Small Pointwise Projective Modules

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Abstract

Let R be a ring and let M be a left R-module. In this work we present a small pointwise projective module as generalization of pointwise projective module, we also introduce the notations of small pointwise projective hollow modules, amply supplemented small pointwise projective module, small pointwise projective module M with finite spanning dimension, small pointwise hereditary module and we study their basic properties.

Keyword : projective , pointwise projective , small projective

المقاسات الاسقاطية نقطيا من النوع الصغير

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الخلاصة

لتكن R حلقة ولتكن M مقاس ايسر معرف على R. قدمنا في هذا البحث مفهوم المقاس الاسقاطي النقطي من النوع الصغير بصفته تعميما لمفهوم المقاس الاسقاطي النقطي . كذلك قدمنا مفاهيم المقاسات الاسقاطية النقطية المجوفة من النوع الصغير. المقاسات الاسقاطية النقطية المكملة باسهاب من النوع الصغير المقاسات الاسقاطية النقطية المنتهية البعد من النوع الصغير. المقاسات الاسقاطية الوراثية النقطية من النوع الصغير ودرسنا بعض الخواص الاساسية.

الكلمات المفتاحية : المقاس الاسقاطي - المقاس الاسقاطي النقطي - المقاس الاسقاطي من النوع الصغير

Introduction

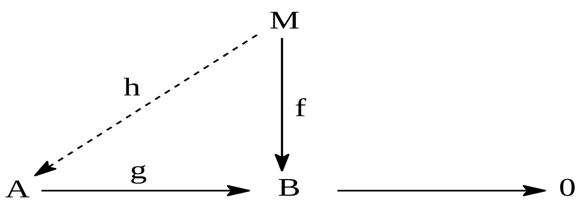
Let R be a ring and M be a left R-module. A submodule N of an R-module M is called small submodule of M if N + L = M for any submodule L of M implies L = M [1]. An epimorphism $g : A \rightarrow B$ is called small provided ker g is small submodule in B [2]. An R-module M is called small projective module if for each small epimorphism $g : A \rightarrow B$ where

A and B are any R-modules and for each homomorphism f : M → B there exists a homomorphism h : M → A such that g ∘ h = f [2]. An R-module M is called pointwise projective module if for each epimorphism g : A → B where A and B are any R-modules and for each homomorphism f : M → B then for every m ∈ M there exists a homomorphism h : M → A such that g ∘ h(m) = f(m) [3]. In this paper we introduce the concept of small pointwise projective module as follows : An R-module M is called small pointwise projective module if for each small epimorphism g : A → B where A and B are any R-modules and for each homomorphism f : M → B then for every m ∈ M there exists a homomorphism h : M → A such that g ∘ h(m) = f(m). A non zero module M is hollow if for every proper submodule is small in M [1]. we study the endomorphism ring of a small pointwise projective hollow modules. A submodule V of M is called supplemented of a submodule U of M if V is a minimal element in the set of submodules L of M with U + L = M [1]. An R-module M is called supplemented if for every submodule of M has supplemented in M [1]. An R-module M is called amply supplemented if for every two submodules U , V of M such that M = U + V, U contains a supplemented of V in M [4]. we study amply supplemented small pointwise projective module. Finally, recall that an R-module M is hereditary if for every submodule of M is projective [1]. In this paper we introduce the concept of small pointwise hereditary module and some properties.

§1 Characterization Of Small Pointwise Projective Modules

In this section, we give the definition of small pointwise projective modules and we give some characterization of this concept.

Let R be a ring and M be a left R-module. M is called small pointwise projective module if for each small epimorphism g : A → B where A and B are any R-modules and for each homomorphism f : M → B then for every m ∈ M there exists a homomorphism h : M → A such that g ∘ h(m) = f(m), i.e., the following diagram commutes:



The following proposition gives a characterization for small pointwise projective modules.

Proposition (1.1) The following are equivalent for a module M

1) M is small pointwise projective module.

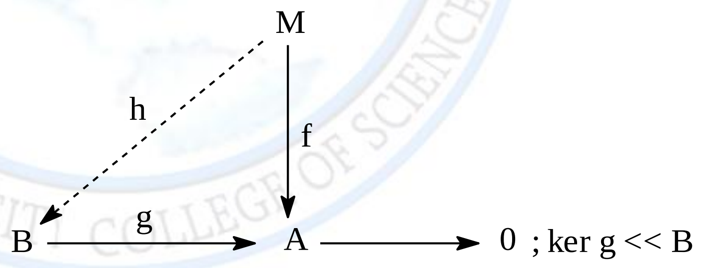
2) For each small epimorphism $f : N \to K$, the homomorphism $Hom(I , f) : Hom(M , N) \to Hom(M , K)$ is an epimorphism.

3) For any small epimorphism $g : B \to A$, $g \circ Hom(M , B) = Hom(M , A)$.

<u>Proof:</u> 1→2) Let $f : N \to K$ be a small epimorphism and $g \in Hom(M , k)$. Since M is small pointwise projective, for each $m \in M$ there exists a homomorphism $h : M \to N$ such that $f \circ h(m) = g(m)$. Therefore $(Hom(I , f) \circ h)(m) = g(m)$ where $h \in Hom(M , N)$.Thus $Hom (I , f)$ is an epimorphism.

