Diyala Journal for Pure Science

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Wadhah Abdulelah Hussein and Huda Amer Abdul Ameer

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Wadhah Abdulelah Hussein* and Huda Amer Abdul Ameer

Department of Mathematics – College of Science – University of Diyala

wadhahabdulelah@uodiyala.edu.iq

Received: 27 March 2022 **Accepted: 24 April 2022**

DOI: https://dx.doi.org/10.24237/djps.1803.583B

Abstract

The new generalized operator $F^{m}_{\nu,\delta}$, is a conjunction between Unconstrained optimization and Univalent Harmonic Functions.We derived some properties by this conjunction like, coefficient inequality, convex set, apply Bernardi operator and determine the extreme points such that $\sum_{n=1}^{\infty} (\omega_n + \vartheta_n) = 1, (\omega_n \geq 0, \vartheta_n \geq 0)$. In particular, the extreme points of $N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$ are $\{h_n\}$ and $\{g_n\}$.

Keywords: Coefficient inequality, Integral operator, Optimization, Harmonic, Univalent, Generalized operator and Bernardi operator.

الامثلية الغير مقيدة للدوال التوافقية أحادية التكافؤ

وضاح عبد الإله حسين *و هدى عامر عبد الأمير

قسم الرياضيات – كلية العلوم – جامعة ديالى

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الخلاصة

العامل الجديد المعمم $F^{m}_{\nu,\delta}$، هو اقتران بين الامثلية غير المقيد والوظائف التوافقية أحادية التكافؤ اشتققنا بعض الخصائص من خلال هذا الاقتران مثل، معامل عدم المساواة، مجموعة محدبة، تطبيق عامل برناردي ، تحديد النقاط القصوى بحيث ان $(\omega_n \geq 0, \vartheta_n \geq 0)$ $\sum_{n=1}^{\infty} (\omega_n + \vartheta_n) = 1,$ عمليا. $N\delta^{*}_{u}(\beta,\gamma,\mu;n,\lambda)$ are $\{h_n\}$ and $\{g_n\}$

الكلمات المفتاحية: معامل عدم المساواة، عامل متكامل، أمثلية، متناسق، أحادي التكافؤ، عامل معمم، عامل بيرناردي.

Introduction

Between Univalent Harmonic Functions we derived some properties by this conjunction like, coefficient inequality, convex set, apply Bernardi operator, determine the extreme points and Unconstrained optimization, also the BFGS formula H -version is given by:

$$H^{*} = H + \left[1 + \frac{\underline{y}_k^T H \underline{y}_k}{\underline{s}_k^T \underline{y}_k}\right] \frac{\underline{s}_k \underline{s}_k^T}{\underline{s}_k^T \underline{y}_k} - \frac{\underline{s}_k \underline{y}_k^T H + H \underline{y}_k \underline{s}_k^T}{\underline{s}_k^T \underline{y}_k}$$

Lemma 1[14]: Let the current Hessian matrix be a symmetric and positive definite, then the next Hessian matrix defined by BFGS update is positive definite if and only if $\underline{y}_k^T \underline{s} > 0$.

Lemma 2: If $\beta \in R$, and $\underline{a} \in R^n$ such that $\underline{a}^T \underline{a} \neq 0$, then the minimum value of $\underline{x}^T \underline{x} = ||\underline{x}||^2$, $\underline{x} \in R^n$, subject to $\underline{a}^T \underline{x} = \beta$ is $\underline{x} = \beta \frac{\underline{a}}{\underline{a}^T \underline{a}}$.

If row i, (i = 1, 2, …, n) is set as

$\beta = z_1$, $\underline{a} = [s_1\ s_2\ \dots\ s_n]^T$, $\underline{x} = [v_{i1}\ v_{i2}\ \dots\ v_{in}]^T$

By imposing the constraints $v_{ij} = 0$, for all $j < i$.

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This update does not preserve the symmetric property and could lead to failure if $v_{ii} = -u_{ii}$.

Now by $\Omega = \Omega$ $(\mathfrak{E})$ the class of analytic functions in the open unit disk $\mathfrak{E} = \{z \in \mathbb{C} : |z| < 1\}$. For n is a positive integer and $a \in \mathbb{C}$,

let Ω [a, n] be the subclass of the function $f \in \Omega$ of the form:

$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots (a \in \mathbb{C}, n \in N = \{1,2,3\ldots\})$.

A continuous function $f = u + iv$ be a complex valued harmonic function in a complex domain $\mathbb{C}$, if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in D see (Clunie and Sheil – Small [1]), (also [2] and [3]), defined the harmonic function. Denote by δ_u the class of functions $f = h + \bar{g}$ that are harmonic univalent in the unit disk $\mathfrak{E} = \{z \in \mathbb{C}: |z| < 1\}$, for which $f(0) = f'(0) - 1 = 0$.

where h and g are given by:

$$\boldsymbol{h}(\boldsymbol{z}) = \boldsymbol{z} + \sum_{\boldsymbol{k}=2}^{\infty} \boldsymbol{a_n} \ \boldsymbol{z^n}, \boldsymbol{g}(\boldsymbol{z}) = \sum_{\boldsymbol{k}=1}^{\infty} \boldsymbol{b_n} \ \boldsymbol{z^n} \ |\boldsymbol{b_1}| < \mathbf{1} \qquad (\mathbf{1})$$

So $f = h + \bar{g} \in \delta_u$.

