On A Class of Analytic Univalent Function Defined By Ruscheweyh Derivative

A.R.S .Juma *, H. H.Ebrahim And SH. I. Ahmed****

*Department Of Mathematics, Alanbar University

**Department Of Mathematics, Tikrit University

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Abstract

Making use the Ruscheweyh derivative operator, we introduce and study a certain $\mathcal{AH}_n(\delta,\alpha,\lambda)$ of univalent analytic function with negative coefficient .In this paper, we obtain coefficient estimates, distortion theorem, radii of starlikness, convexity and modified Hadamard product of functions belonging to the class $\mathcal{AH}_n(\delta,\alpha,n)$.

Key words: Analytic function, univalent function, Hadmard product, starlike function , convex functions, Ruscheweyh derivative. AMS subject classifications: 30C45.

على اصناف الدوال التحليلية أحادي التكافؤ معرفة بواسطة المشتقة الرشويه

عبد الرحمن سلمان جمعة , قسم الرياضيات، جامعة الانبار, الرمادي - العراق
حسن حسين ابراهيم, قسم الرياضيات , جامعة تكريت, تكريت-العراق
شامل ابراهيم احمد , قسم الرياضيات، جامعة تكريت ,تكريت-العراق

الملخص

مما يجعل استخدام المؤثر المشتقة رشويه، قدمنا ودرسنا صنف معين $\mathcal{AH}_n(\delta,\alpha,\lambda)$ من الدالة التحليلية أحادية التكافؤ مع معامل سلبي. في هذا البحث، حصلنا على معامل التقدير، نظرية التشويه ، قطر ال starlikness، التحدب الدوال والضرب الدوال التي تنتمي الى صنف $\mathcal{AH}_n(\delta,\alpha,\lambda)$.

الكلمات الرئيسية: الدوال التحليلية، الدالة الأحادية التكافؤ، الضرب الدالي، دالة الجمة، الدالة المحدبة، المشتقة رشويه.

Introduction

Let $\mathcal{A}(n)$ denotes the class of functions normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k Z^k \quad (k \in \mathbb{N}), \qquad (1)$$

which are analytic and univalent in the unit disk $\mathcal{U}= \{Z\in\mathbb{C}: |z|<1\}$.

Let $\mathcal{A}^{+}(n)$ be subclass of $\mathcal{A}(n)$, consisting of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (k \in \mathbb{N}, a_k \geq 0). \qquad (2)$$

For functions $f(z) \in \mathcal{A}(n)$ given by (1) and $g(z) \in \mathcal{A}(n)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (k \in \mathbb{N}),$$

The convolution or Hadmard product of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad , \; z \in \mathcal{U} \; . \qquad (3)$$

A function $f(z) \in \mathcal{A}$ (n) is said to be univalent starlike of order α if satisfies the inequality:

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha \quad (z \in \mathcal{U}\,; 0 \leq \alpha < 1).$$

denotes the class of all univalent starlike functions of order α. Also $f(z) \in \mathcal{A}$ (n) is $S^*(\alpha)$ said to be univalent convex of order α if satisfies the inequality:

$$Re\left(1 + \frac{zf''(z)}{f(z)}\right) \geq \alpha \quad (z \in \mathcal{U}\,; 0 \leq \alpha < 1).$$

denotes the class of all univalent convex functions of order α[see for example Duren [1] $C(\alpha)$ and Goodman [3]].

The classes $S^*(\alpha)$ and $C(\alpha)$ are studied Owa [6].

$D^{\lambda} : \mathcal{A}(n) \rightarrow \mathcal{A}(n)$ represent the Ruscheweyh derivative of order λ defined by [7]:

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) , \qquad \lambda > -1$$

$$= \frac{z\left(z^{\lambda-1} f(z)\right)^{\lambda}}{\lambda!}, f(z) \in \mathcal{A}(n)$$

$$= z + \sum_{k=2}^{\infty} B_K(\lambda) a_k \, z^k \quad ,$$

$$where\ \lambda > -1 \qquad , \ B_K(\lambda) = \frac{(\lambda+1)(\lambda+2)\ldots\ldots(\lambda+k-1)}{(k-1)!} \quad \text{also see } [4],[5].$$

We aim to study the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$ which consists of functions $f \in S$ and satisfy

$$\left|\frac{z\left(D^{\lambda}f(z)\right)' + \delta z^2\left(D^{\lambda}f(z)\right)''}{(1-\delta)D^{\lambda}f(z) + \ \delta z(D^{\lambda}f(z))'}\right| > \alpha \ , \mathcal{Z} \ \in \mathcal{U}. \quad \ldots\ldots..(*)$$

For $|z|<1$, $0 \le \delta < 1$, $0 \le \alpha < 1$, $\lambda > -1$.