2→3) Let $g : B \to A$ be a small epimorphism.By2) $Hom(I , g) : Hom(M , B) \to Hom(M , A)$ is an epimorphism. Now to show that $g \circ Hom(M , B) = Hom(M , A)$.Let $f \in Hom(M , A)$ so there exists $f_1 \in Hom(M , B)$ such that $Hom(I , g) \circ f_1 = f$. i.e.; $g \circ f_1 = f$.Thus $f \in g \circ Hom(M , B)$, so $Hom(M , A) \leq g \circ Hom(M , B)$. Clearly $g \circ Hom(M , B) \leq Hom (M , A)$. Therefore $g \circ Hom(M , B) = Hom(M , A)$.

3→1) Consider the following diagram



where $g : B \to A$ is small epimorphism and $f : M \to A$ is any homomorphism.

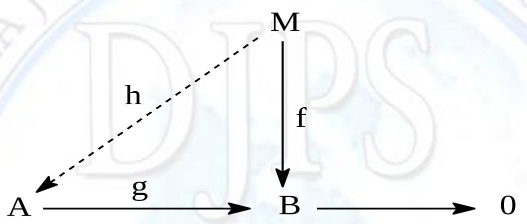
Let $m \in M$, since $g \circ Hom(M , B) = Hom(M , A)$ and $f \in Hom(M , A)$ there exists $h \in Hom(M , B)$ such that $g \circ h = f$ and hence $g \circ h(m) = f(m)$. Thus M is small pointwise projective module. Every small projective module is small pointwise projective module since every projective module is pointwise projective module .Every projective module is small pointwise projective module since every projective module is small projective module. Every pointwise projective module is small pointwise projective module since every projective module is small projective module.

A ring R is called cosemisimple if Rad (M) = 0, for each R-module M [2].

Proposition (1.2) Let R be a cosemisimple ring, every module over R is small pointwise projective module.

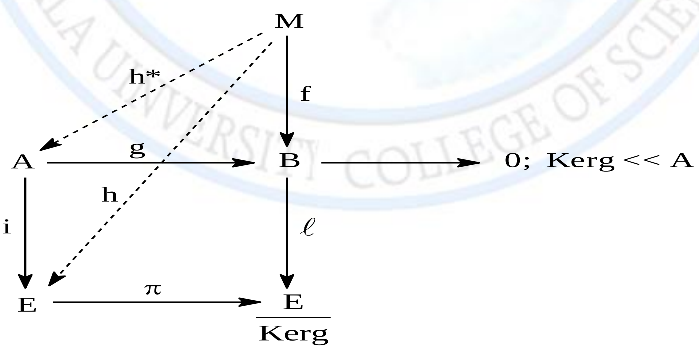
Proof: Every module over cosemisimple ring is small projective module [1] and hence small pointwise projective module .The next proposition is another characterization of small pointwise projective module.

Proposition (1.3) An R-module M is small pointwise projective module if and only if for every homomorphism $f : M \to B$ where B is an R-module and every small epimorphism $g : A \to B$ from an injective module A, for each $m \in M$ there exists a homomorphism $h : M \to A$ such that $g \circ h(m) = f(m)$, i.e., the following diagram commutes:



Proof : ⇒) Clear

⇐) Let $g : A \to B$ be any small epimorphism and $f : M \to B$ be any homomorphism. Consider the following diagram



Since every module can be imbedded in an injective module [5], then there exists an injective module E , $i : A \to E$ be the inclusion homomorphism and $\pi : E \to \frac{E}{\ker g}$ be the natural epimorphism. Define $\ell : B \to \frac{E}{\ker g}$ by $\ell(b) = a + \ker g$ for all $b \in B$ where $g(a) = b$. It is clear that ℓ is well define and homomorphism. For each $m \in M$ there exists a homomorphism $h : M \to E$ such that $\pi \circ h(m) = \ell \circ f(m)$.Let $w \in h(M)$,there exists $m_1 \in M$ such that

$w = h(m_1)$,$\pi \circ h(m_1) = \ell \circ f(m_1)$,where $g(a) = f(m_1)$,thus $h(m_1) - a \in \ker g$ and hence $h(m_1) \in A$. Thus $h(M) \leq A$. Define $h^*: M \to A$ by $h^*(x) = h(x)$ for all $x \in M$. Now, $\ell \circ f(m) = \pi \circ h(m) = \pi \circ i \circ h^*(m) = \ell \circ g \circ h^*(m)$ .Thus $g \circ h^*(m) = f(m)$ for each $m \in M$ and hence M is small pointwise projective module.

Proposition (1.4) Let M be an R-module , the following statements are equivalent

1) M is a small pointwise projective module.

2) Each small epimorphism $g : A \to M$ where A is any R-module is a pointwise split i.e. for each $m \in M$, there exists a homomorphism $h : M \to A$ such that $g \circ h(m) = m$.

Proof: clear

An R-module, M is called Z-regular if for each $x \in M$, there exists $h \in M^*$ such that $x = h(x)\, x$ [6].

Proposition (1.5) Every Z-regular module is a small pointwise projective R-module.

Proof: Every Z-regular module is pointwise projective R-module [3] and hence small pointwise projective R-module.

A submodule N of an R-module M is called pure submodule if for every finitely generated ideal I of R, $IM \cap N = IN$ [7].

An R-module M is called F-regular if every submodule of M is pure [8].

Proposition (1.6) Every Z-regular R-module is F-regular small pointwise projective module.

Proof: It follows from proposition (1.5) and [8]

It is known that every module over a regular ring is F-regular [9].Thus we have

Proposition (1.7) Let R be a regular ring, then every small pointwise projective R-module is F-regular,

Now we ready to give example of small pointwise projective module that is not projective.