Also, denote by $N\delta_u$ the subclass of δ_u containing of all functions $f = h + \bar{g}$, where h and g are given by

$$\boldsymbol{h}(\boldsymbol{z}) = \boldsymbol{z} - \sum_{\boldsymbol{n}=2}^{\infty} |\boldsymbol{a_n}| \ \boldsymbol{z^n}, \boldsymbol{g}(\boldsymbol{z}) = -\sum_{\boldsymbol{n}=1}^{\infty} |\boldsymbol{b_n}| \ \boldsymbol{z^n}, |\boldsymbol{b_1}| < \mathbf{1} \qquad (\mathbf{2})$$

new generalized operator $F^m_{\nu,\delta}$ on Class of analytic functions $\mathbb{A}$.

as follows: For $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$ and ν a real number with $\nu + \delta > 0$. Then for $f \in \mathbb{A}$, we define the operator $F^{m}_{\nu,\delta}$ by

$F^{0}_{\nu,\delta} f(z) = f(z)$,

$$F^{1}_{\nu,\delta} f(z) = \frac{\nu f(z) + \delta z f'(z)}{\nu + \delta},$$

⋮

$F^{m}_{\nu,\delta} f(z) = F_{\nu,\delta}\left(F^{m-1}_{\nu,\delta} f(z)\right)$.

We observe that $F^{m}_{\nu,\delta} : \mathbb{A} \longrightarrow \mathbb{A}$ is a linear operator and

$$\mathbf{F}^{m}_{\nu,\delta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\nu + n\delta}{\nu + \delta}\right)^{m} a_n z^n \qquad (3)$$

It follows

$F^{m}_{\nu,0} f(z) = f(z)$,

$(\nu + \delta) F^{m+1}_{\nu,\delta} f(z) = \nu F^{m}_{\nu,\delta} f(z) + \delta z \left(F^{m}_{\nu,\delta} f(z)\right)', \delta > 0,$

and

$F^{m_1}_{\nu,\delta}\left(F^{m_2}_{\nu,\delta} f(z)\right) = F^{m_2}_{\nu,\delta}\left(F^{m_1}_{\nu,\delta} f(z)\right)$, for all $m_1, m_1 \in N_0$.

We note that

1) $F^{m}_{\nu,1} f(z) = F^{m}_{\nu} f(z)$, $\nu > -1$ (see Cho and Srivastava [4]and Cho and Kim [5]).

2) $F^{m}_{1-\delta,\delta} f(z) = D^{m}_{\delta} f(z), \delta \geq 0$ (see Al-oboudi [6]).

3) $F^{m}_{l+1-\delta,\delta} f(z) = F^{m}_{l,\delta} f(z), l > -1, \delta \geq 0$ (see Catas [7]).

4) $F^{m}_{1,\delta} f(z) = N^{m}_{\delta} f(z)$, is an operator defined by (see [8]).

$N^{m}_{\delta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+n\delta}{1+\delta}\right)^{m} a_n z^n, (f \in \mathbb{A})$.

Patel [9] defined an integral operator $I^{m}_{\nu,\delta}$ on $\mathbb{A}$ as follows:

For $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$ and ν a real number with $\nu + \delta > 0$.

Then for $f \in \mathbb{A}$, we define the operator $I^{m}_{\nu,\delta}$ by

$I^{0}_{\nu,\delta} f(z) = f(z)$

$$\mathrm{I}^{1}_{v,\delta}f(\mathrm{z}) = \left(\frac{v+\delta}{\delta}\right) z^{1-\left(\frac{v+\delta}{\delta}\right)} \int_{0}^{z} t^{\left(\frac{v+\delta}{\delta}\right)-2} f(t)dt\,,\ z \in \mathfrak{E}.$$

$$\mathrm{I}^{2}_{v,\delta}f(\mathrm{z}) = \left(\frac{v+\delta}{\delta}\right) z^{1-\left(\frac{v+\delta}{\delta}\right)} \int_{0}^{z} t^{\left(\frac{v+\delta}{\delta}\right)-2}\, \mathrm{I}^{1}_{v,\delta} f(t)dt\,,\ z \in \mathfrak{E}.$$

$\vdots$

$$\mathrm{I}^{\mathrm{m}}_{v,\delta}f(\mathrm{z}) = \left(\frac{v+\delta}{\delta}\right) z^{1-\left(\frac{v+\delta}{\delta}\right)} \int_{0}^{z} t^{\left(\frac{v+\delta}{\delta}\right)-2}\, \mathrm{I}^{\mathrm{m}-1}_{v,\delta} f(t)dt\,,\ z \in \mathfrak{E}.$$

$$= \underbrace{\mathbf{I}^{1}_{v,\delta}\left(\frac{z}{1-z}\right) * \mathbf{I}^{1}_{v,\delta}\left(\frac{z}{1-z}\right) * \ldots * \mathbf{I}^{1}_{v,\delta}\left(\frac{z}{1-z}\right) * \boldsymbol{f}(\mathbf{z})}_{m-times}. \qquad (4)$$

We observe that $\mathrm{I}^{\mathrm{m}}_{v,\delta}\ \mathbb{A} \longrightarrow \mathbb{A}$ is an integral operator

$$\mathrm{I}^{\mathrm{m}}_{v,\delta}f(\mathrm{z}) = \mathrm{z} + \sum_{\mathrm{n}=2}^{\infty}\left(\frac{v+\delta}{v+\mathrm{n}\,\delta}\right)^{\mathrm{m}} \mathrm{a_n}\, z^{n}, (z \in \mathfrak{E}).$$

It follows

$$\mathrm{I}^{\mathrm{m}}_{v,0}f(\mathrm{z}) = f(\mathrm{z}),$$

$$(v+\delta)\mathrm{I}^{\mathrm{m}}_{v,\delta}f(\mathrm{z}) = v\mathrm{I}^{\mathrm{m}+1}_{v,\delta}f(\mathrm{z}) + \delta\mathrm{z}\left(\mathrm{I}^{\mathrm{m}+1}_{v,\delta}f(\mathrm{z})\right)'.$$

We note that

1) $\mathrm{I}^{\mathrm{m}}_{1,1}f(\mathrm{z}) = \mathrm{T}^{m}f(\mathrm{z})$ (see [9]) .