The investigation here is motivated by M. Darus [2].

Remark:

1) when $\delta = 0,$ then form $(*)$ we get

$$Re\left\{\frac{z\left(D^{\lambda}f(z)\right)'}{D^{\lambda}f(z)}\right\} > \alpha \ , \mathcal{Z} \ \in \mathcal{U}$$

which is the class of starlike of order α.

2) when $\delta = 0,$ λ =1, we have the class

$$Re\left\{\frac{zf(z)'}{f(z)}\right\} > \alpha \ ; 0 \le \alpha < 1,$$

which is the class of starlike of order α studied by Owa [6] and Yamakawa [8]

3) when $\delta = 1$, we have

$$Re\left\{1 + \frac{z\left(D^{\lambda}f(z)\right)''}{(D^{\lambda}f(z))'}\right\} > \ \alpha \, ; 0 \le \alpha < 1$$

which is the class of convex of order α.

4) when $\delta = 1$, λ =1 we have

$$Re\left\{1+\frac{zf(z)''}{f(z)'}\right\} > \alpha\,;0 \le \alpha < 1$$

which is the class of convex of order α studied by Owa [6] and Yamakawa [8].

Next we characterize the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$ by proving the coefficient inequality.

2. Coefficient Inequality

Let the function $f(z)$be of the from (2).Then $f \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ if and only **Theorem 2.1**: if

$$\sum_{k=2}^{\infty}(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)a_k \le 1-\alpha, \qquad (4)$$

where $0 \le \alpha<1,\ 0 \le \delta<1,\ \lambda > -1,\ k\epsilon\mathbb{N}_0$,

$$B_K(\lambda) = \frac{(\lambda+1)(\lambda+2)\ldots\ldots(\lambda+k-1)}{(k-1)!}.$$

Proof: Assume inequality (4) is true then

$$\left|z\left(D^{\lambda}f(z)\right)'+\delta z^2\left(D^{\lambda}f(z)\right)''\right|-\alpha\left|(1-\delta)D^{\lambda}f(z)+\ \delta z(D^{\lambda}f(z))'\right|$$

$$\le\left|1+\sum_{k=2}^{\infty}[1+\delta(k-1)]k\,B_K(\lambda)a_kz^k\right|-\alpha\left|1+\sum_{k=2}^{\infty}[1-\delta+\delta k]B_K(\lambda)a_kz^k\right| \qquad (5)$$

$$\le\sum_{k=2}^{\infty}(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)a_k\ +(1-\alpha)$$

≤ 0 by maximum modules principle.

Therefore $f \in \mathcal{AH}_n(\delta,\alpha,\lambda)$.

Conversly ,*let* $f \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ .Then

$$\left|\frac{z\left(D^{\lambda}f(z)\right)'+\delta z^2\left(D^{\lambda}f(z)\right)''}{(1-\delta)D^{\lambda}f(z)+\ \delta z(D^{\lambda}f(z))'}\right| > \alpha\ ,z\ \in \mathcal{U}$$

that is

$$\frac{\left|1+\sum_{k=2}^{\infty}[1+\delta(k-1)]k\,B_K(\lambda)\,a_k z^k\right|}{\left|1+\sum_{k=2}^{\infty}[1-\delta+\delta k]B_K(\lambda)a_k z^k\right|}>\alpha \quad , \qquad (6)$$

using the fact that $|\mathrm{Re}\, f(z)| \le |f(z)|$

$$\left|Re\left\{\frac{1+\sum_{k=2}^{\infty}[1+\delta(k-1)]k\,B_K(\lambda)\,a_k z^k}{1+\sum_{k=2}^{\infty}[1-\delta+\delta k]B_K(\lambda)a_k z^k}\right\}\right|>\alpha \quad .$$

Choose z on real axis and allowing $z \to 1-$

$$\frac{1+\sum_{k=2}^{\infty}[1+\delta(k-1)]k\,B_K(\lambda)\,a_k}{1+\sum_{k=2}^{\infty}[1-\delta+\delta k]B_K(\lambda)a_k}>\alpha$$

$$\sum_{k=2}^{\infty}(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)a_k+(1-\alpha)>0 \;.$$

Thus the proof is complete.

