Spectral properties of the Helmholtz problem with spectral parameter Dependent conditions.

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Abstract:

We consider Helmholtz problems containing a spectral parameter both in the equation and in the boundary conditions.
we prove that the system of corresponding eigen functions forms an orthonormal basis in some adequate Hilbert spaces. The oscillation properties as completeness, minimality and basic properties are investigated for the eigenfunction of the Helmholtz operator equation in the triple of adequate Hilbert spaces. Asymptotic formula for eigenvalue and eigenfunction are deduced.

Keywords: Helmholtz equation ,boundary conditions , operator , Eigenvalue , Eigenfunction , Basis property.

الخواص الطيفية لمسالة هلمهولتز مع وجود المعلمة الطيفية في الشروط الحدودية

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الملخص

في هذا البحث تناولنا مسالة هلمهولتز (معادلة تفاضلية جزئية من الرتبة الثانية) مع وجود المعلمة الطيفية في الشروط الحدودية ، هذه المسالة تتحول باستخدام طريقة فصل المتغيرات الى مسالة قيم ذاتية من الرتبة الثانية مع وجود القيمة الذاتية في المعادلة وشرط حدودي واحد ، برهنا ان انظمة الدوال الذاتية للمؤثرات التفاضلية الاعتيادية المكافئة لمسائل القيم الذاتية المتولدة ومؤثر معادلة هلمهولتز تكون اساس متعامد في فضاءات هلبرت الموسعة. وكذلك برهنا ان الدوال الذاتية لمؤثر هلمهولتز في فضاء هلبرت الثلاثي تشكل نظام اصغر ما يمكن ومتكامل. وتم استنتاج صيغة تقاربية للقيم الذاتية والدوال الذاتية.

الكلمات المفتاحية: المعادلة هيلمهولتز، الشروط الحدية، العامل القيمة الذاتية، اساس الاحتمالية.

1-Introdaction

The scalar Helmholtz equation

$$\nabla^2 \omega(x,y,z) + \varphi^2 \omega(x,y,z) = 0 \ , \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (1\text{-}1)$$

Where $\omega(x,y,z)$ is a complex scalar function (potential) defined at a spatial point $(x,y,z) \in R^3$ and φ is some real or complex constant (eigenvalue) , takes its name from Hermann Von Helmholtz (1821-1894), the famous German scientist, whose impact on acoustics, hydrodynamics, and electromagnetics is hard to overestimate. This equation naturally appears from general conservation laws of physics and can be interpreted as a wave equation for monochromatic waves (wave equation in the frequency domain). The Helmholtz equation it can also be derived from the heat conduction equation, Schrodinger equation , telegraph and other wave-type , or evolutionary, equations . From a mathematical point of view it appears also as an eigenvalue problems for the Laplace operator ∇^2.

Helmholtz equation is an equation of the elliptic type, for which if is usual to consider boundary value problems. Boundary conditions follow from particular physical laws (conservation equation) formulated on the boundaries of the domain in which a solution is required. This domain can be finite (internal problems) or infinite (external problems). For infinite domains, the solutions should satisfy some conditions at the infinity. These conditions also have a physical origin. For the Helmholtz equation that arise as a transform of the wave equation into the frequency domain, the boundary conditions should be understood in the context of the original wave equation.

The Helmholtz equation was solved for many basic shapes in the 19th century , the rectangular membrane by Simian Denis poisson in 1829,the equilateral triangle by Gabriel in 1852, and the circular membrane by Alfred Clebsch in 1862. The elliptical drumhead was studied by Emile Mathieu, leading to Mathieus differential equation.[1]

This paper presents a study. For Helmholtz problem with spectral parameter dependent conditions:

$$\omega_{xx} + \omega_{yy} + \omega_{zz} + \varphi^2\omega = 0 \ ,0 \le x \le 1 ,0 \le y \le 1 ,0 \le z \le 1 \qquad (1\text{-}2)$$

$$\omega(0,y,z) = \omega(x,0,z) = \omega(x,y,0) = 0$$

$$\omega_x(1,y,z) = \varphi^2\omega(1,y,z)$$

$$\omega_y(x,1,z) = \varphi^2\omega(x,1,z)$$

$$\omega_z(x,y,1) = \varphi^2\omega(x,y,1)$$

The Helmholtz problem (1-2) , (2-2) when the boundary conditions contain a spectral parameterθ^2 , this problem can't interpreted an eigenvalue – Eigen function problem in the Hilbert space $L_2(0,1)$.[10]

In order to motivate the subject of this paper we recall that the generalized Regge problem is realized by a second order differential operator which depends quadrafically on the eigenvalue parameter and which has eigenvalue parameter dependent boundary conditions, see [2]. The particular feature of the Regge problem is that coefficient operators of the corresponding quadratic operator pencil are self-adjoin,[12]. Applying the separation of variables to the boundary value problems associated with Helmholtz equation:

Assume that $\omega(x,y,z) = u(x)v(y)w(z)$ (1-4)

Then the equation (1-2) becomes

$$vw\frac{d^2u}{dx^2} + uw\frac{d^2v}{dy^2} + uv\frac{d^2w}{dz^2} + \varphi^2uvw = 0 \qquad (1\text{-}5)$$

Dividing (1-5) by uvw , we obtain:

$$\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{v}\frac{d^2v}{dy^2} + \frac{1}{w}\frac{d^2w}{dz^2} + \varphi^2 = 0 \qquad ,u(x) \ne 0, \forall x \ \ ,\frac{1}{v} \ne 0 \ \ ,\forall y \qquad (1\text{-}6)$$

