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Space of Stone-Čech Compactification $\beta\mathbb{N}$

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Abstract

In this paper, the definition of Stone-Čech defined on the discrete topological space of the natural numbers is used, which is the set of all ultra-filters on $\mathbb{N}$ and this is the largest compact space created from space. The algebraic and topological properties of $\mathbb{N}$ will be used to understand and study some of the algebraic and topological properties of it. We have expanded the semi-group process $(\mathbb{N},+)$, naturally to the Stone-Čech compactification $(\beta\mathbb{N},+)$. We use this to illustrate the structure, emphasizing on the smallest ideal $\beta\mathbb{N}+\mathcal{P}$ where $\mathcal{P}\in\beta\mathbb{N}$. We show that $\beta\mathbb{N}+\mathcal{P}$ containing a minimal left ideal and has the smallest ideal $M(\beta\mathbb{N})$, which is also an ideal.

Keywords: Smallest ideal, Dense set, Compact space, Left ideal, Embedding.

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الفضاء ستون سيج التراص $\beta\mathbb{N}$

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الخلاصة

في هذا البحث تم استخدام مفهوم ستون سيج المعرف على مجموعة الاعداد الطبيعية معرف عليها فضاء التبولوجي المتقطع, وهي مجموعة الالترا فلتر وهذا أكبر فضاء مرصوص يتم الحصول عليه من الفضاء المعرف على $\mathbb{N}$. من خلال استخدام الخصائص الجبرية والتبولوجية لـ $\mathbb{N}$ لفهم ودراسة بعض الخصائص الجبرية والتبولوجية لفضاء المرصوص ستون سيج $\beta\mathbb{N}$. ومن خلال توسيع عماية الثنائية + على الشبه الزمرة($\mathbb{N}, +$) لتكون معرفه على الفضاء المرصوص ستون سيج $\beta\mathbb{N}$ ومن خلال هذا الفضاء الموسع ($\beta\mathbb{N}, +$) يمكن الحصول على اصغر مثالي وهو $P + \beta\mathbb{N}$. ثم برهنة على $P + \beta\mathbb{N}$ يحتوي على اصغر مثالي يساري. وهذا المثالي أصغري ($M(\beta\mathbb{N}$ يكون مثالي أيضا.

الكلمات المفتاحية: مثالي ألاصغري، مجموعة الكثيفة، فضاء المرصوص، مثالي اليساري، التضمين.

Introduction

That $\beta\mathbb{N}$ has interesting properties in its application to topological dynamics topology and combinatorial number theory. Ramsey's theory is one example of the applications of $\beta\mathbb{N}$, where Ramsey theory has played an important role in mathematics over the past century and it has been an open problem for several decades. Many mathematicians, including (Hilbert) in [2]. In this paper, we will expand the space $\mathbb{N}$ into larger space called the stone-Čech compactification denoted by $\beta\mathbb{N}$ which is one of the basic and functional concepts in algebra and topology. In general, it is a function of technique from one of the spaces and results in the largest compact space generated from the space in which it is located, and thus we obtain the largest submerged mathematical space by expanding the original space as given [1],[6]. Therefore, we applied its conditions to space $\beta\mathbb{N}$, and this was demonstrated by some theories that we have proven. We also got some new results regarding the space $\beta\mathbb{N}$. The definition of dense which is a subset A

of topological space X such that every point $x \in X$ is either belong to A or is a limit point of A as give in [7]. Once we've been proven the set $\mathcal{B}_{q} = \{\overline{f[B]}: B \in q\}$ has the finite intersection property Such that $f: \mathbb{N} \to W$ is continuous, W is a compact space. Finally, we recalled two fundamental notions related to our work, the pair $(e, \beta N)$ which represent the stone-Čech compactification of $\mathbb{N}$ for $e: \mathbb{N} \to \beta\, \mathbb{N}$, *and* left ideal of βN in [3], [5].