Of the form of (2) be in the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$ **Corolary 2.2 :** Let the function $f(z)$ .Then

$$a_k \le \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)} \qquad for\; k=2,3,\dots$$

with equality for

$$f(z)=z+\frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}z^k \quad ,z=2,3,\dots .$$

3. Growth And Distortion Theorem

Let the function $f(z)$ of the form of (2) be in the class **Theorem 3.1 :** $\mathcal{AH}_n(\delta,\alpha,\lambda)$ .Then

$$|z| - \frac{(1-\alpha)}{(2-\alpha)\,[1+\delta\,]B_K(\lambda)}|z|^2 \le |f(z)| \le |z| + \frac{(1-\alpha)}{(2-\alpha)\,[1+\delta\,]B_K(\lambda)}|z|^2 .$$

The result is sharp for

$$f(z) = zz + \frac{(1-\alpha)}{(2-\alpha)\,[1+\delta\,]\,B_K(\lambda)} z^2 \quad z = 2,3,\ldots .$$

Proof: we have

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

$$|f(z)| \le |z| + \sum_{k=2}^{\infty} a_k |z|^k$$

$$|f(z)| \le |z| + \frac{(1-\alpha)}{(2-\alpha)\,[1+\delta\,]B_K(\lambda)}|z|^2 . \quad (7)$$

Similarly

$$|f(z)| \ge |z| - \sum_{k=2}^{\infty} a_k |z|^k$$

$$\ge |z| - \frac{(1-\alpha)}{(2-\alpha)\,[1+\delta\,]B_K(\lambda)}|z|^2 . \quad (8)$$

Combining (7) and (8) we get the result.

Theorem 3.2 : Let the function $f(z)$of the form of (2)be in the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$.

Then

$$1 - \frac{2\,(1-\alpha)}{(2-\alpha)\,[1+\delta\,]B_K(\lambda)}|z| \le |f'(z)| \le 1 + \frac{2\,(1-\alpha)}{(2-\alpha)\,[1+\delta\,]B_K(\lambda)}|z| .$$

The result is sharp for

$$f(z) = z + \frac{(1-\alpha)}{(2-\alpha)\,[1+\delta\,]\,B_K(\lambda)}|z|^2 \ .$$

4. Radii Of Starlikness And Convexity

Theorem 4.1 : Let the function $f(z)$ of the form of (2) be in the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$.

Then $f(z)$ is starlike in $|Z| < R$ where

$$R = \inf_K \left\{\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(k+s-2)(1-\alpha)}\right\}^{\frac{1}{k-1}} \quad k = 2,3, \dots \ .$$

The estimate is sharp for the function

$$f(z) = z + \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)} z^k \quad \text{for some k} \, .$$

Proof: f is starlike of order S, $0 \le S < 1$ if

$$Re\left(z\frac{f'(z)}{f(z)}\right) \ge S \qquad (9)$$

that is if

$$\left|z\frac{f'(z)}{f(z)} - 1\right| \le 1 - S \qquad (10)$$

which simplifies to

$$\sum_{k=2}^{\infty} \frac{(k+s-2)a_k|z|^{k-1}}{1-s} \le 1 \qquad (11)$$

by (4) we have

$$a_k \le \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)} z^k \quad k \ge 2. \qquad (12)$$

Using (11) and (12) we get

$$|z|^{k-1} \le \frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(k+s-2)(1-\alpha)}$$

thus

$$|z| < R = \inf_K \left\{\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(k+s-2)(1-\alpha)}\right\}^{\frac{1}{k-1}}.$$

Theorem 4.2 : Let the function $f(z)$ of the form of (2) be in the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$.

then $f(z)$ is convex of order C, $0 \le C < 1$ in $|z| < R$ where

$$R = \inf_K \left\{ \frac{(1-c)\,[1+\delta(k-1)]\,B_K(\lambda)}{(k+c-2)(1-\alpha)} \right\}^{\frac{1}{k-1}}, \quad k = 2,3,\ldots$$

estimate is sharp for

$$f(z) = z + \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)} z^k \quad \text{for some k .}$$