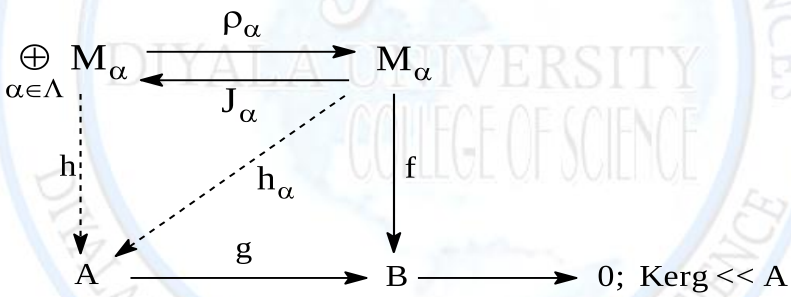
Example (1.8) Let k be a field, let $i \in I$ where I is an infinite countable set and let $k_i = k$. Put $R = \prod_{i \in I} k_i$ with the usual operations, R is a ring. Since k is a field, then k is a regular ring and hence R is regular ring. Let $p = \oplus_{i \in I} k_i$. Clearly, p is an ideal of R, also p is a z-regular R-module [6]. By proposition (1.5) p is a small pointwise projective R-module.We claim that p is not projective R-module. In fact, it can be easily shown that p is not direct summand of R, thus it is not a direct summand of $\oplus_{i \in I} R_i$; $R_i = R$. Therefore p is not direct summand for any free R-module. Hence by [5 , p.256] p is not projective.

§2 Some Properties Of Small Pointwise Projective Modules

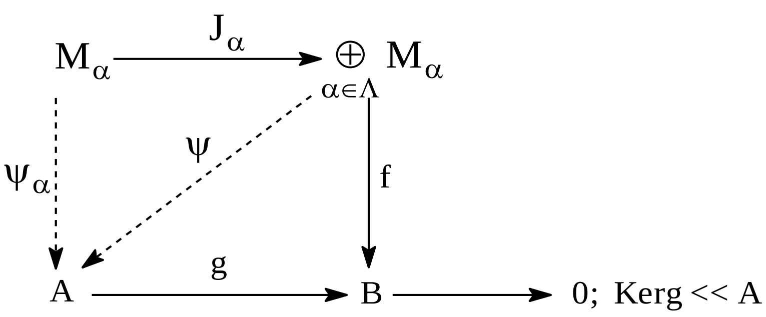
In this section, we give new properties of small pointwise projective module.

Proposition (2.1) $\oplus_{\alpha \in \Lambda} M_\alpha$ is small pointwise projective module if and only if M_α is small pointwise projective module for each $\alpha \in \Lambda$.

Proof: Assume $\oplus_{\alpha \in \Lambda} M_\alpha$ is small pointwise projective module. Consider the following diagram



where $g : A \to B$ is any small epimorphism, $f : M_\alpha \to B$ is any homomorphism, $\rho_\alpha : \oplus_{\alpha \in \Lambda} M_\alpha \to M_\alpha$ is the projection homomorphism and $J_\alpha : M_\alpha \to \oplus_{\alpha \in \Lambda} M_\alpha$ is the injection homomorphism. Since $\oplus_{\alpha \in \Lambda} M_\alpha$ is small pointwise projective module for each $m \in M$ there exists a homomorphism $h : \oplus_{\alpha \in \Lambda} M_\alpha \to A$ such that $g \circ h(m) = f \circ p_\alpha(m)$. Define $h_\alpha : M_\alpha \to A$ by $h_\alpha = h \circ j_\alpha$, $g \circ h_\alpha(m) = g \circ h \circ j_\alpha(m) = f \circ p_\alpha \circ j_\alpha(m)$. Thus M_α is small pointwise projective module .Suppose M_α is small pointwise projective module. Consider the following diagram



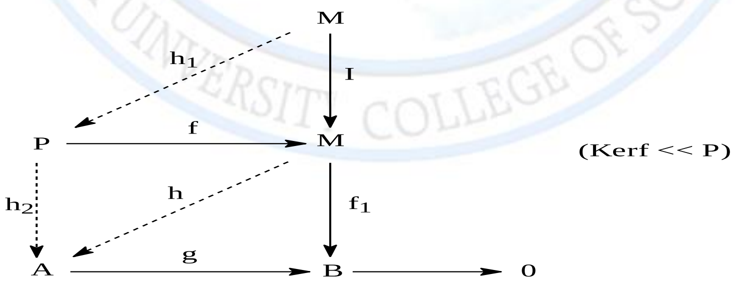
where g : A → B is any small epimorphism, f : $\oplus_{\alpha\in\Lambda} M_\alpha$ → B is any homomorphism and J_α : $M_\alpha \to \oplus_{\alpha\in\Lambda} M_\alpha$ is the injection homomorphism. Since M_α is small pointwise projective module for each m ∈ M there exists a homomorphism $\psi_\alpha : M_\alpha \to A$ for all α ∈ Λ such that $g\circ\psi_\alpha(m) = f\circ j_\alpha(m)$. Define $\psi : \oplus_{\alpha\in\Lambda} M_\alpha \to A$ by $\psi(h) = \sum_{\alpha\in\Lambda}\psi_\alpha(h(\alpha))$ for each $h \in \oplus_{\alpha\in\Lambda} M_\alpha$.It is clear that ψ is well define and homomorphism.$[(g\circ\psi)(h)](m) = [g(\sum_{\alpha\in\Lambda}\psi_\alpha(h(\alpha)))](m) = [\sum_{\alpha\in\Lambda}(g\circ\psi_\alpha)(h(\alpha))](m) = [\sum_{\alpha\in\Lambda}(f\circ j_\alpha)(h(\alpha))](m) = f(\sum_{\alpha\in\Lambda} j_\alpha(h(\alpha)))(m) = (f(h))(m)$.