1. Preliminaries

Definition 1.1 [2]: Let K be a non-empty set, a filter on a set K is a non-empty set μ with the following properties:

1. $\emptyset \notin \mu$.
2. $\mathcal{P}, q \in \mu$ and $\mathcal{P} \cap q \in \mu$.
3. If $\mathcal{P} \in \mu$ and $\mathcal{P} \subseteq q \subseteq K$ then $q \in \mu$.

Definition 1.2 [7]: A filter μ on a set K is called an ultra-filter if it is not properly contained in any other filter on K.

Definition 1.3 [8]: The stone-Čech Compactification of Discrete topological space $\mathbb{D}$ is a pair $(\varphi, \mathbb{Y})$ such that:

1) $\mathbb{Y}$ is a compact space.

2) $\mathbb{D}$ Embedding into $\mathbb{Y}$ by φ.

3) $\varphi[\mathbb{D}]$ is Dense in $\mathbb{Y}$, and

4) Given any compact space W and any continuous function $f: \mathbb{D} \longrightarrow W$ there is a continuous function $g: \mathbb{Y} \longrightarrow W$ such that $g \ o\ \varphi = f$.

$$\begin{tikzcd} & \mathbb{Y} \arrow[dr, "g"] & \\ \mathbb{D} \arrow[ur, "\varphi"] \arrow[rr, "f"'] & & W \end{tikzcd}$$

Remark 1.4 [2]:

1) Any two Stone-Čech Compactification of the same topological space $\mathbb{D}$ are homeomorphism.

2) The topology induced on $\mathbb{D}$ as a subset of $\mathbb{Y}$ is the original topology of $\mathbb{D}$.

Note 1.5: From now we will concentrate our work on a special case when $\mathbb{D} = \mathbb{N}$, and next theorem show that $\beta\mathbb{N}$ is the stone-Čech compactification corresponding to $\mathbb{N}$.

2. Stone-Čech compactification $\boldsymbol{\beta}\mathbb{N}$

The next proposition can be found in [2] as a problem and will be given a proof for it. The importance of this proposition is one of the properties for $\beta\mathbb{N}$ related to the finite intersection property.

Proposition 2.1: Let $f: \mathbb{N} \to W$ be continuous function where W be a compact space. then $\mathcal{B}_{q}$ = { $\overline{f[B]}: B \ \in \ q$} has finite intersection property for any $q \in \beta\ \mathbb{N}$.

Proof:

Pick $B_1, B_2, B_3, \ldots, B_n \in q$. If we can show that $\cap_{j=1}^{n} \overline{f(B_j)} \neq \emptyset$ then we are finished. Because $B_1, B_2, B_3, \ldots, B_n \in q$ hence $\emptyset \neq f(\cap_{j=1}^{n} B_j)$

$\subseteq \cap_{j=1}^{n} f(B_j)$

Diyala Journal for Pure Science

$\subseteq \overline{\cap_{j=1}^{n} f(B_j)}$ since $f(B_j) \subseteq \overline{f(B_j)}$.

Theorem 2.2: For the Discreet space $\mathbb{N}$. Then the pair $(e, \beta N)$ is the stone-Čech compactification of $\mathbb{N}$.

Proof:

We need to achieve the stone-Čech compactification conditions which are:

1) $\beta\mathbb{N}$ is compact

Proof: Now for compactness of $\beta\mathbb{N}$ we need to show every collection of closed sets of $\mathbb{N}$ satisfies the finite intersection property has non- empty intersection. Note that the set of the form $\widehat{M}$ is the stone set M which acts as both an open and closed sets bases because $(\widehat{\mathbb{N} / M}) = \beta\mathbb{N} / \widehat{M}$. To prove $\beta\mathbb{N}$ is compact we will show that the family $\mathcal{H} = \{$the stone set $\widehat{M}$ with finite intersection property $\}$has non-empty intersection. Let $\mathcal{B} = \{M \subseteq \mathbb{N}: \widehat{M} \in \mathcal{H}\}$. If $F \in \rho_{f(\mathcal{B})} = \{F: \emptyset \neq F \subseteq \mathcal{B}$, and F is finite$\}$, this mean for each $M \in F, \widehat{M}_i \in \mathcal{H}$. From the definition of $\mathcal{H}$ there is some $\mathcal{P} \in \cap_{M \in F} \widehat{M}$, and by definition of $\widehat{M}$ we get $\cap F \in \mathcal{P}$. Thus $\cap F \neq \emptyset$ and hence $\mathcal{B}$ has the finite intersection property. State Theorem 3.8 [2] in section 1 there is an ultra-filter $q \in \beta\mathbb{N}$ such that $\mathcal{B} \subseteq q$, and so $q \in \cap \mathcal{H}$. Therefore $\beta\mathbb{N}$ is compact.