Proof: $f \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ is convex of order C, $0 \le C < 1$ *if*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge c$$

which is equivalent to $\left|\frac{zf''(z)}{f'(z)}\right| \le 1 - c$

that is $$\left|\frac{\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} k\, a_k z^{k-1}}\right| \le 1 - c \ .$$

Using the arguments similar to Theorem (3.2), we get

$$|z| < R = \inf_K \left\{ \frac{(1-c)\,[1+\delta(k-1)]\,B_K(\lambda)}{(k+c-2)(1-\alpha)} \right\}^{\frac{1}{k-1}}$$

5. Extreme Points

Let $f_1(z) = z$, **Theorem 6**:

$$f_k(z) = z + \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)} z^k, \quad k = 2,3,\ldots.$$

then $f \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ If f it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z) \quad where \quad \lambda_k \geq 0 \; and \quad \sum_{k=1}^{\infty}\lambda_k = 1 .$$

Proof: suppose

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$$

$$f(z)=\sum_{k=1}^{\infty}\lambda_k\left(z+\frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}z^k\right)$$

$$f(z)=z+\sum_{k=2}^{\infty}\lambda_k \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}z^k . \quad (9)$$

Now $f\in \mathcal{AH}_n(\delta,\alpha,\lambda)$ since

$$\sum_{k=2}^{\infty}\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}\cdot\frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}\lambda_k=\sum_{k=2}^{\infty}\lambda_k = 1-\lambda_1 \leq 1.$$

Conversely, suppose $f\in \mathcal{AH}_n(\delta,\alpha,\lambda)$ then by (5)

$$a_k \leq \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)} \quad , \quad k=2,3,\ldots$$

Setting

$$\lambda_k \leq \frac{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)}{(1-\alpha)} \quad , \quad k=2,3,\ldots.. \quad (10)$$

and $\lambda_1 = 1-\sum_{k=2}^{\infty}\lambda_k$ we notice that $f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$. Hence the result .

6. Hadmard Production

Theorem 6.1: Let $f,g\in \mathcal{AH}_n(\delta,\gamma,\lambda)$, then

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k \qquad \in \mathcal{AH}_n(\delta,\alpha,\lambda)$$

$$for \quad f(z)=z+\sum_{k=2}^{\infty}a_k\,z^k \qquad ,g(z)=z+\sum_{k=2}^{\infty}b_k\,z^k$$

where

$$\gamma=1-\frac{(k-1)(1-\alpha)^2}{B_K(\lambda)(k-\alpha)^2[1+\delta(k-1)]-(1-\alpha)^2}$$

Proof: $f,g\in \mathcal{AH}_n(\delta,\alpha,\lambda)$ and so

$$\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}a_k\le 1 \qquad (11)$$

$$\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}b_k\le 1 \qquad (12)$$

We need to find largest number γ such that

$$\frac{(k-\gamma)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\gamma)}a_kb_k\le 1\,.$$

By Cauchy schwas z inequality we have

$$\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}\sqrt{a_kb_k}\le 1 \qquad (13).$$

Thus it enough to show that

$$\frac{(k-\gamma)}{(1-\gamma)}a_kb_k<\frac{(k-\alpha)}{(1-\alpha)}\sqrt{a_kb_k} \qquad (14)\,.$$

Or equivalently, that

$$\sqrt{a_kb_k}\le\frac{(k-\alpha)(1-\gamma)}{(1-\alpha)(k-\gamma)} \qquad (15).$$

From (13)

$$\sqrt{a_k b_k} \leq \frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)} \qquad (16).$$

Therefor in view of (15) and (16) it is enough to show that

$$\frac{(1-\alpha)}{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)} \leq \frac{(k-\alpha)(1-\gamma)}{(1-\alpha)(k-\gamma)} \quad .$$

Which simplifies to

$$\therefore\ \gamma = 1 - \frac{(k-1)(1-\alpha)^2}{B_K(\lambda)(k-\alpha)^2[1+\delta(k-1)]-(1-\alpha)^2}$$

7. Closuer Theorem

Theorem 7.1: Let $f_j \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ j = 1,2, … ..

$$g(z) = \sum_{j=1}^{\ell} C_j f_j \in \mathcal{AH}_n(\delta,\alpha,\lambda)$$

where $\sum_{j=1}^{\ell} C_j = 1$ *and* $f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k$.

