Traveling Wave Solutions by using Extended Tanh Function Method with special values

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Abstract

The Extended hyperbolic tangent (tanh) method is proposed for building more general exact solutions of the nonlinear evolution equations, Burgers equation and KdV equation. As a result of this, when the equation parameters are taken as special values, we find more general exact solution for traveling wave, which include solitary wave solutions. In this work that the extended tanh method give some new and more general results which are very simple, flexible and faster to compute by general mathematical software, like Maple and Mathematica. The results obtained were compared with tanh method.

Keywords: Extended Tanh method, Traveling wave solutions, Burgers equation, KdV equation.

حلول الموجات المتنقلة باستخدام طريقة الظل الزائدي الموسعة مع القيم الخاصة

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الخلاصة

استخدمت طريقة الظل الزائدي الموسعة للحصول على حلول دقيقة اكثر عمومية للمعادلات الغير خطية، معادلة Burgers و معادلة KdV . في النتيجة، عندما معلمات المعادلة تاخذ قيما خاصة نحصل على بعض الحلول الجديدة للموجات المتنقلة والتي تشمل على الموجة الانفرادية . لقد وجدنا في هذه الدراسة بان طريقة الظل الزائدي الموسعة تعطي بعض الحلول الجديدة والتي تكون بسيطة ومرنة و اسرع للحساب من قبل البرامج الرياضية. ولقد تم مقارنة النتائج مع طريقة الظل الزائدي.

الكلمات المفتاحية : طريقة الظل الزائدي الموسعة ، الموجات المتنقلة ، معادلة Burgers ، معادلة kdv

Introduction

In recent years, nonlinear evolution equations (NLEEs) have become a very active area for describing various branches of nonlinear sciences. Traveling wave solutions is a special class of analytical solutions for NLEEs, become a useful tool for describing most of the phenomena that arise in mathematical physics and engineering. Many powerful methods have been created and successfully developed to find for exact solution of (NLEEs), such as the tanh-coth method [1,2], sine-cosine method [3], homogeneous balance method [4,5], exp-function method [6], first-integral method [7], Jacobi elliptic function method [8], and (G'/G)-Expansion method [9,10,11]. The hyperbolic tangent (tanh) method presented by Malfliet [12] is a powerful technique to compute traveling waves solutions of nonlinear evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction-diffusion phenomena. In this work, we will employ the extended tanh method to find the traveling wave solutions of Burgers equation [13]. The Burgers equation is a nonlinear partial differential equation of second order of the form

$$u_t + uu_x = vu_{xx} \tag{1}$$

Where v is the viscosity coefficient. The Burgers equation appears in different areas of applied mathematics, such as modelling of fluid dynamics, turbulence, boundary layer behaviour, shock wave formation, and traffic flow. The KdV equation can be written as

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

Where x and t are the scaled space and time, respectively, and u describes the propagation of long one-dimensional, which arises in many physical problems such as surface water waves and ion-acoustic waves in plasma [14]. Solitary waves are localized travelling waves travelling with constant speeds and shape, asymptotically zero at large distances. Solitons are special kinds of solitary waves. A soliton has properties that set it apart from the general class of solitary waves.

Our paper is arranged as follows. In Section 2, we give the description of the summary of the Extended tanh method for finding travelling wave solutions. In Section 3, we clarify the application of the Extended tanh method for Burgers equation and Korteweg–de Vries equation (KdV) . In the last Section, conclusions are given.

The Extended Tanh Function Method

In this section, we describe the extended tanh method for finding traveling wave solutions of NLEEs. Suppose that the given nonlinear equation PDE $u(x,t)$, say in two independent variables, x and t, is defined by

$$P(u, u_x, u_t, u_{xt}, u_{xx}, u_{tt,} \dots) = 0 \tag{3}$$

Where p is a polynomial in $u = u(x,t)$, and $u = u(x,t)$ is an unknown function dependent to x, t variables and it's various partial derivatives. The main purpose of extended tanh method can be presented in the following seven steps.

Step1. Looking for traveling wave solutions of (3), we assume that the wave variable.

$$u(x,t) = u(\zeta), \qquad \zeta = (x - ct) \tag{4}$$

The constant c is termed the wave velocity. Substituting (4) into (3), we obtain the following ordinary differential equations (ODE) :

$$P(u, cu', cu'', c^2u'', u'', \dots) = 0 \tag{5}$$

Step2. In some times we integrate (5) and set the constants of integration to be zero for simplicity.