A pair (p , f) is called a projective cover for a module M, if p is projective module and f is an epimorphism of p onto M with ker f ≪ p [10, p.199].

Proposition (2.2) A small pointwise projective module which has projective cover is projective module.

Proof : Let M be small pointwise projective module . Let (p , f) be projective cover for M.

Consider the following diagram



where g : A → B is any epimorphism, f_1 : M → B is any homomorphism and I : M → M is the identity. Since M is small pointwise projective module for each m ∈ M there exists a homomorphism $h_1 : M \longrightarrow P$ such that $f\circ h_1(m) = I(m)$. But P is a projective module then there exists a homomorphism $h_2 : p \longrightarrow A$ such that $g \circ h_2 = f_1 \circ f$. Define $h : M \rightarrow A$ by $h = h_2 \circ h_1$, $g \circ h = g \circ h_2 \circ h_1 = f_1 \circ f \circ h_1 = f_1 \circ I = f_1$.Therefore M is projective module.

A module M is called S.F, if zero is the only small submodule in M [2].

Example

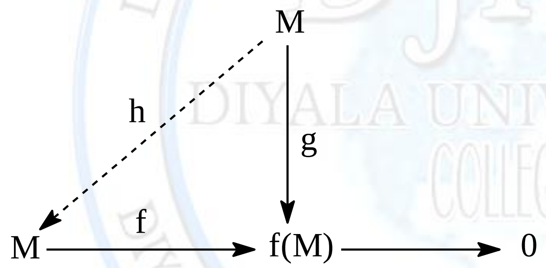
1. Every simple module is S.F.
2. Z as Z-module is S.F.

Proposition (2.3) Let R be a ring. If all R-modules over R are S.F then they are small pointwise projective module.

The next result gives a property for small pointwise projective modules.

Proposition (2.4) Let M be a small pointwise projective module. If $\ker f \ll M$ where $f \in End(M)$ then any epimorphism $g : M \rightarrow f(M)$ can be extended to an epimorphism in End(M).

Proof : Consider the following diagram



Since M is small pointwise projective module,for each $m \in M$,there exist a homomorphism $h : M \rightarrow M$ such that $f \circ h(m) = g(m)$. we claim that h is an epimorphism.Let $m_1 \in M$, then $f(m_1) = g(y)$ for some $y \in M$, $f(m_1) = f \circ h(y)$ and this implies that $m_1 - h(y) \in \ker f$ hence $M = \ker f + h(M)$, but $\ker f \ll M$ then $M = h(M)$.

A submodule N of M is called M-cyclic submodule if it is the image of an element of End(M) [11].

A module N is called M-principally injective if for any endomorphism ψ of M, and every homomorphism from $\psi(M)$ into N, can be extended to a homomorphism from M to N [11].

Before, we give the next proposition, we will introduce the following definition:

An R-module M is called small factor of a module N if there exists a small epimorphism from N to M.

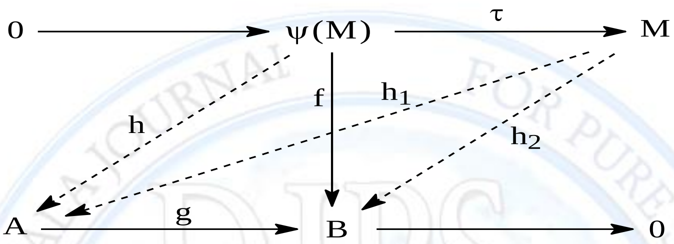
Proposition (2.5) Let M be a small projective module. The following are equivalent

1) Every M-cyclic submodule of M is small pointwise projective module.

2) Every small projective module of an M-principally injective module is M-principally injective module.

3) Every small factor module of an injective module is M-principally injective module.

Proof: 1→2) Let g : A → B be a small epimorphism where A is M-principally injective module. Consider the following diagram



Where f : ψ(M) → B is any homomorphism, ψ ∈ End(M) and τ : ψ(M) → M is the inclusion homomorphism. By 1) ψ(M) is small pointwise projective module,for every m ∈ M, there exists a homomorphism h : ψ(M) → A such that g ∘ h(m) = f(m). Since A is M-principally injective module there exists a homomorphism h_1 : M → A such that $h_1 \circ k = h$. Define h_2 : M → B by $h_2 = g \circ h_1$, $h_2 \circ k = g \circ h_1 \circ k = g \circ h = f$.

2→3) Clear

3 →1) By proposition 1.3.

Proposition (2.6) Let M be a module and A ≤ M , then for every direct summand B of M such that A∩B ≪ A and A + B is small pointwise projective module, we have A∩B = {0}.

Proof: Consider the following natural epimorphism $\pi_1 : A \to \frac{A}{A\cap B}$, $\pi_2 : A + B \to \frac{A+B}{A}$. By second isomorphism theorem $\frac{A}{A\cap B} \cong \frac{A+B}{B}$.Since B is a direct summand of M, so M ≅ B⊕ k_1, where $k_1 \leq$ M,by modular law M ∩ (A + B) = (B ⊕ k_1) ∩ (A + B), so A + B = B⊕(k_1 ∩(A+B)), so B is a direct summand of A + B, by proposition (2.1) k_1 ∩ (A + B) is small pointwise projective module and hence $\frac{A+B}{B}$ is small pointwise projective module and so $\frac{A}{A\cap B}$. we get A ∩ B = {0}.

