# Approximation of Unbounded Functions by Trigonometric Polynomials

**Alaa A. Auad[1] and Mousa M. Khrajan[2]**

[1]University of Al-Anbar -Faculty of Education of Pure Science- Department of Mathematics.

[2]University of Al-Muthana- Science College- Department of Mathematics and Computer Applications.

[1]alaa.adnan66.aa@gmail.com
[2]mmkrady@gmail.com

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## Abstract

The purpose of this paper is present a brief survey of know on estimates the rate for best approximation of unbounded functions by suitable trigonometric polynomials of one variable in weighted space $L_{p,\alpha}(X)$. Moreover we studied concerning the degree of best trigonometric approximation of $f^{(k)}$ with $k$ non-integer in $L_{p,\alpha}(X)$.

**Keywords:** Unbounded functions, trigonometric polynomials, weight space.

# تقريب الدوال الغير مقيدة بواسطة متعددات الحدود المثلثية

**علاء عدنان عواد[1] و موسى مكي خريجان[2]**

[1]جامعة الانبار – كلية التربية للعلوم الصرفة – قسم الرياضيات
[2]جامعة المثنى – كلية العلوم – قسم الرياضيات وتطبيقات الحاسوب

## الخلاصة

الغرض من هذا البحث هو عرض دراسة موجزة لمعرفة تخمين درجة افضل تقريب للدوال الغير مقيدة في الفضاء الموزون بواسطة متعددات الحدود المثلثية ذات المتغير الواحد. اضافة الى ذلك المبرهنات الثالثة والرابعة تتعلق بالتقريب المثلثي للدوال $f^{(k)}$ في فضاء الموزون حيث ان k عدد غير صحيح.

**الكلمات المفتاحية :** الدوال الغير مقيدة ، متعددات الحدود المثلثية ، فضاء الوزن

## Introduction

In 1977 Taberski R. see [10 ] and in 1979 Andreev, Popov, Sendov see [2] introduced the best approximation of functions by trigonometric polynomials of fractional order. This notion could considered as direct generalization of classical trigonometric approximation in weighted space $L_{p,\alpha}(X)$ and it is more natural to use it for number of problems of approximation functions ( see, for example [3],[5],[6], [9]). The important problem of approximation theory and theory of Fourier series is the problem of description of best approximation of functions by trigonometric polynomials see [7] and [10].One can consider this problem from the viewpoint of description of Roman Taberski of trigonometric approximation. In this paper, we obtain the description of some estimates for the best trigonometric approximation of unbounded functions in $L_{p,\alpha}(X)$ and the order approximation of $f^{(k)}$ by suitable trigonometric polynomials.

Let $X = [-\pi, \pi]$, we denote the $L_p(X)$-norm $(1 \le p < \infty)$,

$$\|f\|_p = (\int_X |f(x)|^p \; dx)^{\frac{1}{p}} < \infty \quad \dots\dots\dots\dots\dots\dots(1)$$

and define for suitable $W(\alpha, x)$ of all weight function on closed interval $X$ such that $|f(x)| \le M\alpha(x)$, where $M$ is positive real number and $\alpha: X \to \mathbb{R}^+$ weight function.

We shall denote $L_{p,\alpha}(X)$ the space of all unbounded functions on $X$ which are equipped with the following norm

$$\|f\|_{p,\alpha} = (\int_X \left|\frac{f(x)}{\alpha(x)}\right|^p dx)^{\frac{1}{p}} < \infty \quad \dots\dots\dots\dots\dots\dots(2).$$

Let $H_n$ be the set of all $2\pi -$ periodic trigonometric polynomials of order less than or equal $n, n \in \mathbb{N} = \{1,2,3, \dots\}$.

The degree of best trigonometric approximation of an arbitrary function in weighted space $L_{p,\alpha}(X)$ is define by

$$E_n(f)_{p,\alpha} = \inf \{\|f - t_n\|_{p,\alpha} \;,\; t_n \in H_n\} \quad \dots\dots\dots\dots\dots\dots(3).$$

Denote by $\omega(f, \delta)_{p,\alpha}$ the modulus of continuity with respect to the $L_{p,\alpha}$ $-$norm i.e.

$$\omega(f,\delta)_{p,\alpha} = \underbrace{sup}_{h\in[0,\delta]} \|\Delta_h^1 f\|_{p,\alpha} \quad (0 \le \delta < \infty) \quad \ldots\ldots\ldots\ldots\ldots(4)$$

where $\Delta_h^1 f(x) = |f(x) - f(x+h)|$ .