Step3. We suppose that Eq. (5) has the following formal solution:

$$u(\zeta) = S(Y) = a_0 + \sum_{i=1}^{m} a_i\,(d+y)^i + b_i(d+y)^{-i} \tag{6}$$

Where m is a positive integer, and a_0, a_i, b_i , c and d are constants, while Y is given by

$$Y = \tanh(\zeta) \tag{7}$$

The independent variable (7) leads to the following derivatives:

$$\frac{d}{d\zeta} = (1-y^2)\frac{d}{dy}\ ,$$

$$\frac{d^2}{d\zeta^2} = (1-y^2)\left\{(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy}\right\}, \tag{8}$$

$$\frac{d^3}{d\zeta^3} = 2(1-y^2)(3y^2-1)\frac{d}{dy} - 6y(1-y^2)^2\frac{d^2}{dy^2} + (1-y^2)^3\frac{d^3}{dy^3}\ ,$$

and so on.

Step4. Usually, can be determine the positive integer m by balance between the linear terms of highest order derivatives and non-linear terms appearing in (5). In other words, we define the degree of $u(\zeta)$ as $D[u(\zeta)] = m$, which gives the degree of other expressions as follows

$$D\left[\frac{d^q u}{d\zeta^q}\right] = m + q,$$

$$D\left[u^r\left(\frac{d^q u}{d\zeta^q}\right)^s\right] = mr + s(q+m) \tag{9}$$

So, we can find the value of m in (6)

Step5. Replace (6) and (8) into (5) produces an algebraic equation involving powers of $(d+y)$ then, equating the coefficients of each power to zero, yields a set of algebraic equations for a_0, a_i, b_i, c and d

Step6. Solving these algebraic equations by Maple or Mathematica, we get the values of a_0, a_i, b_i, c and d.

Step7. Substituting these values into (6) and (4), we can obtain the exact traveling wave solutions of (3).

Applications of The Extended Tanh Method

Burgers equation.

We can get the solitary wave solution of (1), by using the transformations

$$\zeta = x - ct \tag{10}$$

Eq. (1) becomes

$$-cu' + uu' - vu'' = 0 \tag{11}$$

Integrating (11) with respect to ζ and for simplicity, equating the integration constant equal to zero, we have

$$-cu + \frac{1}{2}u^2 - vu' = 0 \tag{12}$$

Now, using the balancing procedure between u^2 with u' in (12), we get $m = 1$. Therefore, we can write the solution of (12) in the form:

$$u = a_0 + a_1(d+y) + \frac{b_1}{(d+y)} \tag{13}$$

Where a_0, a_1, d and $b_1$are constants which are unknown. Substituting (13) into (12), with computerized symbolic computation, equating to zero the coefficients of all power y^i $(i = 0,1,2,3,4)$, we derive a set of algebraic equations for a_0, a_1, b_1, d, v and c

y^4: $a_1^2 + 2va_1 = 0,$

y^3: $2a_0a_1 - 2ca_1 + 4a_1^2d + 4vda_1 = 0,$

y^2: $6a_1^2d^2 - 2vb_1 + a_0^2 + 6a_0a_1d - 6cda_1 - 2ca_0 - 2va_1 + 2a_1b_1 + 2va_1d^2 = 0,$

$y^1: -6ca_1d^2 - 4cda_0 - 2cb_1 + 4da_1b_1 + 2da_0^2 + 6a_0a_1d^2 + 4a_1^2d^3 - 4vda_1 + 2a_0b_1 = 0,$

$y^0: b_1^2 + 2a_0a_1d^3 - 2ca_0d^2 - 2cd\,b_1 - 2va_1d^2 - 2ca_1d^3 + 2da_0b_1 + a_0^2d^2 + a_1^2\,d^4 + 2a_1d^2b_1 + 2vb_1 = 0.$

Solving this system by software Maple, we have the following six sets of solutions

Case1:

$$a_0 = -2v - 2vd, \quad a_1 = 0, \quad b_1 = 2v(1+d)(-1+d), \quad c = -2v, \quad d = d$$

Case2:

$$a_0 = 2v - 2vd, \quad a_1 = 0, \quad b_1 = 2v(1+d)(-1+d), \quad c = 2v, \quad d = d$$