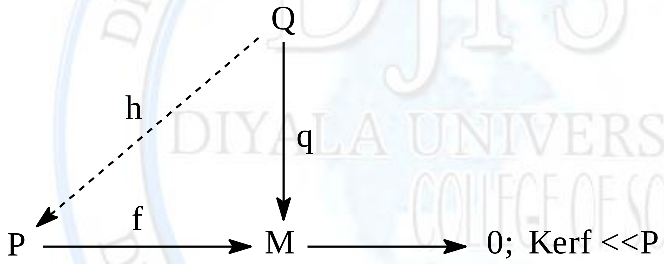
Proposition (2.7) If p_1 is a pointwise projective module and p_2 is a small pointwise projective module, then $p_1 \otimes p_2$ is a small pointwise projective module.

<u>Proof:</u> Let $f : N \to K$ be an epimorphism,By proposition (1.1) $Hom(I , f) : Hom(p_2 , N) \to Hom(p_2 , K)$ is an epimorphism. Since p_1 is a pointwise projective module, by [10 , proposition 17(1 , 2)]**.** we have $Hom(I , Hom(I , f)) : Hom(p_1 , Hom(p_2 , N)) \to Hom(p_1 , Hom(p_2 , K))$ is anepimorphism, using[10, proposition(20.6)].we get $Hom(p_1 \otimes p_2 , N) \to Hom(p_1 \otimes p_2 , K)$ is an epimorphism, By proposition (1.1), we get $p_1 \otimes p_2$ is a small pointwise projective module**.**

Proposition (2.8) Let M be an R-module has projective cover (p , f). If Q is a small pointwise projective module and $f_1 : Q \to M$ is an epimorphism, then there exists a decomposition $Q \cong p_1 \oplus p_2$ such that 1) $p_1 \cong p$

2) $p_2 \leq \ker f_1$, $f_1|_{p_1} : p_1 \to M$ is projective cover for M.

<u>Proof:</u> Consider the following diagram



Since Q is a small pointwise projective module for every $m \in M$ there exists a homomorphism $h : Q \to P$ such that $f \circ h(m) = f_1(m)$.we claim that h is an epimorphism. Let $x \in P$, $f(x) = f_1(y)$ for some $y \in Q$, so $f(x) = f(h(y))$ which implies $x - h(y) \in \ker f$, hence $p = \ker f + h(M)$. But $\ker f \ll p$. Thus $p = h(M)$. Now, $h : Q \longrightarrow p$ splits by [10, proposition 17(3)], therefore there exists a homomorphism $g : p \longrightarrow Q$ such that $h \circ g = I_p$ .Hence $Q = Ker\ h + Im\ g$. Also $Ker\ h \cap Im\ g = \{0\}$.Therefore $Q = \ker h \oplus Im\ g$. let $p_1 = Im\ g$ and $p_2 = \ker h$, since g is monomorphism, thus $p_1 \cong p$. Now, let $x \in p_2 = \ker h$, so $h(x) = 0$, $f(h(x)) = q(x) = 0$ and hence $x \in \ker f_1$, consequentially, $p_2 \leq \ker f_1$. Now , $f_1(p_1) = f \circ h(p_1) = f \circ h \circ g(p) = f(p) = M$, thus $f_1\ {}_{p_1} : p_1 \to M$ is onto. But $p_1 \cong p$ implies that p_1 is projective module. Let $g^* = f_1\ {}_{p_1}$.It is easy to show that $f = f_1 \circ g$. we claim that $\ker g^* \leq g(\ker f)$. let $w \in \ker g^*$, so $g^*(w) = 0$, $f_1(w) = 0$ and $w = g(y)$ for some $y \in P$, hence $f_1 \circ g(y) = 0$, so $f(y) =$

0 implies that y ∈ ker f, so w ∈ g(ker f).Since Ker f << P by [10, proposition (5.18)] , we get g(Ker f) << P_1 and hence Ker g* << P_1, Therefore, (p_1, g^*) is projective cover for M.

§3 The Endomorphism Ring of a Small Pointwise Projective Hollow Module

In this section we discuss the endomorphism ring of a small pointwise projective hollow modules. A non zero module M is hollow if every proper submodule is small in M [12].

A ring R is called local if for every r ∈ R, either r or 1- r is invertible [1].

Remark Every small epimorphism N $\to$ M $\to$ 0, where M is small pointwise projective, splits and consequently an isomorphism.

Proposition (3.1) If S is the endomorphism ring of a small pointwise projective hollow module, then S is a local ring.

Proof: Let f ∈ S = End (M), we have two cases:

Case 1: f is onto, since M is hollow, Ker f << M and by remark the sequence $M \xrightarrow{f} M \longrightarrow 0$ splits. This implies that Ker f = 0, i.e., f is invertible.

Case2: f is not onto, since M is hollow, f(M) << M. But f(M) + (I - f)(M) = M, therefore (I - f)(M) = M, hence (I - f) is onto and by a similar way as in case 1, we get (I - f) is invertible.

