Compactness in probabilistic Hilbert space

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Abstract

In this paper we focus our study on weakly, strongly and uniformly convergence of operators in probabilistic Hilbert space (PH- space) and the relations between these convergence of operators. Also, we introduce the definition of operator in (PH- space).

Keyword: compact operator, PH- space, weakly convergence, strongly, convergence, bounded operator.

التراص في فضاء هلبرت الاحتمالي

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الخلاصة

في هذا البحث تركزت دراستنا حول تقارب المؤثرات في فضاء هلبرت الاحتمالي (التقارب الضعيف، القوي، المنتظم) والعلاقة بين المؤثرات المتقاربة. كذلك قدمنا المؤثر المتراص في فضاء هلبرت الاحتمالي.

الكلمات المفتاحية: المؤثر المتراص، فضاء هلبرت الاحتمالي، التقارب الضعيف، التقارب القوي، المؤثر المحدود

Introduction

The concept of a probabilistic inner product space (PIP- space) which is based on the modern concept of a PH- space that was newly introduced in [01,02,03]. The concept of PH- space was also introduced and studied in 2007 by S.Y.; wany, x, : and Ga, J. [04] . Before we proceed we must state some definitions. Known details and results to be used in the complement the concepts used are those of [01,05]

Definition 1 [01]

In probability theory, a distribution function (D.F) is a mapping $G: R \to [0,1]$ check the following:

1- G is non-decreasing.
2- G is left-continuous.
3- $\inf_{x \in R} G(X) = 0$.
4- $\sup_{x \in R} G(x) = 1$

Note :

1- D: The set of all D.F's .
2- $H(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x > 0 \end{cases}$ (1-1)
3- $H \in D$ (special element)

Definition 2 [04]

A PIP-space is a triple $(E, G, *)$ whese: E is a real linear space, And G is a mapping of $E \times E \to D$ ($G_{u,v}$ will denote the D.F), where: $G_{u,v}(x)$ will represent the value $G_{u,v}$ at $x \in R$ satisfies the following conditions:

1- $G_{u,u}(0) = 0$, $\forall u \in E.$ (PI-1)
2- $G_{u,v}(x) = G_{v,u}(x),$ $\forall u, v \in E.$ (PI-2)

3- $G_{u,v}(x) = H(x)$ $if\ and\ only\ if$ $u = 0$ (PI-3)

4- For $u, v \in E$ and $\gamma \in R$

$$G_{\gamma u,v}(x) = \begin{cases} G_{u,v}\left(\frac{x}{\gamma}\right) & \gamma > 0 \\ H(x) & \gamma = 0 \\ 1 - G_{u,v}\left(\frac{x}{\gamma}+\right) & \gamma < 0 \end{cases} \quad \text{(PI-4)}$$

Where $G_{u,v}(x+) = \lim_{x_1 \to x+} G_{u,v}(x_1)$

5- If u and v is linearly independent then

$$G_{u+v,w}(x) = \left(G_{u,w} * G_{w,v}\right)(x) \quad \text{(PI-5)}$$

Where $(G_{u,v} * G_{W,v}) = \int_{-\infty}^{+\infty} G_{u,W}(x - t)\ dG_{w,v}(t)$ (1-2)

Definition 3 [04]

A sequence u_n in E is said be $\tau - converges$ to $u \in E$ if $\forall \epsilon > 0$ and $\propto > 0$ there must exists $N(\epsilon, \propto)$ appositive integer s.t

$$G_{u_n - u, u_n - u}(\in) > 1 - \propto \text{ whenever } n > N \quad \text{(1-3)}$$

Definition 4 [04]

A sequence u_n in E is said be $\tau - cauchy$ sequence if $\forall \epsilon > 0$ and $\propto > 0$ there must exists $N(\epsilon, \propto)$ s.t

$$G_{u_n - u_m, u_n - u_m}(\in) > 1 - \propto \text{ whenever } n, m > N \quad \text{(1-4)}$$

Lemma 1 [06]

Let $G(x)$ be d f, Assume $g(x)$ is non decreasing bounded function then

$$\int_{-\infty}^{+\infty} g(x)\, dG(x) = \int_{-\infty}^{+\infty} g(x)\, dG(x+) \quad \text{(1-5)}$$

Definition 5 [06]

A PIP- space $(E, G,*)$ is called with mathematical exaction if

$$\int_{-\infty}^{+\infty} x dG_{u,v}(x) \qquad (1\text{-}6)$$

Is converges for $u, v \in E$.