Suppose that $F(f) = \sum_{m=-\infty}^{\infty} C_m \frac{e^{imx}}{\alpha(x)}$ ……………….(5)

is the Fourier series of $f \in L_{p,\alpha}(X)$ for which the integral over $X$ is zero so that $C_0 = 0$. Given any $a > 0$, we define the a-th integers of $f$ by the identity

$$I_a(f,x) = \sum_{m=-\infty}^{\infty} \frac{C_m}{(im)^a} \frac{e^{imx}}{\alpha(x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where $\frac{1}{(im)^a} = |m|^{-a} \exp(-\frac{1}{2}\pi ia\ sign\ m)$ .

As is well known [12] for $f_a(x) = I_a(f,x)$ exist possibly for almost every $x$, is Lebesgue-integrable and $F[f_a] = f_a(x)$ a.e.

In this case for $a > 0$, the convolution

$$f_a(x) = (f * I_a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-u)I_a(u)}{\alpha(x)} du \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

is of class $L_{p,\alpha}(X)$.

In several cases, the last Fourier series converges for every or a.e. $x$, its sum $f^{(-a)}(x) \equiv I_a[\,f\,](x)$ see [12].

If $0 < a < 1$, the derivative $f(x)$ of $f$ is defined by the formula

$$f^{(a)}(x) = \frac{d}{dx} I_{1-a}(f,x) \quad \ldots\ldots\ldots\ldots\ldots(7)$$

provided the right said exists. We set

$$f^{(a+k)}(x) = (f^{(a)}(x))^k = \frac{d^{k+1}}{dx^{k+1}} I_{1-a}(f,x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

For positive integer $k$.

Let $f$ be a complex-valued function defined in the closed interval $[-\pi,\pi]$. Then

$$V_{p,\alpha}(f,-\pi,\pi) = \underbrace{sup}_{\Pi} \left\{\sum_{j=0}^{n-1} \left|f\left(x_{j+1}\right) - f(x_j)\right|^p\right\}^{\frac{1}{p}} < \infty, (1 \le p < \infty) \ \ldots\ldots\ldots(9)$$

where $\Pi$ denotes the partition $\{-\pi = x_0 < x_1 < \cdots < x_n = \pi\}$ is often called the $p-th$ variation of $f$ in $[-\pi,\pi]$.

The aim of this paper is to present some approximation theorems for unbounded functions in space $L_{p,\alpha}(X)$. Their proofs are based on the suitable result announced in [4] and [11].

## Auxiliary theorem

Let us explicitly formulate direct theorem of the order degree of best approximation of unbounded functions by algebraic or trigonometric polynomials in weighed space.

**Theorem 1:[1]**

Let $f$ be unbounded function in weighted space $L_{p,\alpha}(X)$ and $(1 \le p < \infty)$. For every natural $k$ there exist constant $c(k)$ depending on $k$, such that

$E_n(f)_{p,\alpha} \le c(k)\omega_k(f,\sqrt{n})_{p,\alpha}$ .

## Main results

Considering for any unbounded function $f$ belong to weighted space $L_{p,\alpha}(X)$ such that $\frac{f(x)}{\alpha(x)} \le M$ for all $x \in X$, we shall prove the following:

**Theorem 1**

Suppose that function $f$ in space $L_{p,\alpha}(X)$, $(1 \le p < \infty)$, with a derivative $f^{(k-1)}$ of non-negative integer order $k-1$ and absolutely continuous in $X$. Then

$E_n(f)_{p,\alpha} \le \frac{C(k)}{n^k} E_n(f^k)_{p,\alpha}$ for $n = 0,1,2,\ldots$ .

**Proof :**

Let (5) be the Fourier series of $f \in L_{p,\alpha}(X)$ and $k$ positive integer.

We have

$f^{(k)}(x) = \frac{d^k}{dx^k} I_k[f](x) = \sum_{m=-\infty}^{\infty} C_m (im)^k \frac{e^{imx}}{\alpha^{(k)}(x)}$ for all $x$. Moreover

$F[f^{(k)}](x) = \sum_{m=-\infty}^{\infty} C_m (im)^k \frac{e^{imx}}{\alpha^{(k)}(x)}$ .