Case3:

$$a_0 = 4v, \quad a_1 = -2v, \quad b_1 = -2v, \quad c = 4v, \quad d = 0$$

Case4:

$$a_0 = -4v, \quad a_1 = -2v, \quad b_1 = -2v, \quad c = -4v, \quad d = 0$$

Case5:

$$a_0 = -2v + 2vd, \quad a_1 = -2v, \quad b_1 = 0, \quad c = -2v, \quad d = d$$

Case6:

$$a_0 = 2v + 2vd, \quad a_1 = -2v, \quad b_1 = 0, \quad c = 2v, \quad d = d$$

In view of this, we obtain the following solitons and kink solutions:

$$u_1(x,t) = -2v - 2vd + \frac{2v(1+d)(-1+d)}{(d+\tanh(x+2vt))},$$

$$u_2(x,t) = 2v - 2vd + \frac{2v(1+d)(-1+d)}{(d+\tanh(x-2vt))},$$

$$u_3(x,t) = -2v + 2vd - 2v(d+\tanh(x+2vt)),$$

$$u_4(x,t) = 2v + 2vd - 2v(d+\tanh(x-2vt)),$$

$$u_5(x,t) = 4v - 2v(d+\tanh(x-4vt)) - \frac{2v}{(d+\tanh(x-4vt))},$$

$$u_6(x,t) = -4v - 2v(d + \tanh(x + 4vt)) - \frac{2v}{(d + \tanh(x + 4vt))}.$$

On the other hand, if d=0 and v=1, the traveling wave solution can be written as

$u_7(x,t) = -2[1 + \tanh(x + 2t)]$,

$u_8(x,t) = 2[1 - \tanh(x - 2t)]$,

$u_9(x,t) = -2[1 + \coth(x + 2t)]$,

$u_{10}(x,t) = 2[1 - \coth(x - 2t)]$,

$u_{11}(x,t) = 2[2 - \tanh(x - 4t) - \coth(x - 4t)]$,

$u_{12}(x,t) = -2[2 + \tanh(x + 4t) + \coth(x + 4t)]$.

For the comparison between our solution and that of Wazwaz as given in [12], first we assume c= 2 and $c = 4$, we get the same as that of Wazwaz.

The solitary wave and behaviour of the solutions u_8(x, t) and $u_9(x,t)$ are shown in Figures 1 and 2 respectively for some fixed values of the d=0 and v=1.

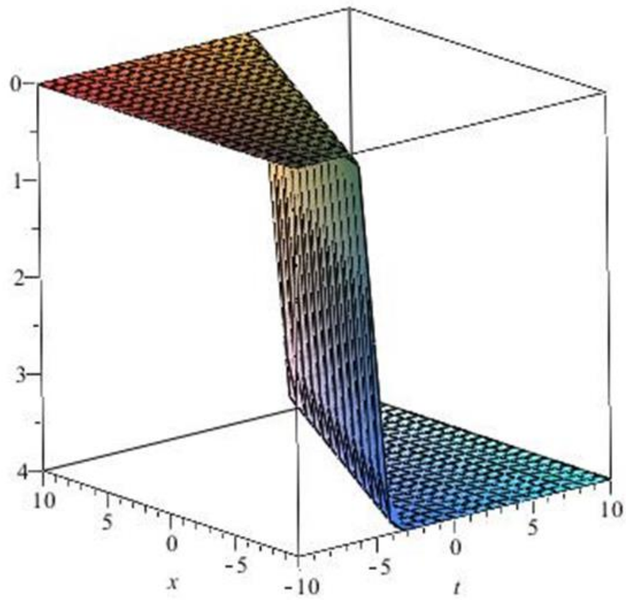

Figure 1 The kink solution of u_8(x, t) for d=0 and v=1.