Remake

$(E, G,*)$ is said to be $(\tau - cauch\ sequence\ in\ E) \Rightarrow (\tau - converges\ to\ some\ point\ in\ E)$

Theorem 1 [06]

Let $(E, G,*)$ be PIP- space with (ME) letting

$$< u, v >= \int_{-\infty}^{+\infty} x dG_{u,v}(x) \qquad (1\text{-}7)$$

$\forall x, y \in E$, Then $(E, <.,.>)$ is IP- space , so that $(E, \|.\|)$ is a nor med space, where $\|u\| = \sqrt{< u, u >}$ for all $u \in E$

Weak convergence

Definition 1

Let G_n and G are dF's $G_n \xrightarrow{weakly} G$ if $G_n(x) \longrightarrow G\,(x)$ for every point at which the timid df G is continuous if

$$C(G) = \{x \in R: G(x) = G(x + 0)\}$$

$\forall x \in C(G)$

$$\lim_{n\to\infty} G_n\,(x) = G(x)\ .$$

Definition 2

Let $(E, G, *)$ be PH- space, and $u_n, u \in E$:

(1) $u_n \xrightarrow{weakly} u$ if $v \in E$, we have $\lim_{n\to\infty} < u_n, v > = < u, v >$

(2) $u_n \xrightarrow{N} u$ (N-convergent) if $\lim_{n\to\infty} G_{u_n-u,u_n-u}(x)$ =1

Remark 1

(1) If $u_n \xrightarrow{w} u$ then $u_n \xrightarrow{m} u$ (m-convergent)

(2) If $u_n \xrightarrow{w} u$ then $u_n \xrightarrow{\tau} u$ (τ-convergent)

Lemma 1

Let $(E, G, *)$ PH- space, and $u_n \in E$. If $\|u_n - u\| \to 0$ then $u_n \xrightarrow{w} u$

<u>Proof</u>

Since $\|u_n - u\| \to 0 \Rightarrow \|u_n - u\|^2 \to 0$

$$\lim_{n\to\infty} \int_0^\infty x dG_{u_n-u,u_n-u}(x) = 0$$

$$\lim_{n\to\infty} G_{u_n-u,u_n-u}(x) = 0$$

$$\lim_{n\to\infty} (u_n - u) = 0$$

$$\lim_{n\to\infty} u_n = u$$

Lemma 2

Let $u_n \xrightarrow{w} u$ then u_n is bounded.

Proof

For any $v \in E$, we have

$\lim_{n\to\infty} < u_n, v > = < u, v > , u \in E$

$\lim_{n\to\infty} \int_0^\infty x dG_{u_n,v}(x) = \int_0^\infty x dG_{u,v}(x)$

Since ever $G_{u_n,v}$ is bounded $0 \le G_{u_n,v}(x) \le 1 \quad \forall x \in \bar{R}$

Therefore the sequence $\{< u_n, v >\}$ is bounded. Now the lemma follows for the principle of uniform boundedness.