Thus $f^{(k)}(x) = \sum_{m=-\infty}^{\infty} C_m (im)^k \frac{e^{imx}}{\alpha^{(k)}(x)} = F[f^{(k)} * \Phi_n](x)$ where

$\Phi_n(x) = \sum_{m=-\infty}^{\infty} \frac{e^{imx}}{(im)^k} = 2\sum_{m=1}^{\infty} \frac{sinmx}{m}$ .

For each trigonometric polynomials $\xi_n(x), \eta_n(x)$ of order $n$ at most, there is a trigonometric polynomial $T_n(x)$ such that

$$f(x) - T_n(x) = \frac{1}{2\pi} \int_X \{f^{(k)}(t) - \xi_n(t)\}\{\Phi_n(x-t) - \eta_n(x-t)\}dt$$

$$= \frac{1}{2\pi} \int_X \{f^{(k)}(x-u) - \xi_n(x-u)\}\{\Phi_n(u) - \eta_n(u)\}du.$$

Thus

$$E_n(f)_{p,\alpha} \leq \|f - T_n\|_{p,\alpha} = \left(\int_X \left|\frac{f(x)-T_n(x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{2\pi}\left\{\left(\int_X \left|\frac{f^{(k)}(x-u)-\xi_n(x-u}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} . \left(\int_X \left|\frac{\Phi_n(u)-\eta_n(u)}{\alpha(u)}\right| du\right)\right\}$$

whence

$$E_n(f)_{p,\alpha} \leq \frac{1}{2\pi} E_n(f)_{p,\alpha} \, . E_n(\Phi_n)_{1,\alpha} \, .$$

We have from direct theorem 2.1

$$E_n(f)_{p,\alpha} \leq C\omega(f, \tfrac{1}{n+1})_{1,\alpha} \; ,$$

from properties of modulus of continuity, we have

$\omega(f,\lambda\delta) = \lambda\, \omega(f,\delta)$ ,

so, $\Phi_n(u) = \begin{cases} \pi - u & if \;\; 0 < u < \pi \\ 0 & if \; u = 0, u = \pi \end{cases}$

i.e. $V(\Phi)(\mathrm{u}) = 2(2\pi) = 4\pi \qquad , \; \mathrm{u} \in [-\pi,\pi]$.

Hence

$$E_n(f)_{p,\alpha} \le \frac{4\pi\, C_1(p)}{n+1} \cdot \frac{1}{2\pi} E_n(f^{(k)})_{p,\alpha} \le \frac{C_2(p)}{n+1} \cdot E_n(f^{(k)})_{p,\alpha} .$$

■

**Theorem 2**

Let $f \in L_{p,\alpha}(X), (1 \le p < \infty)$ and the trigonometric polynomial $T_n(x) = T_n(f,x)$ of order $n$ at most such that

$$\|f - T_n\|_{p,\alpha} \le C(p) E_n(f)_{p,\alpha} \quad , \quad n = 0,1,2,\dots \qquad \dots\dots\dots\dots(10)$$

for all satisfies the conditions of theorem 3.1, with positive integer $k$. Then

$$\left\|f^{(k)} - T_n{}^{(k)}(f)\right\|_{p,\alpha} \le C(p,k) E_n(f^{(k)})_{p,\alpha} \quad , \quad n = 0,1,2,\dots \qquad \dots\dots\dots\dots(11).$$

**Proof :**

Let $F_v(f,x)$ be denote the $v-th$ partial sum of the Fourier series of f and let $V_n(f,x)$ be the Vallee-poussin means of this series, defined by the foemula

$$V_n(f,x) = \frac{1}{n+1} \sum_{v=n}^{2n} F_n(f,x) \;, \quad n = 0,1,2,\dots \; .$$

As is well know, for any $f \in L_{p,\alpha}(X)$

$$V_n(f,x) = \frac{1}{\pi} \int_X \left| \frac{f(x+t)\beta_n(t)}{\alpha(t)} \right| dt \quad , \qquad if \; p = 1$$

where $\beta_n(t) = \frac{1}{\pi}\sum_{v=n}^{2n} \frac{\sin\left(v+\frac{1}{2}\right)t}{2sin\frac{1}{2}t} = \frac{\sin\left(n+\frac{1}{2}\right)t\,.\sin\left(\frac{3}{2}n+\frac{1}{2}\right)t}{2(n+1)(sin\frac{1}{2}t)^2}$ and

$\frac{1}{\pi}\int_X \ \beta_n(t)\,dt = 1\,, \frac{1}{\pi}\int_X \ |\beta_n(t)|\,dt < 2.\frac{4n+1}{2n+2} < 4.$

Moreover

$\psi_n\left(f^{(k)},x\right) = \psi_n{}^{(k)}(f,x)$ , whenever , $f^{(k)} \in L_{p,\alpha}(X)$ .

