we see that the subset $\emptyset^{-1}$D(K) of G is compact. Then G = $\emptyset^{-1}$D (G/D) is compact. On the other hand, if a point gd of G/D has a compact neighborhood V, then the compact set $\emptyset^{-1}$(V) is a neigborhood of g, by continuity of ∅D. So, G is L-compact if G/D is L-compact. But the set {1} is open. So, G is Dis- Top-gp and it is a Hausdorff space. Now G is group with Hasusdorff property yields G is a Hausdorff group. But G is L-compact, then it is a locally-Hausdorff group, any D $\subseteq$G is a suitable set.

Corollary 1.5. Let G be a Top-gp. If the following conditions are true:

(1) Locally-group is totally disconnected,

(2) D in G is closed-Dis-Top,

(3) $D \cup \{e\}$ is closed,

(4) Clo(≺D≻)=G. Then D is suitable set.

Proof. From condition (1), G is a totally disconnected. So is T_1-space. But G is a Top-gp. Then G is T_1-Top-group. Hence the set $\{e\}$ is closed. Now conditions (2), (3) and (4) with closed property of $\{e\}$, the proof is completed (see Proposition 1.4).

Lemma 1.6. Let (X, τ) be a regular space. If (X, τ) is T_0-space, then it is T_2-space (Hausdorff).

Proof. Let x, y in X $\ni x \neq y$ and (X, τ) is T_0-space. Therefore, x not in Clo{y} or y not in Clo{x}. Let x not in Clo{y}. Since Clo{y} is a closed and (X, τ) is a regular, then $\exists$ $U, V \in \tau$ $\ni$ $x \in U$, (note that U and V are open sets). Clo{y} $\subseteq V$, $U \cap V = \emptyset$. So (X, τ) is Hausdorff space.

Corollary 1.7. Let G be a Top-gp and G/D be L-compact. If D is a closed-subgroup of a Top-gp G, then D is suitable set.

Corollary 1.8. Let G be a Top-gp. If the following are true:

(1) D is a Hausdorff,

(2) D is closed set,

(3) $D \leq G$. Then (D is suitable set).

In the following theorem, we study suitable set on T_0- topological group and discrete topology

Theorem 1.9. Let G be a Top-gp and Dis-space. If G is T_0-space, then D subset of G is a suitable set.

Proof. Let $d_1 \neq d_2$ be in G. Since G is T_0, so without loss of generality d_2 not in $\{d_1\}$. Then there exists a symmetric open e-neighborhood U of G $\ni$ d_1 not in Ud_2. Choose a symmetric open e-neighborhood W of G $\ni$ $W^2 \subset U$. Now it is easy to see that $Wd_1 \cap Wd_2 = \phi$ are open neighborhoods of d_1 and d_2 respectively. Therefore, G is Hausdorff (see Lemma 1.6). But G is Dis-space, then it is L-compact over Top-gp, G has suitable set.

The next theorem explains the relation between Euclidean space and topological group.

Theorem 1.10. Let G be a Top-gp and Euclidean space. If the diagonal in $G \times G$ is closed, then the subset D of G is a suitable set.

Proof. Since G is a Top-gp, then it is a Top-space. But the diagonal in G×G is closed. Therefore, G is a Hausdorff space. We have G is a Top-gp. So, G is a group, with Hausdorff property we obtain G is Hausdorff group. But G is a Euclidean space. Then it is L-compact with Hausdorff property yield G has suitable set.

Corollary 1.11. If every metric space (X, d) is L-compact, then any subset of X is a suitable set. (in particular R^n has subset is suitable set) (for n > 1).

Proof. Let (X, d) be a metric space and let x, y ∈ X with x ≠y. Let r = d (x, y). Let U = B (x; r/2) and V = B (y; r/2). Then x ∈ U and y ∈ V. We claim that the intersection U∩V≠∅ and this means there is no z in U∩V. But d (x, z) < r/2 and d (z, y) < r/2, so we get r=d(x, y)d(x, z)+d(z, y) < r/2 + r/2=r (i.e. r < r, a contradiction). Hence U∩V = ∅ and X is Hausdorff. But X is L-compact. Then X has suitable set.

Theorem 1.12. Let X be a Hausdorff and let Z in X is Cantor set. Then Z is a suitable set in X.

Proof. Let x, y ∈ Z. We have X is a Hausdorff. So, ∃ U, V ∈ X with x∈U and y ∈ V∋ U∩V= ∅, where U, V are open-sets. So, U^*= U∩Z and V^*=V∩Z are open in Z. Moreover; x ∈ U^* and y ∈ V^*. So, $U^*\cap V^*$= (U∩Z) ∩(V∩Z) = (U∩V) ∩ Z = ∅, Z is Hausdorff. But Z is Cantor set in X. So, it is L-compact in X. Thus, X has suitable set.

The unit interval is a complete metric space. As a Top-space and satisfy many other properties. In addition, many copies of the unit interval give Hilbert cube, therefore next theorem determine the strong conditions about suitable set.

Theorem 1.13. Let G be a Top-gp. If D≤G and the following assertions are true,

(1) G/D is a Top-gp,

(2) D is connected component of e in G,

(3) D is closed and normal of e in G,

(4) If D is a unit interval, then D has suitable set.