Let us write (1-6) as :

$$\frac{1}{u}\frac{d^2u}{dx^2} = -\frac{1}{v}\frac{d^2v}{dy^2} - \frac{1}{w}\frac{d^2w}{dz^2} - \varphi^2 \qquad (1\text{-}7)$$

Now we have a paradox. The LHS of (1-7) depends only on the x-variable while the RHS of (1-7) depends on y & z- variables. One way to avoid this paradox is to say $-U^2_1$. Continuing a similar process, we separate Helmholtz equation into three ordinary differential equations:

$$\frac{1}{u}\frac{d^2u}{dx^2} = -\mu_1^2 \qquad (1\text{-}8)$$

$\frac{1}{v}\frac{d_2 v}{dy^2} = -\mu_2^2$ (1-9)

$\frac{1}{w}\frac{d^2 w}{dz^2} = -\mu_3^2$ (1-10)

Where $\varphi^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$ (1-11)

and the equation (1-3) becomes:

$u(0) = v(0) = w(0) = 0$ (1-12)

$\acute{u}(1) = (\mu_1^2 + \mu_2^2 + \mu_3^2)u(1)$ (1-13)

$\acute{v}(1) = (\mu_1^2 + \mu_2^2 + \mu_3^2)v(1)$ (1-14)

$\acute{w}(1) = (\mu_1^2 + \mu_2^2 + \mu_3^2)w(1)$ (1-15)

We obtain three second order eigenvalue problems:

$\acute{\acute{u}}(x) + \mu_1^2 u(x) = 0$ (1-16)

$u(0) = 0$

$\acute{u}(1) - (\mu_2^2 + \mu_3^2)u(1) = \mu_1^2 u(1)$ (1-17)

and

$\acute{\acute{v}}(y) + \mu_2^2 v(y) = 0$ (1-18)

$v(0) = 0$

$\acute{v}(1) - (\mu_1^2 + \mu_3^2)v(1) = \mu_2^2 v(1)$ (1-19)

And

$\acute{\acute{w}}(z) + \mu_3^2 w(z) = 0$ (1-20)

$w(0) = 0$

$\acute{w}(1) - (\mu_1^2 + \mu_2^2)w(1) = \mu_3^2 w(1)$ (1-21)

This paper presented a study the properties as completeness, minimafity and basis property are investigated for eigenfunction of the spectral problem (1-2)-(1-3) in adequate Hilbert space.

2-An operators formulation in the adequate Hilbert space

Define Adequate Hilbert space H by :

$H=H_1*H_2*H_3$ where

$H_1 = L_{21}(0,1) \oplus c = \{\tilde{u} = (u,a): u \in L_{21}(0,1), a \in c\}$ s.t (c is complex number, v &w are constants, where $L_{21}(0,1)$ is standee).

$H_2 = L_{22}(0,1) \quad c = \{\tilde{v} = (v,b): v \in L_{22}(0,1), b \in c\}$ s.t (u & w constant).

$H_3 = L_{23}(0,1) \quad c = \{\tilde{w} = (w,c): w \in L_{23}(0,1), c \in c\}$ s.t (u & v constant).

and the inner product by :

$\langle \tilde{U}\tilde{V}\tilde{W}, \tilde{U}\tilde{V}\tilde{W} \rangle = \langle \tilde{u},\tilde{u} \rangle + \langle \tilde{v},\tilde{v} \rangle_2 + \langle \tilde{w},\tilde{w} \rangle_3$............(2-1)

$$= \int_0^1 u(x)\bar{u}(x)dx + a\bar{a} + \int_0^1 v(y)\bar{v}(y)dy + b\bar{b} + \int_0^1 w(z)\bar{w}(z)dz + c\bar{c}$$

$$\|.\|^2 = <.,.>_{\cdot 2}$$

$$= \|u\|^2 + |a|^2 + \|v\|^2 + |b|^2 + \|w\|^2 + |c|^2$$

and denote by L_1 , L_2 and L_3 the operators in the H_1 , H_2 and H_3 respectively:

$L_1\tilde{U} = L_1(u,a) = \left(-u^{11}, u^1(1) - (\mu_2^2 + \mu_3^2)u(1)\right)$ $for\ \tilde{u} \in H_1$.....(2-2)

$L_2\tilde{V} = L_2(v,b) = \left(-v^{11}, v^1(1) - (\mu_1^2 + \mu_3^2)v(1)\right)$ $for\ \tilde{v} \in H_2$.....(2-3)

$L_3\widetilde{w} = L_3(w,c) = \left(-w^{11}, w^1(1) - (\mu_1^2 + \mu_2^2)w(1)\right)\ for\ \widetilde{w} \in H_3$...(2-4)

And defined the operator L in adequate Hilbert space H by :

$$L(\tilde{u}\tilde{v}\widetilde{w}) = \begin{bmatrix} vwL_1u \\ vwL_1a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} uwL_2v \\ 0 \\ uwL_2b \\ 0 \end{bmatrix} + \begin{bmatrix} uvL_3w \\ 0 \\ 0 \\ uvL_3c \end{bmatrix} \text{...........(2-5)}$$

for $\tilde{u}, \tilde{v}, \&\ \widetilde{w} \in H$.

and its domains D(L_1), D(L_2), D(L_3) and D(L) of all elements $(u,a)\ (v,b)\ (w,c) \in H$ satisfying the conditions:

1. D(L)=D(L_1)*D(L_2)*D(L_3).
2. u , u^1 , v , v^1 and w, w^1 absolutely continuous on (0,1)
3. a=u(1)
4. b=v(1)
5. c=w(1)
6. u(0)=v(0)=w(0)=0

we can easily obtains the boundary problems (1-16)-(1-17),(1-18)-(1-19) and (1-20)-(1-21) are equivalent to the spectral problems:

$L_1\widetilde{U} = \mu_1^2\widetilde{U}\ \ for\ \ \tilde{u} \in H_1$.......(2-6)

$L_2\widetilde{V} = \mu_2^2\widetilde{V}\ \ for\ \ \tilde{v} \in H_2$........(2-7)

$L_3\widetilde{W} = \mu_3^2\widetilde{W}\ \ for\ \ \widetilde{w} \in H_3$......(2-8)

Remark 2.1: the operator L (2-5) describes the eigenvalue problem (1-2) ,(1-3).

Lemma 2.2: The domains D(L_1) , D(L_2) and D(L_3) are dense in the spaces H_1 , H_2 and H_3 respectively.

proof:[3]

lemma 2.3: the eigenvalues $\mu_{01}^2\ ,\mu_{02}^2\ and\ \mu_{03}^2$ of the problems(1-16)-(1-17) , (1-18)-(1-19) and (1-20)-(1-21) respectively, with multiplicity coincide with eigenvalues of the operators L_1 L_2 and L_3 resp.

proof : let $\widetilde{u_\iota} \in D(L_1)\ ,\widetilde{v_\iota} \in D(L_2)\ and\ \widetilde{w_\iota} \in D(L_3)$ be the eigenfunctions corresponding to the eigenvalues $\mu_{01}^2\ ,\mu_{02}^2\ ,and\ \mu_{03}^2$ of the operators L_1, L_2 and L_3 resp. then:

$L_1\widetilde{U_\iota} = \mu_{01}^2\widetilde{U_\iota}$

$L_2\widetilde{V_\iota} = \mu_{02}^2\widetilde{V_\iota}$

$L_3\widetilde{W_\iota} = \mu_{03}^2\widetilde{W_\iota}$

$\left(-\acute{u_\iota}, u_i^1(1) - (\mu_{02}^2 + \mu_{03}^2)u_i(1)\right) = \mu_{01}^2(u_i, u_i(1))$

$\left(-v_i^{11}, v_i^1(1) - (\mu_{01}^2 + \mu_{03}^2)v_i(1)\right) = \mu_{02}^2(v_i, v_i(1))$

$\left(-w_i^{11}, w_i^1(1) - (\mu_{01}^2 + \mu_{02}^2)w_i(1)\right) = \mu_{03}^2(w_i, w_i(1))$

We can obtain

$-u_i^{11} = \mu_{01}^2 u$ and $u_i^1(1) - (\mu_{02}^2 + \mu_{03}^2)u_i(1) = \mu_{01}^2 u_i(1)$

$-v_i^{11} = \mu_{01}^2 v_i$ and $v_i^1(1) - (\mu_{01}^2 + \mu_{03}^2)v_i(1) = \mu_{02}^2 v_i(1)$

$-w_i^{11} = \mu_{03}^2 w_i$ and $w_i^1(1) - (\mu_{01}^2 + \mu_{02}^2)w_i(1) = \mu_{03}^2 w_i(1)$

Then

$u_i^{11} + \mu_{01}^2 u_i = 0$

$u_i(0) = 0$

$u_i^1(1) = (\mu_{01}^2 + \mu_{02}^2 + \mu_{03}^2)u_i(1)$

and

$v_i^{11} + \mu_{02}^2 v_i = 0$

$v_i(0) = 0$

$v_i^1(1) = (\mu_{01}^2 + \mu_{02}^2 + \mu_{03}^2)v_i(1)$

and

$w_i^{11} + \mu_{03}^2 w_i = 0$

$w_i(0) = 0$

$w_i^1(1) = (\mu_{01}^2 + \mu_{02}^2 + \mu_{03}^2)w_i(1)$

The lemma is proved .

Lemma 2.4: the eigenvalues $\varphi_0^2 = \mu_{01}^2 + \mu_{02}^2 + \mu_{03}^2$ of the problem (1-2)- (1-3) with multiplicity coincide with the eigenvalues of the operator L. the similar is true for the associated functions.

Proof : let $\widetilde{u_\iota}\widetilde{v_\iota}\widetilde{w_\iota} \in D(L)$ the eigenfunctions corresponding to the eigenvalues φ_0^2 of the operator L then

$$L\widetilde{u_\iota}\widetilde{v_\iota}\widetilde{w_\iota} = \varphi_0^2\widetilde{u_\iota}\widetilde{v_\iota}\widetilde{w_\iota}$$

$$\begin{bmatrix} v_i w_i L_1 u_i \\ v_i w_i L_1 a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_i w_i L_2 v_i \\ 0 \\ u_i w_i L_2 b \\ 0 \end{bmatrix} + \begin{bmatrix} u_i v_i L_3 w_i \\ 0 \\ 0 \\ u_i v_i L_3 c \end{bmatrix} =$$

$$\begin{bmatrix} \mu_{01}^2 u_i v_i w_i \\ \varphi_0^2 u_i(1) v_i w_i \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_{02}^2 u_i v_i w_i \\ 0 \\ \varphi_0^2 u_i v_i(1) w_i \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_{03}^2 u_i v_i w_i \\ 0 \\ 0 \\ \varphi_0^2 u_i v_i w_i(1) \end{bmatrix}$$