2) To show e is an embedding.

i) We claim that $e: \mathbb{N} \to \beta\mathbb{N}$ is injective. Let $t \neq d \in \mathbb{N}$. By definition $e(t)$ and $e(d)$ are two ultra-filters generated by t and d respectively. Then $\{t\}^c = \mathbb{N} \setminus \{t\} \in e(d) \setminus e(t)$. Hence $e(t) \neq e(d)$ i.e we have a one to one condition.

ii) Obviously, e is continuous because $\mathbb{N}$ is discreet space.

Diyala Journal for Pure Science

iii) If we can show that e is a closed map we are done. Suppose $B \subseteq \mathbb{N}$ be a closed sub set and since $\hat{B} \in e[B]$ then $e[B] \cap \hat{B} \subseteq e[B]$. Also $e[B] \subseteq e[B] \cap \hat{B}$, since if $t \in e[B] \Rightarrow t = (b') = e(b')$ where $b' \in B \Rightarrow B \in t \Rightarrow t \in \hat{B}$. Implies $e[B] = e[B] \cap \hat{B}$.

3) To show $e[\mathbb{N}]$ is a dense. We need to show $e[\mathbb{N}]$ has its point and a limit point of $e[\mathbb{N}]$. So we will try to show if p a point in $\beta\mathbb{N}$ is a limit point of $e[\mathbb{N}]$ if every neighborhood of p contains at last one point of $e[\mathbb{N}]$ deferint from p itself. Let $\hat{A}$ be a non-basic open subset of $\beta\mathbb{N}$, then $A \neq \emptyset$, any $a \in A$ satisfy $e(a) \in e[\mathbb{N}] \cap \hat{A}$ and so $e[\mathbb{N}] \cap \hat{A} \neq \emptyset$. i.e $e[\mathbb{N}]$ has its limit point. Hence $e[\mathbb{N}]$ is a dense in $\beta\mathbb{N}$.

4) Given a compact space W and let $f: \mathbb{N} \longrightarrow W$ be continuous, to show there

is a continuous function $g: \beta\mathbb{N} \longrightarrow W$ such that its commutative diagram.

$$\begin{tikzcd} & \beta\mathbb{N} \arrow[dr, "g"] & \\ \mathbb{N} \arrow[ur, "e"] \arrow[rr, "f"'] & & W \end{tikzcd}$$

First, we need to define the function g. For each $q \in \beta\mathbb{N}$, let $\mathcal{A}_q = \{c\ell_W\, f[B]: B \in q\}$. Then for each $q \in \beta\mathbb{N}$, by Proposition 2.1 $\mathcal{A}_q$ has finite intersection property, and since W is compact, so $\mathcal{A}_q$ has non-empty intersection. Choose $g(q) \in \cap\, \mathcal{A}_q$.

Secondly, to show the diagram is commutative. Let $n \in \mathbb{N}$ then $\{n\} \in e(n) = \{B \subseteq \mathbb{N}: n \in B\}$.

So $g\big(e(n)\big) \in c\ell_W\, f[\{n\}] = c\ell_W\, [\{f(n)\}]$

$= \{f(n)\}$ since singleton is closed.