Proof: suppose

$$g(z) = \sum_{j=1}^{\ell} C_j \left(z + \sum_{k=2}^{\infty} a_{k,j} z^k \right)$$

$$= z + \sum_{k=2}^{\infty} \left(\sum_{k=2}^{\ell} a_{k,j}\, C_j z^k \right) \qquad (17)$$

$$= z + \sum_{k=2}^{\infty} e_k z^k \qquad (18)$$

where $e_k = \sum_{j=1}^{\infty} a_{k,j} C_j$.

Since $f_j \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ by (5)

$$\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}a_k \le 1\,. \qquad (19)$$

In view of (18) $g(z) \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ if

$$\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}e_k \le 1.$$

Now

$$\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}e_k = \frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}\sum_{j=1}^{\infty} a_{k,j}C_j$$

$$= \sum_{j=1}^{\ell}\sum_{k=2}^{\infty}\frac{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)}{(1-\alpha)}a_k$$

$$\le \sum_{j=1}^{\ell} C_j \quad \text{using } (19)$$

$= 1.$

Thus $g(z) \in \mathcal{AH}_n(\delta,\alpha,\lambda)$.

Theorem 7.2: Let $f,g \in \mathcal{AH}_n(\delta,\alpha,\lambda)$, then

$h(z) = z + \sum_{k=2}^{\infty}({a_k}^2 + {b_k}^2)\,z^k$ *is in* $\mathcal{AH}_n(\delta,\alpha,\lambda)$ and where

$$\zeta = p - \frac{2(k-1)(1-\alpha)^2}{(k-\alpha)^2\,[1+\delta(k-1)] - 2(1-\alpha)^2}$$

Proof: $f,g \in \mathcal{AH}_n(\delta,\alpha,\lambda)$ and hence

$$\sum_{k=2}^{\infty}\left[\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}\right]^2 {a_k}^2 \le \left[\sum_{k=2}^{\infty}\frac{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)}{(1-\alpha)}a_k\right]^2 \le 1 \qquad (20)$$

$$\sum_{k=2}^{\infty}\left[\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}\right]^2 b_k{}^2 \leq \left[\sum_{k=2}^{\infty}\frac{(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)}{(1-\alpha)}b_k\right]^2$$

$$\leq 1 \quad . \quad (21)$$

Adding (20) and (21) we get

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(k-\alpha)\,[1+\delta(k-1)]\,B_K(\lambda)}{(1-\alpha)}\right]^2\left(a_k{}^2+b_k{}^2\right) \leq 1\,. \quad (22)$$

We must show that $h \in \mathcal{AH}_n(\delta,\alpha,\lambda)$.

Therefore, we have fined largest such that

$$\frac{(k-\zeta)}{(1-\zeta)} \leq \frac{B_K(\lambda)(k-\alpha)^2[1+\delta(k-1)]}{2(1-\alpha)^2} \quad .$$

That is, that

$$\zeta \leq p - \frac{2(k-1)(1-\alpha)^2}{(k-\alpha)^2\,[1+\delta(k-1)] - \,2(1-\alpha)^2} \quad .$$

8. Integral Operator

Theorem 8.1: Let function defined by (2) in the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$, and let c be a real number such that $c > -1$ then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} + \int_0^z t^{c-1} f(t)dt \quad .$$

Also belongs the class $\mathcal{AH}_n(\delta,\alpha,\lambda)$.

Proof. For the representation of $F(z)$,it follows that

$$F(z) = z + \sum_{k=2}^{\infty} d_k$$

$$where \qquad d_k = \left(\frac{c+1}{c+k}\right) a_k.$$

Therefor

$$\sum_{k=2}^{\infty}(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)d_k = \sum_{k=2}^{\infty}(k-\alpha)\,[1+\delta(k-1)]B_K(\lambda)\left(\frac{c+1}{c+k}\right)a_k$$

$$\leq \sum_{k=2}^{\infty}(k-\alpha)\ [1+\delta(k-1)]B_K(\lambda)a_k \leq 1-\alpha.$$

Since $f(z) \in \mathcal{AH}_n(\delta,\alpha,\lambda)$. Hence by Theorem (2.1) $F(z) \in \mathcal{AH}_n(\delta,\alpha,\lambda)$.

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