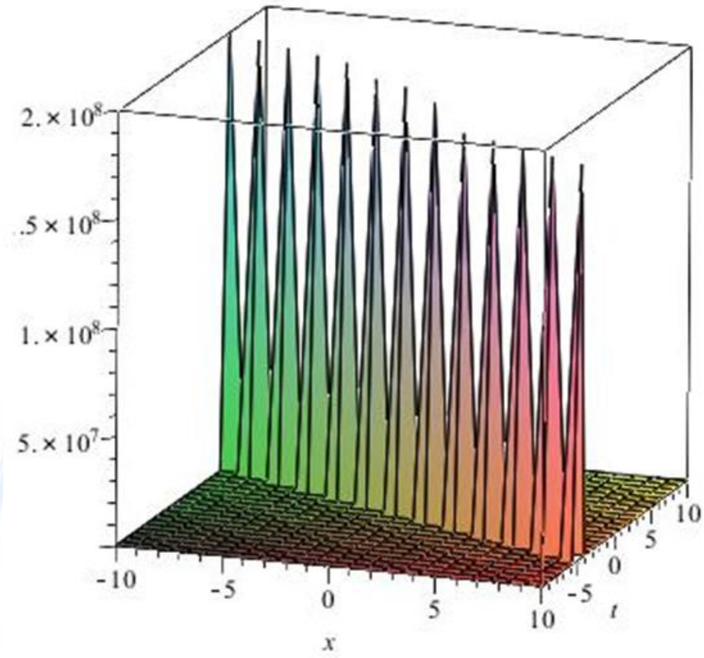

Figure 2 The soliton solution of u_9(x, t) for d=0 and v=1.

Kdv equation.

Finally, we would like to obtain the traveling wave solution of (2), we let $\zeta = x - ct$ carries Eq. (2) into the ODE

$$-cu + 3u^2 + u'' = 0 \tag{14}$$

Obtained after integrating the ODE once and setting the constant of integration equal to zero. Balancing u'' with u^2 in (14) gives m = 2. According the homogeneous balance procedure we obtain

$$u = a_0 + a_1(d + y) + a_2(d + y)^2 + \frac{b_1}{(d + y)} + \frac{b_2}{(d + y)^2} \tag{15}$$

Where a_0, a_1, a_2, b_1, d and b_2 are constants. Substituting (15) into (14), collecting the coefficients of Y^i we obtain the following system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, d, v$ and c, and solving this system we obtain the eight sets of solutions

Case1:

$$a_0 = -\frac{4}{3},\ a_1 = 0,\ a_2 = -2,\ \ b_1 = 0,\ \ b_2 = -2,\ c = -16,\ d = 0$$

Case2:

$$a_0 = 4,\ a_1 = 0,\ a_2 = -2,\ b_1 = 0,\ b_2 = -2,\ c = 16,\ d = 0$$

Case3:

$$a_0 = -2d^2 + \frac{2}{3},\ a_1 = 4d,\ a_2 = -2, b_1 = 0, b_2 = 0, c = -4, d = d$$

Case4:

$$a_0 = -2d^2 + 2, a_1 = 4d, a_2 = -2, b_1 = 0, b_2 = 0, c = 4, d = d$$

Case5:

$$a_0 = \frac{5}{3}, a_1 = \pm 2i\sqrt{2}, a_2 = -2, b_1 = 0, b_2 = 0, c = -4, d = \pm\frac{1}{2}i\sqrt{2}$$

Case6:

$$a_0 = 3, a_1 = \pm 2i\sqrt{2}\ , a_2 = -2, b_1 = 0, b_2 = 0, c = 4, d = \pm\frac{1}{2}i\sqrt{2}$$

Case7:

$$a_0 = -2d^2 + \frac{2}{3}, a_1 = 0, a_2 = 0, b_1 = -4d + 4d^3, b_2 = 4d^2 - 2d^4 - 2, c = -4, d = d$$

Case8:

$$a_0 = -2d^2 + 2, a_1 = 0, a_2 = 0, b_1 = -4d + 4d^3, b_2 = 4d^2 - 2d^4 - 2,\ c = 4,\ d = d$$

In view of this, we obtain the following solitons and kink solutions:

$$u_1(x,t) = -\frac{4}{3} - 2\tanh^2(x + 16t) - 2\coth^2(x + 16t),$$

$$u_2(x,t) = 4 - 2\tanh^2(x - 16t) - 2\coth^2(x - 16t),$$

$$u_3(x,t) = -2d^2 + \frac{2}{3} + 4d[d + \tanh(x + 4t)] - 2[d + \tanh(x + 4t)]^2,$$

$$u_4(x,t) = -2d^2 + 2 + 4d[d + \tanh(x - 4t)] - 2[d + \tanh(x - 4t)]^2,$$

$$u_5(x,t) = \frac{5}{3} \pm 2i\sqrt{2}[d + \tanh(x + 4t)] - 2[d + \coth(x + 4t)]^2$$

$u_6(x,t) = 3 \pm 2i\sqrt{2}[d + \tanh(x-4t)] - 2[d + \coth(x-4t)]^2,$

$u_7(x,t) = -2d^2 + \frac{2}{3} + \frac{-4d+4d^3}{(d+\tanh(x+4t))} + \frac{4d^3-2d^4-2}{(d+\tanh(x+4t))^2},$