2) $\mathrm{I}^{\mathrm{m}}_{1-\delta,1}f(\mathrm{z}) = \mathrm{T}^{\mathrm{m}}_{\delta}f(\mathrm{z}), \delta > 0$ (see [9]).

3) $\mathrm{I}^{\mathrm{m}}_{v,1}f(\mathrm{z}) = \mathrm{T}^{\mathrm{m}}_{v}f(\mathrm{z}), v > 0$ (see [9]).

Now, we define a new class $N\delta_u\ (\beta,\gamma,\mu; n,\lambda)$of harmonic functions of the form (1) that satisfying the inequality:

$$\boldsymbol{Re}\left\{\boldsymbol{\mu}(\mathbf{1}-\boldsymbol{\beta}) + \frac{\boldsymbol{\mu}(\mathbf{2}\boldsymbol{\beta}+\boldsymbol{\gamma})\left(\mathbf{I}^{\mathbf{m}+\mathbf{1}}_{v,\delta}\boldsymbol{f}(\mathbf{z})\right)' + \boldsymbol{\beta z}\left(\mathbf{I}^{\mathbf{m}+\mathbf{1}}_{v,\delta}\boldsymbol{f}(\mathbf{z})\right)'' + \left(\mathbf{I}^{\mathbf{m}+\mathbf{1}}_{v,\delta}\boldsymbol{f}(\mathbf{z})\right)'}{\left(\mathbf{I}^{\mathbf{m}+\mathbf{1}}_{v,\delta}\boldsymbol{f}(\mathbf{z})\right)' + \boldsymbol{z}\left(\mathbf{I}^{\mathbf{m}+\mathbf{1}}_{v,\delta}\boldsymbol{f}(\mathbf{z})\right)''}\right\}$$

$$> \boldsymbol{\mu}(\mathbf{1}+\boldsymbol{\beta}+\boldsymbol{\gamma}), \qquad (5)$$

where $\lambda \geq 0, 0 < \gamma < \frac{1}{2}, 0 < \beta < 1$ and $0 \leq \mu < 1$,

we further denote by $N\delta_u^* (\beta, \gamma, \mu; n, \lambda)$ the subclass of $N\delta_u(\beta, \gamma, \mu; n, \lambda)$ that satisfy the relation

$N\delta_u^* (\beta, \gamma, \mu; n, \lambda) = N\delta_u \cap \ N\delta_u(\beta, \gamma, \mu; n, \lambda)$.

In the following theorem, we find a sufficient condition for a function f to be in the class $N\delta_u(\beta, \gamma, \mu; n, \lambda)$.

Lemma (3) [9]: Let $\beta \geq 0$. Then $Re(w) > \beta$ if and only if

$|w - (1 + \beta)| < |w + (1 - \ \beta)|$, where w be any complex number.

Theorem (1): Let $f = h + \bar{g}$ (h and g be given by (5.1)). If

$$\sum_{n=2}^{\infty} (n(\mu(n - \beta n + 2\beta + \gamma + \beta) - \beta + 1) - n^2\mu(1 + \beta + \gamma))\big(1 + \lambda(n - 1)\big)\mathrm{I}_{v,\delta}^{m+1}|a_n|$$

$$+ \sum_{n=1}^{\infty} (\mathbf{n}(\boldsymbol{\mu}(\mathbf{n} - \boldsymbol{\beta}\mathbf{n} + 2\boldsymbol{\beta} + \boldsymbol{\gamma} + \boldsymbol{\beta}) - \boldsymbol{\beta} + 1) - \mathbf{n}^2\boldsymbol{\mu}(1 + \boldsymbol{\beta} + \boldsymbol{\gamma}))\big(1 + \boldsymbol{\lambda}(\mathbf{n} - 1)\big)\mathbf{I}_{v,\delta}^{m+1}|\boldsymbol{b}_n| \leq \mathbf{1}, \qquad (6)$$

where $\lambda \geq 0, 0 < \gamma < \frac{1}{2}, 0 < \beta < 1$ and $0 \leq \mu < 1$, then f is harmonic univalent sense-preserving in $\mathfrak{E}$ and $f \in N\delta_u(\beta, \gamma, \mu; n, \lambda)$.