$$\begin{bmatrix} -u_i^{11} v_i w_i \\ v_i w_i u_i^1(1) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -u_i v_i^{11} w_i \\ 0 \\ u_i w_i v_i^1 \\ 0 \end{bmatrix} + \begin{bmatrix} -u_i v_i w_i^{11} \\ 0 \\ 0 \\ u_i v_i w_i^1 \end{bmatrix}$$

$$= \begin{bmatrix} \mu_{01}^2 u_i v_i w_i \\ \varphi_0^2 u_i(1) v_i w_i \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_{02}^2 u_i v_i w_i \\ 0 \\ \varphi_0^2 u_i v_i(1) w_i \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_{03}^2 u_i v_i w_i \\ 0 \\ 0 \\ \varphi_0^2 u_i v_i w_i(1) \end{bmatrix}$$

$$\begin{bmatrix} -u_i^{11} v_i w_i - u_i v_i^{11} w_i - u_i v_i w_i^{11} \\ v_i w_i u_i^1(1) \\ u_i v_i^1(1) w_i \\ u_i v_i w_i^1(1) \end{bmatrix} = \begin{bmatrix} (\mu_{01}^2 + \mu_{02}^2 + \mu_{03}^2) u_i v_i w_i \\ \varphi_0^2 u_i(1) v_i w_i \\ \varphi_0^2 u_i v_i(1) w_i \\ \varphi_0^2 u_i v_i w_i(1) \end{bmatrix}$$

We can obtain

$$u_i^{11} v_i w_i + u_i v_i^{11} w_i + u_i v_i w_i^{11} = -\varphi_0^2 u_i v_i w_i$$

$$v_i w_i u_i^1(1) = \varphi_0^2 u_i(1) v_i w_i$$

$$u_i v_i^1(1) w_i = \varphi_0^2 u_i v_i(1) w_i$$

$$u_i v_i w_i^1(1) = \varphi_0^2 u_i v_i w_i(1)$$

Then

$$\omega_{ixx} + \omega_{iyy} + \omega_{izz} + \varphi_0^2 \omega_i = 0$$

$$\omega_{ix}(1,y,z) = \varphi_0^2 \omega_i(1,y,z)$$

$$\omega_{iy}(x,1,z) = \varphi_0^2 \omega_i(x,1,z)$$

$$\omega_{iz}(x,y,1) = \varphi_0^2 \omega_i(x,y,1)$$

The lemma is proved.

Lemma 2.5 : the operators L_1 , L_2 and L_3 are semi-bounded from below in spaces H_1 , H_2 and H_3 resp.

Proof: [3]

Lemma 2.6 : the operators L_1 , L_2 and L_3 are invertible if and only if $\mu_1^2 = 0$, $\mu_2^2 = 0$ *and* $\mu_3^2 = 0$ are not eigenvalues of L_1 , L_2 and L_3 resp.

Proof: [4]

Lemma 2.7 : there are unboundedly increasing sequences are $\{\mu_{1n}\}_0^\infty$,$\{\mu_{2n}\}_0^\infty$ *and* $\{\mu_{3n}\}_0^\infty$ of eigenvalues of the boundary value problems (1-16)-(1-17),(1-18)-(1-19) and (1-20)-(1-21) respectively:

$$\mu_{11}^2 < \mu_{12}^2 < \mu_{13}^2 < \cdots < \mu_{1n}^2 < \cdots \qquad (2\text{-}9)$$

$$\mu_{21}^2 < \mu_{22}^2 < \mu_{23}^2 < \cdots < \mu_{2n}^2 < \cdots \qquad (2\text{-}10)$$

and

$$\mu_{31}^2 < \mu_{32}^2 < \mu_{33}^2 < \cdots < \mu_{3n}^2 < \cdots \qquad (2\text{-}11)$$

Moreover, the eigenfunctions $U_n(x)$, $V_n(y)$,and $W_n(z)$ corresponding to μ_{1n}^2 , μ_{2n}^2 *and* μ_{3n}^2 respectively, has exactly n simple zeros in the interval [0,1].

Proof:[5,12]

Remarks 2.8:

1. since D(L)=D(L_1)*D(L_2)*D(L_3) and D(L_1) , D(L_2) and D(L_3) dense in H_1 , H_2 and H_3 resp., then D(L) is dense in H.

2. since the operators L_1 , L_2 and L_3 are semi-bounded from below then L is semi-bounded from below in H.

3. addition the equations (2.9)-(2.10) and (2.11) we get: $(\mu_{11}^2 + \mu_{21}^2 + \mu_{31}^2) < (\mu_{12}^2 + \mu_{22}^2 + \mu_{32}^2) < \cdots < (\mu_{1n}^2 + \mu_{2n}^2 + \mu_{3n}^2) < \cdots$ (2-13)

then $\varphi_1^2 < \varphi_2^2 < \cdots < \varphi_n^2 < \cdots$ (2-14)

there is an unboundedly increasing sequence $\{\varphi_n^2\}_0^\infty$ of eigenvalues of the Helmholtz problem (1-2)-(1-3).

Lemma 2.9:

1. the operator L_1 is symmetric with respect to H_1.

2. the operator L_2 is symmetric with respect to H_2.