Diyala Journal for Pure Science

Immediately by definition $g\ o\ e = f$. Finally, to show g is continuous. Let $q \in \beta\mathbb{N}$ and let v be a neighborhood of $g(q)$ in W, and since W is compact Hausdorff space then W is normal space this leads to W is regular. So pick a neighborhood u of $g(q)$ with $c\ell_W\, u \subseteq v$ {by definition of regular we get a closed set}. Let $B = f^{-1}[u] \in \mathbb{N}$, we claim $B \in q$, suppose $\mathbb{N}\backslash B \in q$ then $g(q) \in c\ell_W f[\mathbb{N}\backslash B]$, and since u is a neighborhood of $g(q)$. So $u \cap f[\mathbb{N}\backslash B] \neq \emptyset$ that is a contradiction since $B = f^{-1}[u]$. Hence $B \in q$ then $q \in \hat{B}$ is a neighborhood of q. Claim $g[\hat{B}] \subseteq v$. Let $\mathcal{P} \in \hat{B} = c\ell(B)$ and suppose $g(\mathcal{P}) \notin v$, then $W\backslash c\ell_W\, u$ is a neighborhood of $g(q)$ and $g(q) \in c\ell_W f[B]$, since $\mathcal{P} \in \hat{B} = c\ell(B)$ then $f(\mathcal{P}) \in f(\bar{B}) \subseteq \overline{f(B)}$, so $(W\backslash c\ell_W\, u) \cap f[B] \neq \emptyset$ that is a contradiction since $B = f^{-1}[u]$.

Next proposition wich is founded, as an open problem we found in [2]. The prove we consider is if the two continues maps identify on $e[\mathbb{N}]$ then they will be equal.

Proposition 2.3: Let $\mathcal{B}_q = \{cl_W f[A]: A \in q\}$ be a set belong to W then for each $q \in \beta\mathbb{N}$, then $\bigcap \mathcal{B}_q$ is a singleton.

Proof:

By proposition 2.1 we show that $\mathcal{B}_q$ has a finite intersection property and since W is compact, then every family of closed subsets having the finite intersection property has non-empty intersection. So $\bigcap\mathcal{B}_q \neq \emptyset$, hence there exists $w \in \bigcap\mathcal{B}_q$ such that w is element of all $\overline{f(A)}$ for all $A \in q$. Now to show $\bigcap\mathcal{B}_q$ is singleton.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[d] & W \\ \beta\mathbb{N} \arrow[ur, "g"'] & \end{tikzcd}$$

Choose $w = g(q) \in \bigcap\mathcal{B}_q = \bigcap\{\overline{f(A)}: A \in q\}$. Assume there is another element $m \in \bigcap\mathcal{B}_q$. Define $h: \beta\mathbb{N} \to W$ such that $h(q) = m$ which is the same way how we construct function g.

Note that g and h have same behavior from $\beta\mathbb{N} \to W$ and they will be equal if start from $\mathbb{N}$ because the continues function on dense set they are equal.

Proposition 2.4: Every left ideal in $\beta\mathbb{N}$ the Stone-Čech compactification for the discrete set of natural number $\mathbb{N}$ contained a minimal left ideal.

Proof:

By definition of a left ideal then immediately $\beta\mathbb{N} + \mathcal{P}$ is a left ideal $\forall\, \mathcal{P} \in \beta\mathbb{N}$. Let $q \in \beta\mathbb{N} + \mathcal{P}$ implies $\beta\mathbb{N} + q \subseteq \beta\mathbb{N} + \mathcal{P}$. Note that $\beta\mathbb{N} + q$ *and* $\beta\mathbb{N} + \mathcal{P}$ are both compact since they are image of left translation $\rho_{\mathcal{P}}(\beta\mathbb{N})$ and $\rho_{q}(\beta\mathbb{N})$.