Proof:

$$\left|h'(z)\right| > 1 - \sum_{n=2}^{\infty} n|\ a_n|\ z^{n-1} \ > 1 - \sum_{n=2}^{\infty} n a_n$$

$$> 1 - \sum_{\mathfrak{n}=2}^{\infty} \frac{(\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1+\beta+\gamma))(1+\lambda(\mathfrak{n}-1))\mathrm{I}_{v,\delta}^{m+1}}{\mathfrak{n}} |a_{\mathfrak{n}}|$$

$$> 1 - \sum_{\mathfrak{n}=1}^{\infty} \frac{(\mathfrak{n}(\mu(k - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1+\beta+\gamma))(1+\lambda(\mathfrak{n}-1))\mathrm{I}_{v,\delta}^{m+1}}{\mathfrak{n}} |b_{\mathfrak{n}}|$$

$$> \sum_{\mathfrak{n}=1}^{\infty} \frac{(\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1+\beta+\gamma))(1+\lambda(\mathfrak{n}-1))\mathrm{I}_{v,\delta}^{m+1}}{k} |b_{\mathfrak{n}}||z|^{\mathfrak{n}-1}$$

$$> \sum_{\mathfrak{n}=1}^{\infty} \mathfrak{n} |b_{\mathfrak{n}}||z|^{\mathfrak{n}-1}$$

$|g'(z)|$, sense-preserving.

And let

$$w(z) = \left\{ \mu(1-\beta) + \frac{\mu(2\beta+\gamma)\left(\left(\mathrm{I}_{v,\delta}^{m+1}h(z)\right)' + \overline{\left(\mathrm{I}_{v,\delta}^{m+1}g(z)\right)'}\right) + \beta z\left(\left(D_\lambda^n h(z)\right)'' + \overline{\left(\mathrm{I}_{v,\delta}^{m+1}g(z)\right)''}\right) + \left(\mathrm{I}_{v,\delta}^{m+1}h(z)\right)' + \overline{\left(\mathrm{I}_{v,\delta}^{m+1}g(z)\right)'}}{\left(\mathrm{I}_{v,\delta}^{m+1}h(z)\right)' + \overline{\left(\mathrm{I}_{v,\delta}^{m+1}g(z)\right)'} + z\left(\left(\mathrm{I}_{v,\delta}^{m+1}h(z)\right)'' + \overline{\left(\mathrm{I}_{v,\delta}^{m+1}g(z)\right)''}\right)} \right\}.$$

$$= \boldsymbol{Re} \frac{\boldsymbol{A(z)}}{\boldsymbol{B(z)}} \qquad (7)$$

where

$$A(z) = \mu(1-\beta)\left(\left(I_{v,\delta}^{m+1} f(z)\right)' + \left(I_{v,\delta}^{m+1} f(z)\right)''\right) + \mu(2\beta+\gamma)\left(I_{v,\delta}^{m+1} f(z)\right)'$$
$$+ \beta z \left(I_{v,\delta}^{m+1} f(z)\right)'' + \left(I_{v,\delta}^{m+1} f(z)\right)'$$

and

$$B(z) = 1 + \sum_{n=2}^{\infty} n\big(1+\lambda(n-1)\big) I_{v,\delta}^{m+1} a_n\, z^n + \sum_{n=1}^{\infty} (n^2 - n)\big(1+\lambda(n-1)\big) I_{v,\delta}^{m+1} b_n\, z^n .$$

Using the fact by Lemma (3)

$Re\{w(z)\} > \mu(1+\beta+\gamma)$ if and only if

$$\big|w(z) - \big(1+\mu(1+\beta+\gamma)\big)\big| < \big|w(z) + \big(1-\mu(1+\beta+\gamma)\big)\big|, (8)$$

It is sufficient to show that

$$\big|A(z) - \big(1+\mu(1+\beta+\gamma)\big)B(z)\big| - \big|A(z) + \big(1-\mu(1+\beta+\gamma)\big)B(z)\big| < 0.$$

Substituting for A(z) and B(z) and making use of (4) and (5) and resorting to simple calculation, we find that

$$\big|A(z) - \big(1+\mu(1+\beta+\gamma)\big)B(z)\big| - \big|A(z) + \big(1-\mu(1+\beta+\gamma)\big)B(z)\big|$$

Such that

$$\big|A(z) - \big(1+\mu(1+\beta+\gamma)\big)B(z)\big|$$

$$= \Bigg|(\mu(1+\beta+\gamma)+1)z$$

$$+\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|a_{\mathrm{n}}|z^{\mathrm{n}-1}$$

$$+\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|a_{\mathrm{n}}|z^{\mathrm{n}-1}$$

$$-\big(1+\mu(1+\beta+\gamma)\big)\left(1+\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}a_{\mathrm{n}}\ z^{\mathrm{n}}\right.$$

$$\left.+\sum_{\mathrm{n}=1}^{\infty}(\mathrm{n}^2-\mathrm{n})\ \big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}b_{\mathrm{n}}\ z^{\mathrm{n}}\right)\Bigg| \quad (9)$$

Also

$$\big|A(z)+\big(1-\mu(1+\beta+\gamma)\big)B(z)\big|$$

$$= \Bigg|(\mu(1+\beta+\gamma)+1)z$$

$$+\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|a_{\mathrm{n}}|z^{\mathrm{n}-1}$$

$$+\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|a_{\mathrm{n}}|z^{\mathrm{n}-1}$$

$$+\big(1-\mu(1+\beta+\gamma)\big)\left(1+\sum_{k=2}^{\infty}\mathrm{n}\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}a_{\mathrm{n}}\ z^{\mathrm{n}}\right.$$

$$\left.+\sum_{\mathrm{n}=1}^{\infty}(\mathrm{n}^2-\mathrm{n})\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\ b_{\mathrm{n}}\ z^{\mathrm{n}}\right)\Bigg| \quad (10)$$