Lemma 3

Let $(E, G, *)$ be PH- space, and $u_{n\,,} v_n \in E$ such that:

$u_n \xrightarrow{w} u$ and

$\lim_{n\to\infty} u_n = v, \; v \in E$ then

$\lim_{n\to\infty} < u_n, v_n > = < u, v >$

Or $< u_n, v_n > \xrightarrow{w} < u, v >$

Proof

Since any convergent sequence is bounded, the in equality

$$|< u_n, v_n > - < u, v >| = |< u_n - u, v_n > + < u, v_n - v >|$$

$$\le \|u_n - u\| \, \|v_n\| + \|u\| \, \|v_n - v\|$$

Implies that

$$\lim_{n\to\infty} \int_0^\infty x dG_{u_n,v_n}(x) - \int_0^\infty x dG_{u,v}(x) = 0$$

$$\lim_{n\to\infty} \int_0^\infty x dG_{u_n,v_n}(x) = \int_0^\infty x dG_{u,v}(x) = 0$$

$$\lim_{n\to\infty} < u_n, v_n > = < u, v >$$

Theorem 1

Let $(E, G,*)$ be PH- space with ME, left $T: D(T) \to R(T)$. linear bounded operator, then these exists $\mu > 0$ s.t

$$|< Tu, v >| \le \|Tu\|\|v\|$$

$$\le \mu\|u\|\|v\| \qquad \forall\, u, v \in E$$

Proof

Fix $v \in E$ and $\varphi_v(u) =< Tu, v >, \forall\, u \in E$ φ_v is linear functional on E, i.e $\varphi_v: E \to R$

For any positive real number $\propto$

1) If $Tu - Tv$ with $u - v$ is linearly independent, then $Tu - \propto Iv \neq 0$ then

$$< Tu - \propto Iv, Tu - \propto Iv > = \int_{-\infty}^{+\infty} x dG_{Tu-\propto Iv, Tu-\propto Iv}(x) \ge 0$$

$$= \int_{-\infty}^{+\infty} x dG_{Tu,Tu}(x) - \int_{-\infty}^{+\infty} x dG_{Tu,\propto Iv}(x) - \int_{-\infty}^{+\infty} x dG_{\propto Iv,Tu}(x) + \int_{-\infty}^{+\infty} x dG_{\propto v,\propto v}(x) \ge 0$$

$$= \int_{-\infty}^{+\infty} x dF_{Tu,Tu}(x)\Big| + 2 \propto \int_{-\infty}^{+\infty} x dF_{Tu,v}(x)\Big| + \propto^2 \int_{-\infty}^{+\infty} x dF_{v,v}(x) \ge 0$$

$$\|Tu\|^2 - 2 \propto |< Tu, v >| + \propto^2 \|v\|^2 \ge 0$$

Let $\propto = \frac{|< Tu, v >|}{\|v\|^2}$

$\|Tu\|^2 - 2\frac{|<Tu,v>|^2}{\|v\|^2} + \frac{|<Tu,v>|^2}{\|v\|^2} \geq 0$

$\|Tu\|^2 - \frac{|<Tu,v>|^2}{\|v\|^2} \geq 0$

$|< Tu, v >|^2 \leq \|Tu\|^2\|v\|^2$

$|< Tu, v >| \leq \|Tu\|\|v\|$

Since T bounded in norm

$|< Tu, v >| \leq \mu\|u\|\|v\|$

2) If $Tu - Tv$ with $u - v$is linearly depended and $Tu - \propto Iv \neq 0$, then prove is same in (1).
3) If $Tu - Tv$ with $u - v$ is linearly depended and $Tu - \propto Iv = 0$ then

$< Tu - \propto Iv, Tu - \propto Iv >= \int_{-\infty}^{+\infty} xdG_{Tu-\propto Iv, Tu-\propto Iv}(x)$