3. the operator L_3 is symmetric with respect to H_3

proof:

1. let $\tilde{u}_1$ *and* $\tilde{u}_2 \in D(L_1)$

$$\begin{aligned}\langle L_1\tilde{u}_1, \tilde{u}_2\rangle_1 &= \langle L_1(u_1, a_1), (u_2, a_2)\rangle_1 = \langle L_1(u_1, u_1(1)), (u_2, u_2(1))\rangle_1 \\ &= \langle (-u_1^{11}, u_1^1(1) - (\mu_2^2 + \mu_3^2)u_1(1)), (u_2, u_2(1))\rangle_1 \\ &= -\int_0^1 u_1^{11}(x)\bar{u}_2(x)dx + [u_1^1(1) - (\mu_2^2 + \mu_3^2)u_1]\bar{u}_2(1)\end{aligned}$$

Using two times the integration by parts and the equations (1-17),(1-19) and (1-21) we obtain:

$$\begin{aligned}\langle L\tilde{u}_1\tilde{v}_1\tilde{w}_1, \tilde{u}_2\tilde{v}_2\tilde{w}_2\rangle &= \int_0^1 u_1(x)\bar{u}_2^{11}(x)dx + [\bar{u}_2^1(1) - (\mu_2^2 + \mu_3^2)\bar{u}_2(1)][u_1(1)] \\ &= \langle (u_1, u_1(1), (-u_2^{11}, u_2^1(1) - (\mu_2^2 + \mu_3^2)u_2(1))\rangle_1 \\ &= \langle (u_1, u_1(1), L_1(u_2, u_2(1))\rangle_1 \\ &= \langle (u_1, a_1), L_1(u_2, a_2)\rangle_1 \\ &= \langle \tilde{u}_1, L_1\tilde{u}_2\rangle_1\end{aligned}$$

The operator L_1 is symmetric with respect to $<.,.>$.

The proofs of 2 & 3 are similar to the proof of 1.

Remark 2.10 : the operator L is symmetric with respect to $<.,.>$ in H.

Lemma 2.11: the operators $(L_1 - \mu_1^2 I)^{-1}, (L_2 - \mu_2^2 I)^{-1}$ *and* $(L_3 - \mu_3^2 I)^{-1}$ where I is the unit operator) are compact if $\mu_1^2, \mu_2^2 and \mu_3^2$ are not eigenvalues of L_1 , L_2 and L_3 resp.

Proof: [6,9]

Remark 2.12: since $\mu_1^2, \mu_2^2 and\ \mu_3^2$ are not eigenvalues of L_1 , L_2 and L_3 resp, then $\varphi^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$ is not eigenvalue of the operator L then

is compact in H.

3-Greens Functions of the operators L_1 , L_2 and L_3

_The solutions of the equations (1.16) , (1.18) and (1-20) are given by the functions in the form:

$$U(x) = c_1 cos\mu_1 x + c_2 sin\mu_1 x \quad (3\text{-}1)$$

$$V(y) = c_3 cos\mu_2 y + c_4 sin\mu_2 y \quad (3\text{-}2)$$

$$W(z) = c_5 cos\mu_3 z + c_6 sin\mu_3 z \quad (3\text{-}3)$$

Where c_1,c_2,c_3,c_4 and c_5,c_6 are constants. Let $U_1(x)$ and $U_2(x)$ two solutions of the equation (1-16) such that μ_1^2 is a not eigenvalue of L_1 and satisfying the initial conditions:

$U_1(0) = 0$

$U_1^1(0) = -1$

$U_2(1) = 1$

$U_2^1 = (\mu_1^2 + \mu_2^2 + \mu_3^2)$

Then

$U_1(x) = -\frac{1}{\mu_1} sin\mu_1 x \qquad \mu_1 \neq 0$

$U_2(x) = \frac{1}{\mu_1}[\mu_1 cos\mu_1 - (\mu_1^2 + \mu_2^2 + \mu_3^2) sin\mu_1] cos\mu_1 x + \frac{1}{\mu_1}[\mu_1 sin\mu_1 + (\mu_1^2 + \mu_2^2 + \mu_3^2) cos\mu_1] sin\mu_1 x$

(3-5)

And the solutions of the equations (1-18) and (1-20) ($V_1(y),V_2(y)$ and $W_1(z),W_2(z)$ resp.) are similar to the solution (1-16) , $U_1(x),U_2(x)$ we get

$V_1(y) = -\frac{1}{\mu_2} sin\mu_2 y \qquad \mu_2 \neq 0$ (3-6)

$V_2(y) = \frac{1}{\mu_2}[\mu_2 cos\mu_2 - (\mu_1^2 + \mu_2^2 + \mu_3^2) sin\mu_2] \cos\mu_2 y + \frac{1}{\mu_2}[\mu_2 sin\mu_2 + (\mu_1^2 + \mu_2^2 + \mu_3^2) cos\mu_2] sin\mu_2 y$

(3-7)

And

$W_1(z) = -\frac{1}{\mu_3} sin\mu_3 z \qquad \mu_3 \neq 0$ (3-8)

$W_2(z) = \frac{1}{\mu_3}[\mu_3 cos\mu_3 - (\mu_1^2 + \mu_2^2 + \mu_3^2) sin\mu_3] \cos\mu_3 z + \frac{1}{\mu_3}[\mu_3 sin\mu_3 + (\mu_1^2 + \mu_2^2 + \mu_3^2) cos\mu_3] sin\mu_3 z$

(3-9)

That can be observed

$W(u_1,u_2,0) \neq 0, W(v_1,v_2,0) \neq 0 and\ W(w_1,w_2,0) \neq 0$

For all $x, y\ and\ z\ \in [0,1]$ where W is the wronskian determinate. and this means u_1,u_2 and v_1,v_2 and w_1,w_2 linearly independent.