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$$\leq \sum_{\mathrm{n}=2}^{\infty}(\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma)+\mathrm{n}^2)\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|a_{\mathrm{n}}|$$

$$\sum_{\mathrm{n}=1}^{\infty}(\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma)+\mathrm{n}^2)\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|b_{\mathrm{n}}|$$

$$-1 \leq 0.$$

By inequality (7), which implies that $f \in N\delta_u(\beta,\gamma,\mu;n,\lambda)$.

$$f(z)$$

$$= z + \sum_{\mathrm{n}=2}^{\infty}\frac{x_{\mathrm{n}}}{(\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}}z^{\mathrm{n}}$$

$$+\sum_{\mathrm{n}=1}^{\infty}\frac{\overline{y_k}}{\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}}(\bar{z})^{\mathrm{n}}, (8)$$

where

$$\left(\sum_{\mathrm{n}=2}^{\infty}|x_{\mathrm{n}}| + \sum_{\mathrm{n}=1}^{\infty}|\overline{y_{\mathrm{n}}}| = 1\right),$$

shows that the coefficients bounds given by (7) is sharp.

The function of the form (11) are in the class $N\delta_u(\beta,\gamma,\mu;n,\lambda)$ because in view of (7), we infer that

$$\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\,\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|a_{\mathrm{n}}|$$

$$+\sum_{\mathrm{n}=1}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\,\mathrm{I}_{v,\delta}^{\mathrm{m}+1}|b_{\mathrm{n}}|$$

$$= \sum_{n=2}^{\infty} |\boldsymbol{x_n}| + \sum_{n=1}^{\infty} |\overline{\boldsymbol{y_n}}| = \mathbf{1} \qquad (\mathbf{9})$$

The restriction placed in Theorem (1) on the moduli of the coefficients of $f = h + \overline{g}$ implies that for arbitrary rotation of the coefficients of f, the resulting functions would still be harmonic univalent and $f \in N\delta_u(\beta, \gamma, \mu; n, \lambda)$.

The following theorem shows that the condition (7) is also necessary for the function f to belong to the class $N\delta_u^*(\beta, \gamma, \mu; n, \lambda)$.

Theorem (2). Let $f = h + \overline{g}$ with h and g are given by (2).Then

$f \in N\delta_u^*(\beta, \gamma, \mu; n, \lambda)$ iff

$$\sum_{n=2}^{\infty} n(\mu(n - \beta n + 2\beta + \gamma + \beta) - \beta + 1) - n^2\mu(1 + \beta + \gamma))\big(1 + \lambda(n-1)\big)\, I_{v,\delta}^{m+1}|a_n|$$

$$+ \sum_{n=1}^{\infty} n(\mu(n - \beta n + 2\beta + \gamma + \beta) - \beta + 1) - n^2\mu(1 + \beta + \gamma))\big(1 + \lambda(n-1)\big)\, I_{v,\delta}^{m+1}|b_n|$$

$$\leq 1, \ldots\ldots\ldots\ldots (10)$$

Where $\lambda \geq 0, 0 < \gamma < \frac{1}{2}, 0 < \beta < 1$and $0 \leq \mu < 1$.

Proof: By noting that $N\delta_u^*(\beta, \gamma, \mu; n, \lambda) \subset N\delta_u(\beta, \gamma, \mu; n, \lambda)$, the sufficiency part of Theorem 2 follows at once from Theorem (1).To prove the necessary part, let us assume that

$f \in N\delta_u^*(\beta, \gamma, \mu; n, \lambda)$. Using (3), we get

$$Re\left\{\mu(1-\beta) + \frac{\mu(2\beta+\gamma)\left(\left(I_{v,\delta}^{m+1}h(z)\right)' + \overline{\left(I_{v,\delta}^{m+1}g(z)\right)'}\right) + \beta z\left(\left(I_{v,\delta}^{m+1}h(z)\right)'' + \overline{\left(I_{v,\delta}^{m+1}g(z)\right)''}\right) + \left(I_{v,\delta}^{m+1}h(z)\right)' + \overline{\left(I_{v,\delta}^{m+1}g(z)\right)'}}{\left(D_\lambda^n h(z)\right)' + \overline{\left(I_{v,\delta}^{m+1}g(z)\right)'} + z\left(\left(I_{v,\delta}^{m+1}h(z)\right)'' + \overline{\left(I_{v,\delta}^{m+1}g(z)\right)''}\right)}\right\}$$

$$= Re\{\frac{(\mu(1+\beta+\gamma)+1)z - \sum_{n=2}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)(1+\lambda(n-1))I_{v,\delta}^{m+1}|a_n|z^{n-1}}{1-\sum_{n=2}^{\infty} n^2|a_n|z^{n-1} - \sum_{n=1}^{\infty} n^2|b_n|\bar{z}^{n-1}}$$

$$\frac{-\sum_{n=1}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)(1+\lambda(n-1))I_{v,\delta}^{m+1}|b_k|\bar{z}^{k-1}}{1-\sum_{n=2}^{\infty} n^2|a_n|z^{n-1} - \sum_{n=1}^{\infty} n^2|b_n|\bar{z}^{n-1}}\} > \mu(1+\beta+\gamma)$$

If we choose z to be real and let $z \to 1^-$, we obtain

$$\mu(1+\beta+\gamma)+1-\sum_{n=2}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)(1+\lambda(n-1))I_{v,\delta}^{m+1}|a_n|$$
$$-\sum_{n=1}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)(1+\lambda(n-1))I_{v,\delta}^{m+1}|b_n|$$
$$> \mu(1+\beta+\gamma) - \sum_{n=2}^{\infty}\mu(1+\beta+\gamma)n^2|a_n| - \sum_{n=1}^{\infty}\mu(1+\beta+\gamma)n^2|b_n|.$$

Hence

$$\sum_{n=2}^{\infty}(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma))(1+\lambda(n-1))I_{v,\delta}^{m+1}|a_n|$$
$$+\sum_{n=1}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma))(1$$
$$+\lambda(n-1))I_{v,\delta}^{m+1}|b_n| \le 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

which complete the proof of Theorem (2).