Theorem (3.2) Let M be small pointwise projective hollow module and S be the endomorphism ring of M, then:

(1) J(S) = {α ∈ S | Im α << M};

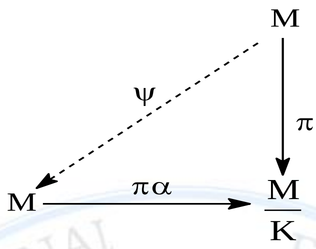
(2) $\frac{S}{J(S)}$ is Von-Neumann regular ring;

(3) Rad M << M if and only if Hom(M , Rad M) = J(S), where J(S) is the Jacobson radical of the ring S.

Proof: (1) Let Λ = {α ∈ S | Im α << M}. Then for every α ∈ Λ, we have (I - ψα)(M) = M for each ψ ∈ S, since ψα(M) << M, by [10,proposition (5.18)], Ker (I - ψα) is a proper submodule of M and by remark Ker(I - ψα) = 0, i.e., (I - ψ_α) is an isomorphism, which implies that α ∈ J(S) by [10, 15.3].Now, let α ∈ J(S) and suppose that Im α + K = M, where K is a proper submodule of M. If π : M → $\frac{M}{k}$ is the natural epimorphism, then we claim that πα : M → $\frac{M}{k}$ is also an epimorphism. To see this , let x

$+ K \in \frac{M}{k}$, then $x = \alpha(y) + w$ where $w \in K$ and $y \in M$. This implies that $\pi(x) = \pi\alpha(y)$. But Ker($\pi\alpha$) is a proper submodule of M and M is hollow, therefore Ker($\pi\alpha$) << M.

Consider the following diagram:



Since M is small pointwise projective, for every $m \in M$ there exists a homomorphism ψ : $M \to M$ such that $\pi\circ\psi(m) = \pi(m)$.Hence $(I - \alpha\psi)(M) \subseteq K$.But $\alpha \in J(S)$ which implies that $(I - \alpha\psi)$ is an isomorphism, i.e., $M \subseteq K$. This is contradicts our assumption that K is a proper submodule of M. Hence Im α << M and consequently $\Lambda = J(S)$.

(2) Let $\alpha \in S$ and $\alpha \notin J(S)$, then $\alpha(M) = M$ and hence Ker(α) << M. Now, $\alpha : M \to M$ is a small epimorphism and since M is a small pointwise projective module, therefore α is an isomorphism, for every $m \in M$ there exists a homomorphism $\beta : M \to M$ such that $\alpha\circ \beta(m) = I(m)$. Thus $\alpha\circ\beta\circ\alpha(m) = I\circ\alpha(m) = \alpha(m)$ for every $m \in M$ and hence $\alpha + J(S) = (\alpha + J(S))(\beta + J(S))(\alpha + J(S))$, which means that $\frac{S}{J(S)}$ is a Von-Neumann regular ring.

(3) Suppose that Hom(M , Rad M) = J(S) and Rad M $\not\ll$ M. Then Rad(M) = M. This contradicts proposition (3.1).Hence Rad(M) << M. Now, suppose that Rad(M) << M and let $\alpha \in$ Hom(M, Rad(M)) so, $\alpha(M) \leq$ Rad(M) << M. By [10, proposition (5.18)] $\alpha(M)$ << M.Consequently $\alpha \in J(S)$.If $\alpha \in J(S)$,then by (1) $\alpha(M)$ << M and so $\alpha \in$ Hom(M,Rad(M)).

§4 Amply Supplemented Small Pointwise Projective Module

In this section, we prove some results on amply supplemented small pointwise projective module.

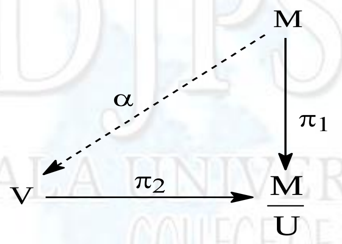
Let A and B be a submodule of M. B is called a supplemented of A, if it is minimal with property M = A + B. A submodule B is called a supplemented in M, if B is a supplemented of some submodule of M [1].

Let A and B be submodules of a module M, then A and B are called mutual supplemented in M, if they are supplemented of each other in M [1].

A module M is called amply supplemented, if for every two submodules U , V of M, such that M = U + V, U contains a supplemented of V in M [4].

Proposition (4.1) Let M = U + V be a small pointwise projective module where U and V are mutual supplemented, then M = U ⊕ V.

Proof: Let $\pi_1 : M \to \frac{M}{U}$ be the natural epimorphism. Define $\pi_2 : V \to \frac{M}{U}$ by $\pi_2(x) = x + U$, for all x ∈ V. Clearly π_2 is an epimorphism with Ker π_2 = U ∩ V which is a small submodule of V. Consider the following diagram:



Since M is a small pointwise projective module For every m ∈ M there exists a homomorphism α : M → V such that $\pi_2 \circ \alpha(m) = \pi_1(m)$, for $m_1 \in M$ we have $\pi_2 \circ \alpha(m_1) = \pi_1(m_1)$. So, $m_1 - \alpha(m_1) \in U$ which implies that M = α(M) + U = α(U) + α(V) + U. It is easy to show that α(U) ≤ U, thus M = α(V) + U. Since α(V) ≤ V and V is a supplemented of U, therefore α(V) = V. Now, let $m_1 \in M$, so $m_1 = u_1 + v_1$ for some $u_1 \in U$ and $v_1 \in V$. Suppose that $\alpha(u_1) = \alpha(v_2)$ for some $v_2 \in V$ so $m_1 = u_1 - v_2 + v_1 + v_2$.Cleary $u_1 - v_2 \in$ Ker α, thus M = Ker α + V. But Ker α ≤ U and U as a supplemented of V, thus Ker α = U. Now, let x ∈ U ∩ V. Since α(V) = V, there exists y ∈ V such that α(y) = x, so $\pi_2 \circ \alpha(y) = \pi_1(y)$ and hence α(y) – y ∈ U. This implies that y ∈ U and hence α(y) = 0, i.e.,x = 0. Consequently, M = U ⊕ V.