By Minkowsk's inequality, we obtain

$$\left\|f^{(k)} - T_n^{(k)}(f)\right\|_{p,\alpha} = \left(\int_X \ \left|\frac{f^{(k)}(x)-T_n^{(k)}(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \left(\int_X \ \left|\frac{f^{(k)}(x)-\psi_n{}^{(k)}(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_X \ \left|\frac{\psi_n{}^{(k)}(f,x)-T_n^{(k)}(\psi_n(f,x))}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$+\left(\int_X \ \left|\frac{T_n^{(k)}(\psi_n(f,x))-T_n^{(k)}(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}.$$

Set $p_n(f,x)$ the trigonometric polynomial of best approximation of $f$ in weighted space $L_{p,\alpha}(X)$ of order $n$, we have

$$\left(\int_X \ \left|\frac{f^{(k)}(x)-\psi_n{}^{(k)}(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} \leq$$

$$\left(\int_X \ \left|\frac{f^{(k)}(x)-p_n(f^{(k)},x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_X \ \left|\frac{p_n(f^{(k)},x)-\psi_n{}^{(k)}(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \left\|f^{(k)} - p_n\left(f^{(k)}\right)\right\|_{p,\alpha} + \left\|\psi_n\left(p_n\left(f^{(k)}\right) - f^{(k)}\right)\right\|_{p,\alpha}$$

$$\leq C_1 E_n\left(f^{(k)}\right)_{P,\alpha} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(12)$$

from Zygmand inequality gives

$$\left(\int_X \left|\frac{T_n^{(k)}(\psi_n(f,x))-T_n^{(k)}(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} \leq n^k\left(\int_X \left|\frac{T_n(\psi_n(f,x))-T_n(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} \ldots\ldots(13)$$

And

$$\left(\int_X \left|\frac{\psi_n{}^{(k)}(f,x)-T_n^{(k)}(\psi_n(f,x))}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} \leq (2n)^k\left(\int_X \left|\frac{\psi_n(f,x)-T_n(\psi_n(f,x))}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq C_2(p)(2n)^k E_n(\psi_n(f))_{p,\alpha} \ldots\ldots.(14)$$

From (13) and (14), we obtain

$$\left(\int_X \left|\frac{T_n(\psi_n(f,x))-T_n(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} \leq \left(\int_X \left|\frac{T_n(\psi_n(f,x))-\psi_n(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$+\left(\int_X \left|\frac{\psi_n(f,x)-f(x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}+\left(\int_X \left|\frac{f(x)-T_n(f,x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq C_3(p)E_n(\psi_n(f))_{p,\alpha}+C_4E_n(f)_{p,\alpha}+C_5(p)E_n(f)_{p,\alpha} \ \ldots..(15)$$

and

$$E_n(\psi_n(f))_{p,\alpha} \leq CE_n(f)_{p,\alpha} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(16).$$

Thus, from (12), (15) and (16), we get

$$\left\|f^{(k)}-T^{(k)}\right\|_{p,\alpha} \leq C_6E_n(f^{(k)})_{p,\alpha}+C_7n^{(k)}(C_8(p)+1)E_n(f)_{p,\alpha}$$

$$+C_9(p)E_n(f)_{p,\alpha}$$

and by theorem 3.1 then (11) follows.

■

**Theorem 3**

Let $f \in L_{p,\alpha}(X)$, $(1 \leq p < \infty)$ and that the derivative $f^{(a)}$ in weighed space with $0 < a < 1$. Then

$E_n(f)_{p,\alpha} \leq \frac{C(a)}{(n+1)^a} E_n(f^{(a)})_{p,\alpha}$ for $n = 0,1,2 \ldots$ ...........(17).

**Proof :**

Putting $(x) = \int_0^x f(u)du$ , we have $G_n(x) = \sum_{m=-\infty}^{\infty} C_m(im)^{-1}(e^{imx})$

$$= f_1(x) - f(0)$$

uniformly in $x \in (-\infty, \infty)$. Consequently $G_n(x) = I_n(x,g) - I_n(0,g) \quad \forall\, x$.