Now, we determine the class $N\delta_u^* \,(\beta, \gamma, \mu; n, \lambda)$ is convex set.

Theorem 3: The class $N\delta_u^* \,(\beta, \gamma, \mu; n, \lambda)$ is convex set.

Proof: Let the function f_s $(S = 1,2)$ defined by

$$f_S(z) = z - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n},1}| z^{\mathfrak{n}} - \sum_{\mathfrak{n}=1}^{\infty} |b_{\mathfrak{n},2}| (\bar{z})^{\mathfrak{n}} \tag{12}$$

be in the class $N\delta_u^* \,(\beta, \gamma, \mu; n, \lambda)$.

It is sufficient to prove that the function

$$C(z) = tf_1(z) + (1-t)f_2 \; (0 < t < 1), \tag{13}$$

is also in the class $N\delta_u^* \,(\beta, \gamma, \mu; n, \lambda)$. Since for $0 < t < 1$,

$$C(z) = z - \sum_{\mathfrak{n}=2}^{\infty} \left(\mathrm{t}|a_{\mathfrak{n},1}| + (1-t)|a_{\mathfrak{n},2}|\right) z^{\mathfrak{n}} - \sum_{\mathfrak{n}=1}^{\infty} \left(\mathrm{t}|b_{\mathfrak{n},1}| + (1-t)|b_{\mathfrak{n},2}|\right) (\bar{z})^{\mathfrak{n}},$$

with the aid of Theorem (2), we have that

$$\sum_{\mathfrak{n}=2}^{\infty} \mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\left(1 + \lambda(\mathfrak{n} - 1)\right) \mathrm{I}_{v,\delta}^{\mathrm{m}+1}\left(\mathrm{t}|a_{\mathfrak{n},1}| + (1-t)|a_{\mathfrak{n},2}|\right)$$

$$+ \sum_{\mathfrak{n}=1}^{\infty} \mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\left(1 + \lambda(\mathfrak{n} - 1)\right) \mathrm{I}_{v,\delta}^{\mathrm{m}+1}\left(\mathrm{t}|b_{\mathfrak{n},1}| + (1-t)|b_{\mathfrak{n},2}|\right)$$

$$= t\left[\sum_{n=2}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma))\big(1+\lambda(n-1)\big)I_{v,\delta}^{m+1}|a_{n,1}|\right.$$

$$\left.+\sum_{n=1}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma))\big(1+\lambda(n-1)\big)I_{v,\delta}^{m+1}|b_{n,1}|\right]$$

$$+(1-t)\left[\sum_{n=2}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma))\big(1+\lambda(n-1)\big)I_{v,\delta}^{m+1}|a_{n,1}|\right.$$

$$\left.+\sum_{n=1}^{\infty} n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma))\big(1+\lambda(n-1)\big)I_{v,\delta}^{m+1}|b_{n,1}|\right]$$

$$\leq t+(1-t)=1.$$

Hence, $C(z) \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic function of the form:

$$\boldsymbol{f(z)=z-\sum_{n=2}^{\infty}|a_n|\,z^n-\sum_{n=1}^{\infty}|b_n|\,(\bar{z})^n} \qquad (14)$$

and

$$\boldsymbol{F(z)=z-\sum_{n=2}^{\infty}|c_n|\,z^n-\sum_{n=1}^{\infty}|d_n|\,(\bar{z})^n} \qquad (15)$$

We define the convolution of two harmonic functions f and F as:

$$(f*F)(z)=f(z)*F(z)=z-\sum_{n=2}^{\infty}|a_n c_n|\,z^n-\sum_{n=1}^{\infty}|b_n d_n|\,(\bar{z})^n.$$

Theorem (4): If f and F are in $N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$, then

$(f * F) \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$

Proof: Let f and F of the forms (14), (15) and

$|c_n| \le 1$ and $|d_n| < 1$. Since $f \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$,

Then we can write by theorem (2)

$$\sum_{n=2}^{\infty}\left(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(n-1)\right) I_{v,\delta}^{m+1}|a_n|$$

$$+\left(\mathbf{n}(\boldsymbol{\mu}(\mathbf{n}-\boldsymbol{\beta}\mathbf{n}+2\boldsymbol{\beta}+\boldsymbol{\gamma}+\boldsymbol{\beta})-\boldsymbol{\beta}+1)-\mathbf{n}^2\boldsymbol{\mu}(1+\boldsymbol{\beta}+\boldsymbol{\gamma})\right)\left(1+\boldsymbol{\lambda}(\mathbf{n}-1)\right)I_{v,\delta}^{m+1}|\boldsymbol{b}_n| \le \mathbf{1}, \qquad (\mathbf{16})$$

and since $F \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$

$$\sum_{n=2}^{\infty}\left(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(n-1)\right) I_{v,\delta}^{m+1}|c_n|$$

$$+\sum_{n=1}^{\infty}\left(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(n-1)\right) I_{v,\delta}^{m+1}|d_n| \le 1. \ (17)$$