Corollary (4.2) for amply supplemented small pointwise projective module M, each supplemented in M is a direct summand of M and consequently small pointwise projective.

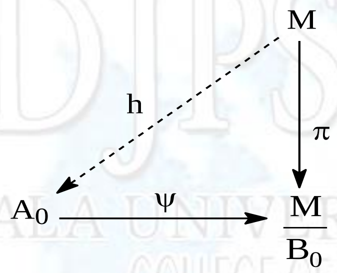
Proof: Let M be an amply supplemented small pointwise projective module. Let N be a supplemented in M.Then there exists a submodule K such that M = N + K with K ∩ N << N.

Also since M is amply supplemented, there exists a submodule K_1 of K such that $M = N + K_1$ and $N \cap K_1 << K_1$.Now, $N \cap K_1 \leq N \cap K << N$, this implies that $N \cap K_1 << N$ and hence N and K_1 are mutual supplemented and hence by proposition (4.1) $M = N \oplus K_1$.

A module M is called π-projective if whenever $M = A + B$, where A and B are submodules of M, there exists $f \in End(M)$ such that $f(M) \leq A$ and $(I - f)(M) \leq B$[1].

Theorem (4.3) Every amply supplemented small pointwise projective module is π-projective.

<u>Proof:</u> Let $M = A + B$ be amply supplemented small pointwise projective module, where A and B are submodules of M. There exist mutual supplemented A_0 , B_0 such that $M = A_0 \oplus B_0$, $A_0 \leq A$, $B_0 \leq B$. Consider the diagram:

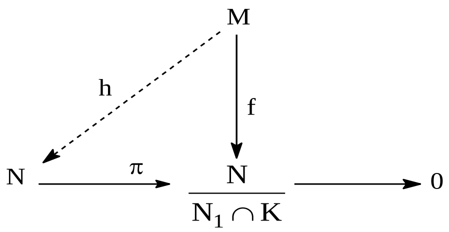


where $\psi : A_0 \rightarrow \frac{M}{B0}$ is the isomorphism defined by $\psi(x) = x + B_0$, for all $x \in A_0$, and $\pi : M \rightarrow \frac{M}{B0}$ is the natural epimorphism. Since M is small pointwise projective for every $m \in M$ there exists a homomorphism $h : M \rightarrow A_0$ such that $\psi \circ h(m) = \pi(m)$.Let $i : A_0 \rightarrow M$ be the inclusion homomorphism, then $i \circ h \in End(M)$ and $i \circ h(M) \leq A_0 \leq A$.Now, let $w \in (I - i \circ h)(M)$, $w = m_1 - i \circ h(m_1)$,for some $m_1 \in M$. $\psi \circ i \circ h(m_1) = \psi \circ h(m_1) = h(m_1) + B_0 = m_1 + B_0$.This implies that $m_1 - h(m_1) \in B_0$ and hence $w \in B_0$.Therefore,M is π-projective module.

Proposition (4.4) Let M be amply supplemented small pointwise projective module. Then for any non-small submodule N of M, $Hom(M, N) \neq 0$.

<u>Proof:</u> Since N is non-small submodule of M, there exists a proper submodule K of M, such that $M = N + K$. But M is amply supplemented module, thus there exists a submodule N_1 of N, such that $M = N_1+K$, with $N_1 \cap K << N_1$. Consequently $N_1 \cap K << N$. Define f

$: M \to \frac{N}{N_1 \cap K}$ by $f(m) = x + N_1 \cap K$, when $m = x + y$, for some $x \in N_1$ and $y \in K$. Clearly f is well-defined and homomorphism. Consider the following diagram:



Where $\pi : N \to \frac{N}{N_1 \cap K}$ is the natural epimorphism. Since M is a small pointwise projective module for every $m \in M$ there exists a homomorphism $h : M \to N$ such that $\pi \circ h(m) = f(m)$.Suppose that Hom(M , N) = 0.So, h = 0 and hence f = 0. Let $w \in N_1$, then $w \in M$, which implies that $f(w) = N_1 \cap K$.Therefore $w \in N_1 \cap K \leq K$.Hence $N_1 \leq K$,thus M = K which is contradiction.

§5 Small Pointwise Projective Module with Finite Spanning Dimension

Proposition (5.1) [2]: If M has finite spanning dimension and N is a submodule of M which is not small in M, then $\frac{M}{N}$ is Artinian.

Lemma (5.2), [1]: Let $0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$ be a short exact sequence of modules. If N_1 and N_2 are Artinian, then so is N.

A module M is said to be with finite spanning dimension, if for every strictly decreasing sequence $M > M_0 > M_1 > \ldots$ of submodules of M, there exists i such that M_j is a small submodule in M, for every $j \geq i$ [13].

Proposition (5.3) [14]: Every module M with finite spanning dimension is amply supplemented.

Theorem (5.4) A small pointwise projective module M is with finite spanning dimension if and only if it is hollow or Artinian.

<u>Proof:</u>(⇒) Since M is with finite spanning dimension by proposition (5.3) then M is amply supplemented.

Suppose that M is not hollow, then there exists a proper submodules A and B of M, such that M = A + B. But M is amply supplemented module, hence $M = A_0 \oplus B_0$, with $A_0 \leq A$, $B_0 \leq B$.