Therefore, for all real $x$

$G_n(x) = \frac{1}{\pi} \int_X \Phi_n(x-u)g(u)du - f(0)$,

$\int_X \Phi_n(x-u)du = 0$ .

For each trigonometric polynomials $\zeta_n(x)$ and $\eta_n(x)$ of order $n$, there is a trigonometric polynomial $T_n(x)$ such that

$G_n(x) - T_n(x) = \frac{1}{2\pi} \int_X \{g(x-u) - \zeta_n(x-u)\}.\{\Phi_a(u) - \eta_n(u)\}du - I_a(0,g)$.

Assuming $0 < b < \pi$ , we have

$$\left\{\int_X \left|\frac{G(x+b) - T_n(x+b) - (G(x-b) - T_n(x-b))}{2b\ \alpha(x)}\right|^p dx\right\}^{\frac{1}{p}} =$$

$$\frac{1}{2\pi}\left\{\int_X \left|\int_X \frac{g(x+b-u) - \zeta_n(x+b-u) - (g(x-b-u) - \zeta_n(x-b-u))}{2b\ \alpha(x)} .\{\Phi_a(u) - \eta_n(u)\}du\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{2\pi} \int_X |\Phi_a(u) - \eta_n(u)|.\left\{\int_X \left|\frac{g(x+b-u) - \zeta_n(x+b-u) - (g(x-b-u) - \zeta_n(x-b-u))}{2b\ \alpha(x)}\right|^p dx\right\}^{\frac{1}{p}} du$$

we take the best approximation of functions$g(v)$ and $\Phi_a(v)$ they $\zeta_n(u)$ , $\eta_n(u)$ in spaces $L_{p,\alpha}(X)$ and $L_1(X)$ respectively. Then

$$2\pi\left\{\int_X \left|\frac{G(x+b)-G(x-b)}{2b\ \alpha(x)}-\frac{T_n(x+b)-T_n(x-b)}{2b\ \alpha(x)}\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq\left\{\int_X \left|\frac{g(v+b)-g(v-b)}{2b\ \alpha(x)}-\frac{\zeta_n(v+b)-\zeta_n(v-b)}{2b\ \alpha(x)}\right|^p dv\right\}^{\frac{1}{p}}.E_n(\Phi_a)_1 .$$

By theorem 3.2, we have

$$2\pi\left\{\int_X \left|\frac{G_n^{(1)}(x)-T_n^{(1)}(x)}{\alpha(x)}\right|^p dx\right\}^{\frac{1}{p}}\leq\left\{\int_X \left|\frac{g^{(1)}(v)-\zeta_n^{(1)}(v)}{\alpha(v)}\right|^p dv\right\}^{\frac{1}{p}}.E_n(\Phi_a)_1 .$$

$$\leq C(p)E_n(g^{(1)})_{p,\alpha}.E_n(\Phi_a)_1$$

So, $\left\|f-T_n^{(1)}\right\|_{p,\alpha}\leq 2\pi\left\{\int_X \left|\frac{f(x)-T_n^{(1)}(x)}{\alpha(x)}\right|^p dx\right\}^{\frac{1}{p}}$ and consequently

$E_n(f)_{p,\alpha}\leq\frac{1}{2\pi}\ CE_n(f^{(a)})_{p,\alpha}.E_n(\Phi_a)_1$ ,

By direct theorem 2.1, we have

$$E_n(f)_{p,\alpha}\leq\frac{3}{\pi}\ CE_n(f^{(a)})_{p,\alpha}\omega_1(\frac{1}{n+1},\Phi_a)\leq\frac{3}{\pi}\ CE_n(f^{(a)})_{p,\alpha}V_1(\Phi_a).\frac{1}{n+1}$$

$$\leq\frac{C(a)}{(n+1)^a}E_n(f^{(a)})_{p,\alpha} ,$$

the desired assertion is established.

**Theorem 4**

Given any $f\in L_{p,\alpha}(X)$, $(1\leq p<\infty)$, let us consider the trigonometric polynomial $T_n(x)=T_n(f,x)$ of order n such that

$\left\| f^{(k)} - T_n^{(k)} \right\|_{p,\alpha} \leq C(p,k) E_n(f^{(k)})_{p,\alpha}$ , with some positive non-integer $k$.

The proof runs analogously of theorem 3.2.

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