Then

$$\sum_{n=2}^{\infty}\left(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(n-1)\right)I_{v,\delta}^{m+1}|a_n c_n|$$

$$+\sum_{n=1}^{\infty}\left(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(n-1)\right)I_{v,\delta}^{m+1}|b_n d_n|$$

$$\le \sum_{n=2}^{\infty}\left(n(\mu(n-\beta n+2\beta+\gamma+\beta)-\beta+1)-n^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(n-1)\right)|a_n|$$

$$+\sum_{\mathfrak{n}=1}^{\infty}\left(\mathfrak{n}(\mu(\mathfrak{n}-\beta\mathfrak{n}+2\beta+\gamma+\beta)-\beta+1)-\mathfrak{n}^2\mu(1+\beta+\gamma)\right)\left(1+\lambda(\mathfrak{n}-1)\right)\mathrm{I}_{v,\delta}^{m+1}|b_{\mathfrak{n}}| \leq 1 \quad (17)$$

Then $f * F \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$

Definition (1)[11][14]: The Bernardi operator is defined by

$$L_{\mathrm{Q}}(f(z)) = \frac{1+\mathrm{Q}}{z^{\mathrm{Q}}}\int_0^z t^{\mathrm{Q}-1}f(t)dt\,,(-1<\mathrm{Q}) \quad (18)$$

If

$$f(z) = z + \sum_{\mathfrak{n}=2}^{\infty} e_{\mathfrak{n}}\ z^{\mathfrak{n}},$$

then

$$L_{\mathrm{Q}}\big(f(z)\big) = z + \sum_{\mathfrak{n}=2}^{\infty}\frac{1+\mathrm{Q}}{\mathfrak{n}+\mathrm{Q}}e_{\mathfrak{n}}\ z^{\mathfrak{n}}, \quad (19)$$

Remark (1): If $f = h + \bar{g}$, where

$$h(z) = z - \sum_{\mathfrak{n}=2}^{\infty}|a_{\mathfrak{n}}|\ z^{\mathfrak{n}}, g(z) = -\sum_{\mathfrak{n}=1}^{\infty}|b_{\mathfrak{n}}|\ z^{\mathfrak{n}},$$

then

$$L_{\mathrm{Q}}\big(f(z)\big) = L_{\mathrm{Q}}\big(h(z)\big) + \overline{L_{\mathrm{Q}}\big(g(z)\big)} \quad (20)$$

Theorem (5): Let $f \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$Then $L_{\mathrm{Q}}(f)$ $(\mathrm{Q}\in\mathbb{N})$ is also belong to the class $N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$

Proof: By (19) and (20), we get

$$= z - \sum_{\mathfrak{n}=2}^{\infty}\frac{1+\mathrm{Q}}{\mathfrak{n}+\mathrm{Q}}|a_{\mathfrak{n}}|\,z^{\mathfrak{n}} - \sum_{\mathfrak{n}=1}^{\infty}\frac{1+\mathrm{Q}}{\mathfrak{n}+\mathrm{Q}}|b_{\mathfrak{n}}|\,(\bar{z})^{\mathfrak{n}}.$$

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Since $f \in N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$, then by Theorem (2), we have

$$\sum_{\mathrm{n}=2}^{\infty}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(k-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\,|a_{\mathrm{n}}|$$

$$+\sum_{k=1}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta k+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\,|b_{\mathrm{n}}|$$
$$\leq 1.$$

Since $\mathrm{Q} \in \mathbb{N}$, then $\frac{1+\mathrm{Q}}{\mathrm{n}+\mathrm{Q}} \leq 1$, therefore

$$\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)$$
$$-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\left(\frac{1+\mathrm{Q}}{\mathrm{n}+\mathrm{Q}}\right)|a_{\mathrm{n}}|$$

$$+\sum_{\mathrm{n}=1}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)$$
$$-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\left(\frac{1+\mathrm{Q}}{\mathrm{n}+\mathrm{Q}}\right)|b_{\mathrm{n}}|$$

$$\leq\sum_{k=2}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\,|a_{\mathrm{n}}|$$

$$+\sum_{\mathrm{n}=1}^{\infty}\mathrm{n}(\mu(\mathrm{n}-\beta\mathrm{n}+2\beta+\gamma+\beta)-\beta+1)-\mathrm{n}^2\mu(1+\beta+\gamma))\big(1+\lambda(\mathrm{n}-1)\big)\mathrm{I}_{v,\delta}^{\mathrm{m}+1}\,|b_{\mathrm{n}}|\leq 1,$$

and this gives the result.

Next, we determine the extreme points for the class $N\delta_u^*(\beta,\gamma,\mu;n,\lambda)$

are given in the following theorem.

Theorem (6): Let the function f be given by (2). Then

$f \in N\delta_u^* (\beta, \gamma, \mu; n, \lambda)$ if and only if f can be expressed as

$$f(z) = \sum_{n=1}^{\infty} (\omega_n \, h_n(z) + \vartheta_n \, g_n(z)) \text{ , where } z \in n, \tag{21}$$

$$h_1(z) = z,$$

$$h_n(z) = z - \frac{1}{n(\mu(n - \beta n + 2\beta + \gamma + \beta) - \beta + 1) - n^2\mu(1 + \beta + \gamma))(1 + \lambda(n - 1))I_{v,\delta}^{m+1}} z^n,$$

$$(n = 2, 3, \dots) \tag{22}$$

$$g_n(z) = z - \frac{1}{n(\mu(n - \beta n + 2\beta + \gamma + \beta) - \beta + 1) - n^2\mu(1 + \beta + \gamma))(1 + \lambda(n - 1))I_{v,\delta}^{m+1}} (\bar{z})^n,$$

$$(n = 1, 2, \dots) \tag{23}$$

$$\sum_{n=1}^{\infty} (\omega_n + \vartheta_n) = 1, (\omega_n \geq 0, \vartheta_n \geq 0).$$

In particular, the extreme points of $N\delta_u^* (\beta, \gamma, \mu; n, \lambda)$ are $\{h_n\}$ and $\{g_n\}$.