$= \int_{-\infty}^{+\infty} xd\, H(x) = 0$

$|< Tu, v >| \leq \|Tu\|\|v\|$

$\leq \mu\|x\|\|u\|$

$\forall u, v \in E$ and $\mu > 0$

Theorem 2

If T is bounded operator then $\|T\| = sup_{\|u\|=1}|< Tu, u >|$

Proof

For any $u \in E$

$< Tu, u >=< u, uT >$ is real

Now let

$$M = \sup_{\|u\| = 1} |< Tu, u >|$$

By lemma 1

$$|< Tu, v >| \leq \|Tu\| \|u\|$$

$$\leq \|T\| \|u\|^2 = \|T\|$$

For all $u \in E$ such that $\|u\| = 1$. Consequently $M \leq \|T\|$

On the other hand, for any $t, s, \in E$ we have

$$4 < Tt, s >= 4 \int_{-\infty}^{+\infty} x dG_{Tt,s}(x)$$

$$= \int_{-\infty}^{+\infty} x dG_{T(t+s),t+s}(x) - \int_{-\infty}^{+\infty} x dG_{T(t-s),t-s}(x)$$

$$\leq M(\|t + s\|^2 + \|t - s\|^2)$$

$$= 2M(\|t\|^2 + \|s\|^2)$$

Using the parallelogram law. If $Tt \neq 0$ let $s = \frac{\|t\|}{\|Tt\|} Tu$ to obtain, since $\|t\| = \|s\|$ then

$$\|t\| \|Tt\| \leq M \|t\|^2$$

Consequently $\|Tt\| \leq M\|t\|$ (for all u, in clouding these with $Tt = 0$) and $\|T\| \leq M$

Therefore $\|T\| = M$.

Definition 3

Let T_n be a sequence of bounded operator in E.

i) T_n uniformly convergent to T if:

$$\lim_{n\to\infty} \int_0^\infty xdG_{T_n-T,T_n-T}(x)=0$$

ii) T_n strongly convergent to T if :

$$\lim_{n\to\infty} \int_0^\infty xdG_{T_nu-Tu,T_nu-Tu}(x) = 0 \quad , \quad u \in E$$

iii) $T_n \xrightarrow{w} T$ if:

$$\lim_{n\to\infty} \int_0^\infty xdG_{T_nu,v}(x) = \int_0^\infty xdG_{T_nu,v}(x) \quad , \quad \forall u,v \in E$$

Compact Operators

A linear operator T is said to be compact if the image of any bounded sequences has a convergent of subsequences.

Theorem 1

Let $(E,G,*)$ be probabilistic Hilbert space with mathematical expection, let $T: D(T) \to R(T)$. linear bounded operator in $(E,G,*)$, T is compact operator.

<u>Proof</u>

For every sequence $u_n \in E$ such that $\|u_n\| = 1$ then by theorem (2-2)

$$\|T\| = \sup_{\|u\| = 1} |< Tu_n, u_n >|$$

Then

$$\lim_{n\to\infty} |< Tu_n, u_n >| = \|T\|$$

Let $v_n = Tu_n$ has a convergent subsequences related this subsequences as u_n and let

$$v = \lim_{n\to\infty} Tu_n$$

$$\lim_{n\to\infty} < Tu_n, u_n >= \mp\|T\| = \lambda$$

$$\|Tu_n - \lambda u_n\|^2 = \int_{-\infty}^{+\infty} x dG_{T_n u - \lambda u_n, T_n u - \lambda u_n}(x)$$

$$= \int_{-\infty}^{+\infty} x dG_{T_n u, T_n u}(x) - 2\lambda \int_{-\infty}^{+\infty} x dG_{T_n u, \lambda u_n}(x) + \lambda^2 \int_{-\infty}^{+\infty} x dG_{u_n, u_n}(x)$$

Since $\int_{-\infty}^{+\infty} x dG_{u_n, u_n}(x) = \|u_n\|^2 = 1$

$$\|Tu_n - \lambda u_n\|^2 = \|Tu_n\|^2 - 2\lambda < Tu_n, u > +\lambda^2$$

$$\leq 2\lambda^2 - 2\lambda < Tu_n, u_n >$$

Since $< Tu_n, u_n > \to \lambda$

$\|Tu_n - \lambda u_n\|^2 \to 0$

Then

$Tu_n - \lambda u_n \to 0$

On the other hand $Tu_n \to v$ and consequently u_n also converges :

$u_n \to u = \lambda^{-1} v$ u_n convergent to u .

T is compact operator.

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