Clearly $\frac{M}{A0} \cong B_0$. By proposition (5.1), $\frac{M}{A0}$ is Artinian, therefore B_0 is Artinian. Similarly, A_0 is Artinian.

Now, consider the following short exact sequence:

$$0 \longrightarrow A_0 \xrightarrow{J} M \xrightarrow{\rho} B_0 \longrightarrow 0$$

Where J and ρ are the injection and the projection homomorphisms respectively. By lemma (5.2) M is Artinian.

(⇐) Clear.

§6 Small Pointwise Hereditary Modules

In this section, we introduce the concept of small pointwise hereditary module and we discuss some properties of this concept. Recall that a module M is hereditary, if every submodule of M is projective [1] A module M is called small pointwise hereditary if every submodule of M is small point wise projective.

Remark Every module over cosemisimple ring, is small pointwise hereditary.

Let M and N be R-module,N is called M-injective,if for each monomorphism $f : A \to M$ where A is any modules and for each homomorphism $g : A \to N$ there exists a homomorphism $h : M \to N$ such that $h \circ f = g$ [10 , p.184].

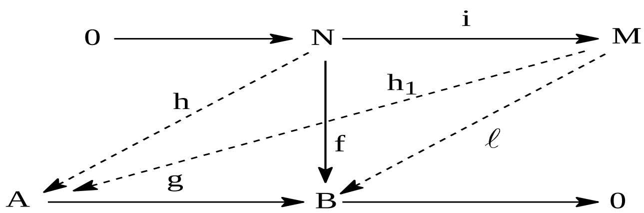
Proposition (6.1) Let M be small pointwise projective, the following statements are equivalent

(1) M is a small pointwise hereditary;

(2) Every small factor of an M-injective module is M-injective;

(3) Every small factor of an injective module is M-injective.

<u>Proof :</u> (1) ⇒ (2) Let B be a small factor for an M-injective module A. Consider the following diagram:



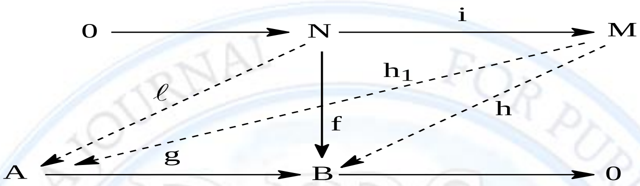
Where $f : N \to B$ is any homomorphisms and N is a submodule of M. Since N is small pointwise projective, for each $m \in M$, there exists a homomorphism $h : N \to A$ such that $g \circ h(m) = f(m)$.

But A is M-injective, thus there exists a homomorphism $h_1 : M \to A$ such that $h_1 \circ i = h$. Define $\ell : M \to B$ by $\ell = g \circ h_1$. Thus $\ell \circ i = g \circ h_1 \circ i = f$.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Let N be a submodule of a small pointwise projective module M. Let $g : A \to B$ be a small epimorphism, where A is an injective module and $f : N \to B$ be any homomorphism where $I : N \to M$ is the inclusion homomorphism.

Consider the following diagram:



By (3), B is M-injective module, then there exists a homomorphism $h : M \to B$ such that $h \circ i = f$. Since M is a small pointwise projective module, for each $m \in M$,there exists a homomorphism $h_1 : M \to A$, such that $g \circ h_1(m) = h(m)$. Define $\ell : N \to A$ by $\ell = h_1 \circ i$.Now, $g \circ \ell = g \circ h_1 \circ i = h \circ i = f$. Therefore by proposition (1.3) N is a small pointwise projective module. Hence M is a small pointwise hereditary.

Proposition (6.2) Let $\{M_\alpha\}_{\alpha \in \Lambda}$ be a family of modules, then $\oplus_{\alpha \in \Lambda} M_\alpha$ is small pointwise hereditary if and only if each M_α is small pointwise hereditary.

Proof :(⇒) Clear.

(⇐) Let M_α be a small pointwise hereditary module, for each $\alpha \in \Lambda$. To show that $\oplus_{\alpha \in \Lambda} M_\alpha$ is a small pointwise hereditary, let $f : Q \to K$ be a small epimorphism, with Q is injective module.By proposition (6.1) K is M_α-injective for each $\alpha \in \Lambda$ and then by [10 , proposition (16.13(1)], K is $\oplus_{\alpha \in \Lambda} M_\alpha$ injective module. Thus M_α is small pointwise hereditary module. Before,we give the last proposition in this section, we need the following two definitions: An R-module M is called cofaithful if there exists a positive integer n, and a monomorphism $\theta : R \to M^n = M \oplus \dots \oplus M$ (n copies) [15] A ring R is called small pointwise hereditary, if R is a small pointwise hereditary as R-module.

Proposition (6.3) Let R be any ring. Then the following are equivalent:

(1) R is a small pointwise hereditary ring;

(2) There exists a cofaithful small pointwise hereditary R-module.

Proof: (1) $\Rightarrow$ (2) Let M = R as R-module. Then M is cofaithful, small pointwise hereditary R-module.

(2) $\Rightarrow$ (1) Let M be a cofaithful, small pointwise hereditary R-module. Then, we obtain an embedding $\theta : R \to M^n$, for some positive integer n. Since M is a small pointwise hereditary, by proposition (6.2) $M^n = M \oplus \dots \oplus M$ (n copies) is a small pointwise hereditary; and hence R is a small pointwise hereditary.

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