Proof: Suppose f is of the form (21). Using (22) and (23), we get

$$f(z) = \sum_{n=1}^{\infty} (\omega_n \, h_n(z) + \vartheta_n \, g_n(z))$$

$$= \sum_{\mathfrak{n}=1}^{\infty} (\omega_{\mathfrak{n}} + \vartheta_{\mathfrak{n}}) z$$

$$- \sum_{k=2}^{\infty} \frac{1}{\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\big(1 + \lambda(\mathfrak{n} - 1)\big)I_{v,\delta}^{m+1}} \omega_{\mathfrak{n}} z^{\mathfrak{n}}$$

$$- \sum_{\mathfrak{n}=1}^{\infty} \frac{1}{\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\big(1 + \lambda(\mathfrak{n} - 1)\big)I_{v,\delta}^{m+1}} \vartheta_{\mathfrak{n}} (\bar{z})^{\mathfrak{n}}$$

$$= \sum_{\mathfrak{n}=1}^{\infty} (\omega_{\mathfrak{n}} + \vartheta_{\mathfrak{n}}) - \omega_1 = (1 - \omega_1) \leq 1,$$

which implies that $f \in N\delta_u^* (\beta, \gamma, \mu; n, \lambda)$.

Conversely, let $f \in N\delta_u^* (\beta, \gamma, \mu; n, \lambda)$. Then

$$|a_{\mathfrak{n}}| \leq \frac{\omega_{\mathfrak{n}}}{\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\big(1 + \lambda(\mathfrak{n} - 1)\big)I_{v,\delta}^{m+1}}$$

$$|b_{\mathfrak{n}}| \leq \frac{\vartheta_{\mathfrak{n}}}{\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\big(1 + \lambda(\mathfrak{n} - 1)\big)I_{v,\delta}^{m+1}}.$$

Set

$$\omega_k = \mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\big(1 + \lambda(\mathfrak{n} - 1)\big)I_{v,\delta}^{m+1}|a_{\mathfrak{n}}|, (\mathfrak{n} = 2,3, \dots)$$

and

$$\vartheta_{\mathfrak{n}} = \mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))\big(1 + \lambda(\mathfrak{n} - 1)\big)I_{v,\delta}^{m+1}|b_{\mathfrak{n}}|(\mathfrak{n} = 1,2,3 \dots).$$

where

$$\sum_{\mathfrak{n}=1}^{\infty} (\omega_{\mathfrak{n}} + \vartheta_{\mathfrak{n}}) = 1,$$

we define

$$\omega_1 = 1 - \sum_{\mathfrak{n}=2}^{\infty} \omega_{\mathfrak{n}} - \sum_{\mathfrak{n}=1}^{\infty} \vartheta_{\mathfrak{n}}.$$

we have

$$f(z) = z - \sum_{\mathfrak{n}=2}^{\infty} |a_{\mathfrak{n}}| z^{\mathfrak{n}} - \sum_{\mathfrak{n}=1}^{\infty} |b_{\mathfrak{n}}| (\bar{z})^{\mathfrak{n}}$$

$$= z$$

$$- \sum_{\mathfrak{n}=2}^{\infty} \frac{1}{\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))(1 + \lambda(\mathfrak{n} - 1))\mathrm{I}_{v,\delta}^{\mathfrak{m}+1}} \omega_{\mathfrak{n}} z^{\mathfrak{n}}$$

$$- \sum_{\mathfrak{n}=1}^{\infty} \frac{1}{\mathfrak{n}(\mu(\mathfrak{n} - \beta\mathfrak{n} + 2\beta + \gamma + \beta) - \beta + 1) - \mathfrak{n}^2\mu(1 + \beta + \gamma))(1 + \lambda(\mathfrak{n} - 1))\mathrm{I}_{v,\delta}^{\mathfrak{m}+1}} \vartheta_{\mathfrak{n}} (\bar{z})^{\mathfrak{n}}$$

$$= \left(1 - \sum_{\mathfrak{n}=2}^{\infty} \omega_k - \sum_{\mathfrak{n}=1}^{\infty} \vartheta_k\right) z + \sum_{\mathfrak{n}=2}^{\infty} \omega_{\mathfrak{n}} h_{\mathfrak{n}}(z) + \sum_{\mathfrak{n}=1}^{\infty} \vartheta_{\mathfrak{n}} g_{\mathfrak{n}}(z)$$

$$= \omega_1 h_1(z) + \sum_{\mathfrak{n}=2}^{\infty} \omega_{\mathfrak{n}} h_{\mathfrak{n}}(z) + \sum_{\mathfrak{n}=1}^{\infty} \vartheta_{\mathfrak{n}} g_{\mathfrak{n}}(z)$$

$$= \sum_{\mathfrak{n}=1}^{\infty} (\omega_{\mathfrak{n}} h_{\mathfrak{n}}(z) + \vartheta_{\mathfrak{n}} g_{\mathfrak{n}}(z)).$$

That is the required representation.

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