Moreover, both are closed since $\beta\mathbb{N}$ is T_2- space and every compact subset of T_2-space is closed. We will try to show $\beta\mathbb{N} + q$ is a minimal left ideal. Consider $\mathcal{R} = \{\, \beta\mathbb{N} + q_i$ a left closed ideal on $\beta\mathbb{N}$ and $\beta\mathbb{N} + q^{`} \subseteq \beta\mathbb{N} + \mathcal{P}\}$. We have $\beta\mathbb{N} + q \in \beta\mathbb{N}$, so $\mathcal{R} \neq \emptyset$ and is partially ordered by inclusion such that $\{K_1 \subseteq K_2$ then $K_1 \leq K_2\}$. Define $\mathcal{C} = \{\beta\mathbb{N} + q_1 \supseteq \beta\mathbb{N} + q_2 \supseteq \cdots\cdots\cdots\}$ be a chain. By finitely intersection property $\cap\, \beta\mathbb{N} + q_i \neq \emptyset$ which is a left closed ideal. Denote $S = \cap\, \beta\mathbb{N} + q_i$ which is a lower bound of $\mathcal{C}$. By Zorn's lemma $\mathcal{R}$ has a minimal left ideal $\beta\mathbb{N} + S$ amony left closed ideals in $\beta\mathbb{N} + \mathcal{P}$, i.e if $\mathcal{F} \subseteq \beta\mathbb{N} + S$ and $\mathcal{F}$ is left closed ideal then $\mathcal{F} = S$. To complete the proof, we need to show this S is a minimal corresponding to all space. Claim every left ideal contains a left closed ideal. Let L be a left ideal and $d \in L$. There for $\rho_d(\beta\mathbb{N}) = \beta\mathbb{N} + d \subseteq L$ which is an image of compact space. Furthermore, is a closed since it's a compact subset of Hausdorff space. So there exist $\mathcal{F}^{`}$ a left closed ideal subset of $L$. Now $\mathcal{F}^{`} \subseteq L \subseteq S$, i.e $\mathcal{F}^{`} \subseteq S$ implies $\mathcal{F}^{`} = S$.

Lemma 2.5: The set of $M(\beta\mathbb{N})$ is an ideal in $\beta\mathbb{N}$ in fact which is a smallest ideal corresponding to $\beta\mathbb{N}$.

Proof:

$M(\beta\mathbb{N}) \neq \emptyset$ since from proposition 3.4 $\beta\mathbb{N}$ has a minimal left ideal. Let $\mathcal{P} \in M(\beta\mathbb{N})$, $\mathcal{P} \in L$ which is one a minimal left ideal in $M(\beta\mathbb{N})$, then $\beta\mathbb{N} + \mathcal{P} \subseteq L \subseteq M(\beta\mathbb{N})$, and so $M(\beta\mathbb{N})$ is a left ideal. Similarly, for the right ideal. Hence $M(\beta\mathbb{N})$ is an ideal. Finally, to show $M(\beta\mathbb{N})$ is smallest. First we need to show $M(\beta\mathbb{N})$ is a minimal ideal. Let an ideal $\mathcal{F} \subseteq M(\beta\mathbb{N})$ to show $\mathcal{F} = M(\beta\mathbb{N})$ suppose L be any minimal left ideal subset of $M(\beta\mathbb{N})$. Then $\mathcal{F} \cap L \neq \emptyset$. To show $\mathcal{F} \cap L$ is a left ideal in $\beta\mathbb{N}$. Let $x \in \mathcal{F} \cap L$. Since $\beta\mathbb{N} + x \subseteq \mathcal{F}$ *and* $\beta\mathbb{N} + x \subseteq L$ then $\beta\mathbb{N} + x \subseteq \mathcal{F} \cap L$. Since $\mathcal{F} \cap L$ is a left ideal. By minimality of L and since $\mathcal{F} \cap L \subseteq L$ then $\mathcal{F} \cap L = L$

$$\Rightarrow L \subseteq \mathcal{F}$$

$$\Rightarrow \mathcal{F} = M(\beta\mathbb{N})$$

So $M(\beta\mathbb{N})$ is a minimal ideal.

Conclusion: the Stone-Čech compactification $\beta\mathbb{N}$ is an important to will for stadey same algebraic and topological properties. This give an important applications in number theory and set theory. For like the finite same theory or the finite same concepts and Ramssy theory in number theory.

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