$u_8(x,t) = -2d^2 + 2 + \frac{-4d+4d^3}{(d+\tanh(x-4t))} + \frac{4d^2-2d^4-2}{(d+\tanh(x-4t))^2},$

The various known results can be rediscovered, if d taken as special values. For example: if d=0, the traveling wave solution can be written as

$u_9(x,t) = \frac{2}{3} - 2\tanh^2(x+4t)$

$u_{10}(x,t) = 2[1 - \tanh^2(x-4t)]$

$u_{11}(x,t) = \frac{2}{3} - 2\coth^2(x+4t)$

$u_{12}(x,t) = 2[1 - \coth^2(x-4t)]$

This solution is the exact same solution obtained by Wazwaz [13], when c=-4 and c=4. Therefore, some graphs of the traveling wave solutions are represented in the Figures 3 and 4 with the aid of commercial software Maple.

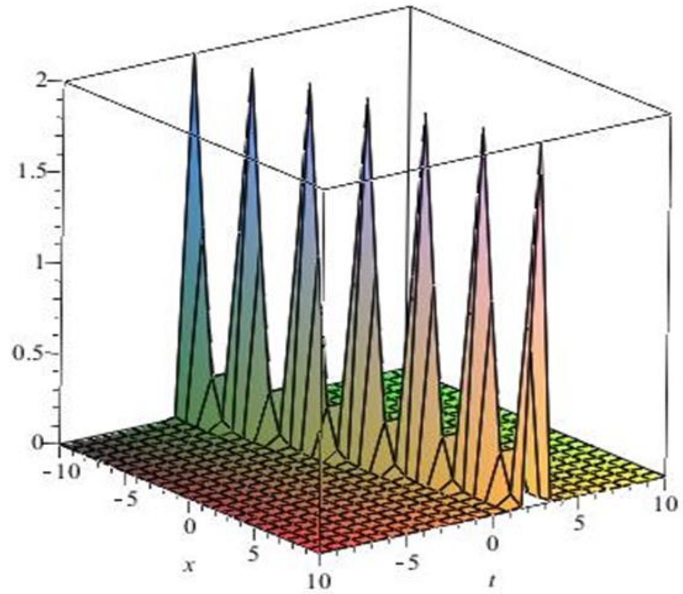

Figure 3 Soliton corresponding to solution u_{10}(x, t) for d=0.

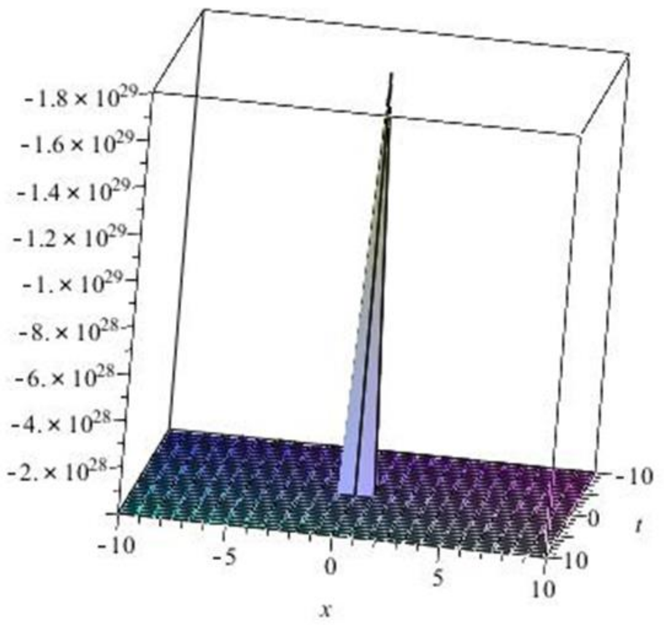

Figure 4 Soliton corresponding to solution u_2(x, t) for d=0.

Conclusion

In this paper, Extended tanh method has been successfully used to obtain several traveling wave solutions of the nonlinear Burgers equation and KdV equation. Furthermore, the reduction in the size of computational domain give this method a capability to compute a nonlinear partial differential equations in different areas of science. This mean that our algorithm is effective and more powerful. Also comparison was made between the solution of the Extended tanh method and the tanh method under special conditions. The solution of Extended tanh method gives many-soliton solutions for NLEEs, this means the tanh method does not have this capability. Also, the solutions contain free parameters. These solutions will be very useful in various physical situations. In the end, we give numerical simulations to complete the study.

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