Proof. First, in topology D is closed, because connected components are closed. We need to show that D ≤ G. We know that the element e in D. Assume that a, b ∈ D. The operation (o) is homemorphism. So Da^{-1} is connected and contains the element e, because a ∈ D. Thus Da^{-1} subset of D. So $b^{-1}a$ ∈ D. Then D ≤ G with condition (3) we obtain D ⊴ G, with condition (3), D is a Hausdorff. But from condition (4), D is a unit interval (D is L-compact). Thus D has suitable set.

Theorem 1.14. Let G be a Top-gp. If X, Y are L-compact and Hausdorff space, then the product of X and Y has suitable set.

Proof. Let w = (x, y) in X × Y. Since X is L-compact, there exists U open and K is a closed and compact with $x \in U \subset K$. Since Y is L-compact, there exists V open, L closed and compact with $y \in V \subset L$. Thus $w=U\times V \subset K\times L$. Moreover; U×V is open, K×L is closed and K×L is a compact. So, X×Y is L-compact. Since X and Y are Hausdorff, then X×Y is also Hausdorff. Thus X×Y has a set w is a suitable.

Corollary 1.15. Let G be a Top-gp. If G is a Dis-space, then G has a suitable set.

Proof. Since G is a Dis-space, then it is L-compact and Hausdorff, G has a suitable set.

Theorem 1.16. Over L-compact-group G and satisfy the following:

(1) $D \leq G$, D is closed-set,

(2) $\alpha: G \rightarrow G/D$ is the law projection,

(3) G/D is a Hausdorff, so G/D has suitable set.

Proof. Let U be $N_e\{1\} \in G$ such that Clo(U) is compact. Take a Neighborhood $\alpha(U)\{1\} \in G/D$. Then α (Clo((U)) $\subseteq$ Clo(α(U)) because α is continuous. Now α(Clo((U))) satisfy the condition of compactness in G/D, which is Hausdorff, and so α(Clo((U))) is closed. Since α(U) is dense in α (Clo((U))), then Clo(α(U)) = α(Clo((U))) = α(U). So, G/D is L-compact. But from condition (3), we have G/D is a Hausdorff quotient group, G/D has suitable set.

Now since the Hilbert cube is homeomorphic to the product of countable infinitely many copies of the unit interval [0, 1], So we will use this concept to highlight the important truth about the suitable set through the relationship of Hausdorff with L-compact.

Theorem 1.17. Let G be a Top-gp. If the following are true:

(1) G is Hausdorff group,

(2) D is a Hilbert cube,

(3) $D \leq G$. Then D has a suitable set.

Proof. Clear that D is Hausdorff, because G is a Hausdorff Top-gp. But D is Hilbert cube, therefore it is L-compact. So, G has suitable set.

Corollary 1.18. Let G be a T_0-Top-gp. If one point of G has a local basis of compact sets, then G has suitable set.

Proof. Let U ∈ N_G. We know multiplication and inversion are continuous. Then ∃ V ∈ N_G such that V V ⊆ U and V^{-1} = V. But Clo(V) ⊆ $V^{-1}V$ = V V ⊆ U. So, by homogeneity, G is a regular at e. Suppose that G is T_0-space, and let g; h ∈ G be distinct. Let U ∈ N_G ∋ h, g ∈ Ug. So, hg^{-1} is not in U, and g is not in $U^{-1}h$. So, G is T_1-space and regular. Hence G is a T_3-space. So is T_2-space (Hausdorff). Since one point in G has local basis. Then G is a L-compact with G is Hausdorff yield G has suitable set.

Theorem 1.19. Let G be a Top-gp. If (X, τ) is a L-compact and any two disjoint compact subsets of X can be separated by disjoint open sets, then G has suitable set.

Proof. Suppose that any two disjoint compact subsets of X can be separated by disjoint open sets. We need to prove that X is a Hausdorff space. Assume x ≠y in X. Then {x} and {y} are compact disjoint subsets of X. By assumption ∃ G, H ∈ τ such that {x}⊆G, {y}⊆ H and G ∩ H=∅. Hence there exists open sets G and H ∋ x ∈ G and y ∈ H. Hence X is a T_2-space. But X is L-compact, G has suitable set.

Remark 1.20. Note that every Top-gp G is com-regular, there is a f: G→[0; 1] such that the image of g equal to zero, f is continuous and image of F equal to {1}. If G is T_0-space, then G is $T_{3\frac{1}{2}}$ (Tychonof). Therefore, we can introduce the following result.

Corollary 1.21. Every Top-gp G satisfy all conditions in Remark 1.20 and has a point with a local basis of compact sets, has a suitable set.

Proof. From Remark 1.20, G is completely-regular and so is $T_{3\ 1/2}$-Top-gp. Hence G is T_3-space and then it is T_2-space (Hausdorff). But G has a point with a local basis imply G is L-compact. Thus, G has suitable set.

Corollary 1.22. Let G be a Top-gp. if the following statements are true:

(1) If the complement of the set {e} is open,

(2) If for all g in G there exists a neighborhood V such that V is compact set.

Then G has suitable set.

Proof. Since the complement of the set feg is open, then the diagonal Δ of $G\times G$ is closed-set, because it is the inverse image of $\{e\}$ under the continuous mapping $(x; y) \to xy^{-1}$. This means G is a T_2-space (Hausdorff). By condition (2), G is L-compact and we finish the proof.

Abbreviations

Discrete space = Dis-space.
Closure = Clo.
Locally compact = L-compact

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