The Greens functions of the operators L_1,L_2 and L_3 such that $\mu_1^2, \mu_2^2\ and\ \mu_3^2$ are not eigenvalues of L_1,L_2 and L_3 resp., are given by the functions in the form:

$$G_1(x,t,\mu_1^2) = \begin{cases} u_1(t)u_2(x) & 0 \le t < x \le 1 \\ u_2(t)u_1(x) & 0 \le x < t \le 1 \end{cases} \tag{3-10}$$

$$G_2(y,t,\mu_2^2) = \begin{cases} v_1(t)v_2(y) & 0 \le t < y \le 1 \\ v_2(t)v_1(y) & 0 \le y < t \le 1 \end{cases} \tag{3-11}$$

$$G_3(z,t,\mu_3^2) = \begin{cases} w_1(t)w_2(z) & 0 \le t < z \le 1 \\ w_2(t)w_1(z) & 0 \le z < t \le 1 \end{cases} \tag{3-12}$$

Where u_1,u_2 and v_1,v_2 and w_1,w_2 defined in the equations (3-4)-(3-5), (3-6)-(3-7) and (3-8)-(3-9).

Theorem 3.1: the operators L_1,L_2 ,L_3 and L are self-adjoint in the spaces H_1 , H_2 ,H_3 and H, resp.

Proof: from lemma (2-9) and remark (2-10) the operators L_1,L_2 ,L_3 and L

$$(L_1 - \mu_1^2)^{-1} H_1 = D(L_1)$$

$$(L_2 - \mu_2^2)^{-1} \mathrm{H}_2 = \mathrm{D}(\mathrm{L}_2)$$

$$(L_3 - \mu_3^2)^{-1} \mathrm{H}_3 = \mathrm{D}(\mathrm{L}_3)$$

$$(L - \varphi^2 I)^{-1} H = \mathrm{D}(L)$$

Let $\tilde{u} = (u(x), a) \in D(L_1)$ and satisfying :

$(L_1 - \mu_1^2)^{-1} \tilde{U} = F_1$ where $F_1=(f_1(x),f_{11})\in H_1$ and μ_1^2 is a not eigenvalue of L_1

Let $\tilde{v} = (v(y), b) \in D(L_2)$ and satisfying:

$(L_2 - \mu_2^2)^{-1} \tilde{V} = F_2$ where $F_2(f_2(y), f_{21}) \in H_2$ and μ_2^2 is a not eigenvalue of L_2.

$\widetilde{w} = (w(z), c) \in D(L_3)$ and satisfying:

$$(L_3 - \mu_3^2)^{-1} \widetilde{W} = F_3 \tag{3-19}$$

where $F_3(f_3(z), f_{31}) \in H_3$ and μ_3^2 is a not eigenvalue of L_3.

Then $\widetilde{\omega} = (\omega, \omega_0) \in D(L)\ and\ \widetilde{\omega} = \tilde{u}\tilde{v}\widetilde{w}\ and\ \omega_0 = abc$ and satisfying:

$$(L - \varphi^2 I)\ \ \widetilde{\omega} = F \tag{3-20}$$

Where $F=F_1F_2F_3\in$ H and $\varphi^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$ is a not eigenvalue of L.

The equations (3-17),(3-18) and (3-19) are nonhomogeneous differential equations have solutions are given by:[7]

$$U(x) = k_1 u_1(x) + k_2 u_2(x) + \int_0^1 G_1(x,t,\mu_1^2) f_1(t) dt \tag{3-21}$$

a=u(1)

and

$$V(x) = k_3 v_1(y) + k_4 v_2(y) + \int_0^1 G_2(y,t,\mu_2^2) f_2(t) dt \tag{3-22}$$

b=v(1)

and

$W(z) = k_5 w_1(z) + k_6 w_2(z) + \int_0^1 G_3(z,t,\mu_3^2) f_3(t)dt$ (3-23)

c=w(1)

where k_1,k_2,k_3,k_4 and k_5,k_6 are constants and $G_1(x,t,\mu_1^2)$, $G_2(y,t,\mu_2^2)$ and $G_3(z,t,\mu_3^2)$ defined in equations (3-10),(3-11) and (3-12).

To find the constants we obtain:

$u^{11}(x) - \mu_1^2 u(x) = f_1(x)$ (3-24)

u(0)=0

$U^1(1) - (\mu_1^2 + \mu_2^2 + \mu_3^2)U(1) = f_{11}$

and

$V^{11}(y) - \mu_2^2 v(y) = f_2(y)$ (3-25)

V(0)=0

$V^1(1) - (\mu_1^2 + \mu_2^2 + \mu_3^2)V(1) = f_{21}$

and

$W^{11}(z) - \mu_3^2 w(z) = f_3(z)$ (3-26)

$W^1(1) - (\mu_1^2 + \mu_2^2 + \mu_3^2)W(1) = f_{31}$

Then

$U(x) = -\frac{f_{11} \sin\mu_1 x}{c_1\mu_1} + \int_0^1 G_1(x,t,\mu_1^2) f_1(t)dt \qquad c_1\mu_1 \neq 0$ (3-27)

a=u(1)

and

$V(x) = -\frac{f_{21} \sin\mu_2 y}{c_2\mu_2} + \int_0^1 G_2(y,t,\mu_2^2) f_2(t)dt \qquad c_2\mu_2 \neq 0$ (3-28)

b=v(1)

and

$W(z) = -\frac{f_{31} \sin\mu_3 z}{c_3\mu_3} + \int_0^1 G_3(z,t,\mu_3^2) f_3(t)dt \qquad c_3\mu_3 \neq 0$ (3-29)

c=w(1)

from [4]:

$\tilde{u} \in (L_1 - \mu_1^2 I)^{-1} H_1$

$\tilde{v} \in (L_2 - \mu_2^2 I)^{-1} H_2$

$\tilde{w} \in (L_3 - \mu_3^2 I)^{-1} H_3$

Then

$D(L_1) \quad (L_1 - \mu_1^2 I)^{-1} H_1$ (3-30)

$D(L_2) \quad (L_2 - \mu_2^2 I)^{-1} H_2$ (3-31)

$D(L_3) \quad (L_3 - \mu_3^2 I)^{-1} H_3$ (3-32)

Since $\mu_1^2, \mu_2^2 \, and \, \mu_3^2$ are not eigenvalues of L_1, L_2 and L_3, for all $F_1 = (f_1(x), f_{11}) \in H_1$, $F_2 = (f_2(y), f_{21}) \in H_2$ and $F_3 = (f_3(z), f_{31}) \in H_3$, there exist $\tilde{U} = (u(x), a)$,$\tilde{V} = (v(y), b) \; and \; \tilde{W} = (w(z), c)$ such that

$(L_1 - \mu_1^2) \quad \tilde{U} = F_1$

$(L_2 - \mu_2^2) \quad \tilde{V} = F_2$

$(L_3 - \mu_3^2)\quad \widetilde{W} = F_3$

We obtain u, u^1, v, v^1 and w, w^1 absolutely continuous on [0,1] and u(0)=v(0)=w(0)=0

Then

$\tilde{U} = \big(u(x), u(1)\big) \in D(L_1)$

$\tilde{V} = \big(v(y), v(1)\big) \in D(L_2)$

$\widetilde{W} = \big(w(z), w(1)\big) \in D(L_3)$

From [4]:

$\tilde{U} = (L_1 - \mu_1^2 I)^{-1} F_1$

$\tilde{V} = (L_2 - \mu_2^2 I)^{-1} F_2$

$\widetilde{W} = (L_3 - \mu_3^2 I)^{-1} F_3$

Then

$(L_1 - \mu_1^2 I)^{-1} F_1 \in D(L_1)$

$(L_2 - \mu_2^2 I)^{-1} F_2 \in D(L_2)$

$(L_3 - \mu_3^2 I)^{-1} F_3 \in D(L_3)$

So

$(L_1 - \mu_1^2 I)^{-1} H_1 \quad D(L_1)$ (3-33)

$(L_2 - \mu_2^2 I)^{-1} H_2 \quad D(L_2)$ (3-34)

$(L_3 - \mu_3^2 I)^{-1} H_3 \quad D(L_3)$ (3-35)

From (3-30)-(3-33) , (3-31)-(3-34) and (3-32)-(3-35) we get

$(L_1 - \mu_1^2 I)^{-1} H_1 = D(L_1)$ (3-36)

$(L_2 - \mu_2^2 I)^{-1} H_2 = D(L_2)$ (3-37)

$(L_3 - \mu_3^2 I)^{-1} H_3 = D(L_3)$ (3-38)

The equations (3-20) is a nonhomogeneous differential equation has solution:

$\tilde{\omega} = \tilde{u}\tilde{v}\tilde{w} = (u(x), a)(v(y), b)(w(z), c) = \Big[\big(\frac{-f_{11}\sin\mu_1 x}{c_1\mu_1} + \int_0^1 G(x,t,\mu_1^2) f_1(t)\,dt, U(1)\big) * \big(\frac{-f_{21}\sin\mu_2 y}{c_2\mu_2} + \int_0^1 G_2(y,t,\mu_2^2) f_2(t)\,dt, V(1) * \big(\frac{-f_{31}\sin\mu_3 z}{c_3\mu_3} + \int_0^1 G(z,t,\mu_3^2) f_3(t)\,dt, W(1)\big)\Big]$

$c_1\mu_1 \neq 0 \,,\; c_2\mu_2 \neq 0$

From [4]

$\tilde{\omega} \in (L - \varphi^2 I)^{-1} H$

Then

$D(L) \quad (L_1 - \mu_1^2 I)^{-1} H$ (3-39)

Since φ^2 a not eigenvalue of L , for all $F = F_1F_2F_3 = (f_1(x), f_{11})\,(f_2(y), f_{21})\,(f_3(z), f_{31}) \in H$ such that

$(L - \varphi^2 I)\tilde{\omega} = F$

We obtain $\omega, \dot{\omega}$ absolutely continuous on [0,1] and $\omega(0) = u(0)v(0)w(0) = 0$

Then

$\tilde{\omega} = (\omega(x,y,z), \omega_0) \in D(L)$

From [4]:

$\tilde{\omega} = (L - \varphi^2 I)^{-1} F$

Then

$(L - \varphi^2 I)^{-1} F \in D(L)$

So

$(L-\varphi^2 I)^{-1}F \quad D(L)$ (3-40)

From (3-39)-(3-40) we get

$(L-\varphi^2 I)^{-1}H = D(L)$

The lemma is proved.

4-The Basis Property of eigenfunctions

Theorem 4.1: The eigenfunction of the operators L_1, L_2, L_3 and L form orthogonal basis in the spaces H_1, H_2, H_3 and H resp.

Proof:

The eigenvalues of the boundary problems (1-16)-(1-17) , (1-18)-(1-19) and (1-20)-(1-21) we must find the intersection of the curves:

1. $\tan\mu_1 \ and \ \frac{\mu_1}{\mu_1^2+(\mu_2^2+\mu_3^2)}$, $\mu_1^2+(\mu_2^2+\mu_3^2)\neq 0$ (figure 1)

2. $\tan\mu_2 \ and \ \frac{\mu_2}{\mu_2^2+(\mu_1^2+\mu_3^2)}$, $\mu_2^2+(\mu_1^2+\mu_3^2)\neq 0$ (figure 2)

3. $\tan\mu_3 \ and \ \frac{\mu_3}{\mu_3^2+(\mu_1^2+\mu_2^2)}$, $\mu_3^2+(\mu_1^2+\mu_2^2)\neq 0$ (figure 3)

It can be easily obtained that the operators L_1, L_2 and L_3 have at most countable eigenvalues $\mu_{1n}^2, \mu_{2n}^2 \ and \ \mu_{3n}^2$ which have the asymptotic from:

$\mu_{1k}^2 = (k\pi)^2 + o\left(\frac{1}{k^2}\right) \quad as \ k\to\infty$ (4-1)

$\mu_{2m}^2 = (m\pi)^2 + o\left(\frac{1}{m^2}\right) \quad as \ m\to\infty$ (4-2)

$\mu_{3n}^2 = (n\pi)^2 + o\left(\frac{1}{n^2}\right) \quad as \ n\to\infty$ (4-3)

Then

$\varphi_j^2 = \mu_{1k}^2 + \mu_{2m}^2 + \mu_{3n}^2$

$\varphi_j^2 = (k\pi)^2 + (m\pi)^2 + (n\pi)^2 + o\left(\frac{1}{k^2 m^2 n^2}\right)$ (4-4)

The operator L has at most countable eigenvalues φ_j which have the asymptotic from in equation (4-4) . From the lemmas (2-5),(2-11) and theorem(3.1), the operators L_1, L_2, L_3 and L : compact , selfadjoint and bounded. Applying the Hilbert-Schmidt theorem [8,11], to operators L_1, L_2, L_3 and L we obtain that the eigenfunctions of this operators from an orthogonal basis in the spaces H_1, H_2, H_3 and H resp.

Theorem 4.2: let k_0, m_0 and n_0 be an arbitrary fixed nonnegative integers. The systems of the eigen-functions: $\{U_k\}_0^\infty \ (k\neq k_0), \{V_m\}_0^\infty \ (m\neq m_0) \ and \{W_n\}_0^\infty \ (n\neq n_0)$, of the boundary problems (1-16)-(1-17) , (1-18)-(1-19) and (1-20)-(1-21) resp., are complete and minimal systems.

Proof: according to theorem(4.1) the eigenfunctions:

$\tilde{U}_k(x) = (u_k(x), a)$

$\tilde{V}_k(y) = (v_k(y), b)$

$\tilde{W}_k(z) = (w_k(z), c)$

Of the boundary problems (1-16)-(1-17) , (1-18)-(1-19) and (1-20)-(1-21) from a basis in :

$H_1 = L_{21} \oplus c$

$H_2 = L_{22} \oplus c$

$H_3 = L_{23} \oplus c$

So, the system: $\{U_k(x)\}_0^\infty$,$\{V_m(y)\}_0^\infty$ *and* $\{W(z)_n\}_0^\infty$ are complete and minimal in the spaces H_1,H_2 and H_3 resp. we denote by P_1,P_2 and P_3 the orthoprojection which is defined by the formulas:

$P_1\tilde{U}_k(x) = U_k(x)$ in H_1

$P_2\tilde{V}_m(x) = U_m(y)$ in H_2

$P_3\tilde{W}_n(x) = U_n(z)$ in H_3

Thus , of course, codim P_1=codim P_2=codim P_3=1. Then, by[10] , the systems:

$\{P_1\tilde{U}_k(x)\}_0^\infty = \{U_k(x)\}_0^\infty$

$\{P_2\tilde{V}_m(y)\}_0^\infty = \{V_m(y)\}_0^\infty$

$\{P_3\tilde{W}_n(z)\}_0^\infty = \{W_n(z)\}_0^\infty$

Whose one element is omined from forms a complete and minimal systems in:

$H_1P_1 = P_1(H_1) = L_{21}(0,1)$

$H_2P_2 = P_2(H_2) = L_{22}(0,1)$

$H_3P_3 = P_3(H_3) = L_{23}(0,1)$

Hence, the eigenfunctions: $\{U_k(x)\}_0^\infty$,$\{V_m(y)\}_0^\infty$ *and* $\{W(z)_n\}_0^\infty$ of the boundary problems (1-16)-(1-17) , (1-18)-(1-19) and (1-20)-(1-21) resp., are complete and minimal in $L_{21}(0,1)$, $L_{22}(0,1)$ and $L_{23}(0,1)$.

Remark 4.3:

According to theorem(4.1) the eigenfunctions of Helemholtz problem (1-2)-(1-3):

$\tilde{\omega}_j(x,y,z) = (\tilde{\omega}_j(x,y,z), \omega_0)$

$\tilde{\omega}_j(x,y,z) = \tilde{\omega}_{kmn}(x,y,z) = \tilde{U}_k(x)\tilde{V}_m(y)\tilde{W}_n(z) = (U_k(x),a)(V_m(y),b)(W_n(z),c)$

From basis in H. so, the systems $\{\omega_j(x,y,z)\}_0^\infty$ are complete and minimal in H.

We denote by $P=P_1P_2P_3$ the orthoprojection which is defined by the formula :

$P\tilde{\omega}_j(x,y,z) = \omega_j(x,y,z)$ in H.

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