On P_β-Open Sets and P_β-Irresolute Functions

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Abstract

The main purpose of this paper is to introduce and study generalization of open sets called P_β-open. This class is weaker than the class $p\theta$-open sets and stronger than the class pre-open sets. We studied the relation between these sets with other types of open sets and we gave several characterization about these sets. Also, we defined and investigated class of functions called P_β-irresolute and P_β^*- Irresolute. Several properties and interesting haracterization concerning of these a new types of functions are obtained.

Key words: pre-open, $p\theta$-open, P_β-open, P_β-irresolute, P_β^*- Irresolute.

حول المجموعات المفتوحة من نمط - P_β والدوال الغير حازمة من نمط - P_β

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الخلاصة

الغرض الرئيسي من هذا البحث هو تقديم تعميم جديد من المجموعات المفتوحة يدعى المجموعات المفتوحة من نمط - P_β. هذه العائلة من المجموعات تكون أضعف من المجموعات المفتوحة من نمط- $p\theta$ و أقوى من المجموعات المفتوحة ألأولية . قدمنا عدة خواص حول هذه المجموعات. كذلك قمنا بدراسة نوع جديد من الدوال اسميناها الدوال الغير حازمة من نمط - P_β و الدوال الغير الحازمة من نمط - P_β^* حيث درسنا العلاقة بين تلك الدوال وقدمنا العديد من الخواص حول هذه الدوال و برهنة العديد من النظريات حول هذا النوع من الدوال.

كلمات مفتاحية: مجموعة مفتوحة اولية، مجموعة مفتوحة من نمط- $p\theta$، مجموعة مفتوحة من نمط P_β، دالة غير حازمة من نمط P_β،دالة غير حازمة من نمط- P_β^*.

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Introduction

In 1982, El-Deeb S.N.[4] and others introduced the concept of pre-open which plays important role in topology. Since that many authors using this definition to study and investigate new topological properties. In same year, pre-irresolute function which is stronger than pre-continuity was studied by Rielly I.L[14]. In 1983, Abd El-Monsef M.E.[1] defined the concept of β-open and β-closed sets in topological space which several properties and theorems had been investigated. Later, Abd El-Monsef and Mahmoud R. A[8], studied β-irresolute and β-topological invariant by using the notion of β-open set . The aim of this work to introduce and study a new class of sets called P_β-open by using these sets we define a new class of irresolute functions called P_β-irresolute and P_β^*- Irresolute.

1.Preliminaries

Throughout this paper, For any subset B of a topological space (X, τ),the interior and closure of B are denoted by $int(B)$ and $cl(B)$, respectively.

Definition 1.1 A subset A of a topological space (X, τ) is said to be

1) Pre-open [4], if $A \subseteq int\ cl\ (A)$.
2) α-open [13], if $A \subseteq int\ cl\ int\ (A)$.
3) β-open [1], if $A \subseteq cl\ int\ cl\ (A)$.
4) Regular open [15], if $A = int\ cl\ (A)$.

Definition 1.2 The complement of pre-open (resp., α-open, β-open, and regular open) is called pre-closed [6] (resp., α-closed [14], β-closed [1], and regular closed[15]).

The family of all pre-open (resp., α-open, β-open, and β-closed) is denoted by $PO(X)$(resp.,$\alpha O(X)$, $\beta O(X)$ and $\beta C(X)$).

Definition 1.3 [6]The intersection of all pre-closed sets of topological space (X, τ)containing a subset A is called pre-closure of A and its denoted by $pcl(A)$.

Definition 1.4[6] The union of all pre-open (resp., β-open) of topological space (X, τ)contained in a subset A is called pre-interior(resp., β-interior) of A and its denoted by $pint(A)$(resp., $\beta int(A)$).

Definition 1.5 [16]A subset A of topological space $(X,\tau)$is called δ-open(resp., θ-open) if for each $x \in A$, there exists an open set G such that $x \in G \subseteq int\ cl(G) \subseteq A$ (resp., $x \in G \subseteq cl(G) \subseteq A$).

The complement of δ-open(resp., θ-open) set is called δ-closed (resp., θ-closed).

Definition 1.6 [12]A subset A of topological space $(X,\tau)$is called $p\theta$-open, if for each $x \in A$, there exists an pre-open set G such that $x \in G \subseteq pcl(G) \subseteq A$.

The complement of $p\theta$-open set is called $p\theta$-closed.

Proposition 1.7 [4] Let $\{A_\lambda : \lambda \in \Delta\}$ be the family of pre-open sets in topological space(X,τ), then $\bigcup_{\lambda\in\Delta} A_\lambda$ is pre-open set in X.

Proposition 1.8[12] every θ-open is $p\theta$-open set.

Definition 1.9 [16] A subset A of topological space $(X,\tau)$is called pre-regular if A is both pre-open and pre-closed.

Proposition 1.10[2] Let (X,τ) be a topological space. If $A \in PO(X)$ and $B \in \tau$, then $A \cap B \in PO(X)$.

Proposition 1.11 [2]Let (Y,τ_Y) be a subspace of topological space (X,τ)

1) If $A \in PO(X,\tau)$and $A \subseteq Y$, then $A \in PO(Y,\tau_Y)$.
2) $A \in PO(Y,\tau_Y)$ and $Y \in PO(X,\tau)$, then $A \in PO(X,\tau)$.

Proposition 1.12[1] Let A and Y be any subsets of topological space (X,τ). If A is β-open set in X and Y is α-open set in X, then $A \cap Y$ is β-open set in Y.

Proposition 1.13 [3]Let A and Y be any subsets of topological space (X,τ) such that $A \subseteq Y \subseteq X$ and Y is α-open set in X, then $A \in \beta O(Y)$ if and only if $A \in \beta O(X)$.

Proposition 1.14 if $H \subseteq Y \subseteq X$ such that $H \in \beta C(Y)$ and $Y \in \alpha O(X) \cap \beta C(X)$, then $H \in \beta C(X)$.

Proof. Let H be β-closed set in Y, then $Y \setminus H$ is β-open in Y. Since Y is α-open set in X , then by Proposition 1.13, $Y \setminus H$ is β-open in X and so $X \setminus (Y \setminus H) = F$ is β-closed set and since Y is β-closed set in X, then $F \cap Y = H$ is also β-closed set in X.

Definition 1.15 [10]A topological space (X,τ) is said to pre-T_1 space if for each two distinct points $x,y \in X$, there exists pre-open set G containing x but not y and pre-open set H containing y but not x.

Proposition 1.16 [10]A topological space (X,τ) is pre-T_1 space if and only if for any point $x \in X$, the singleton $\{x\}$ is pre-closed.

Definition 1.17 [7]A topological space (X,τ) is submaximal if every dense subset of X is open.

Proposition 1.18 [7] in submaximal space, every pre-open is open.

Theorem 1.19[2] A topological space (X,τ) is β-regular, if for any open set G in (X,τ) and each point $x \in G$, there exists a β-open set H such that $x \in H \subseteq \beta cl(H) \subseteq G$.

Definition 1.20 A function $f:(X,\tau) \longrightarrow (Y,\zeta)$ is called

1) Pre-irresolute [5], if the inverse image of every pre-open set in Y is pre-open set in X.
2) β-irresolute [8], if the inverse image of every β-open in Y is β-open in X.
3) Completely pre-irresolute[9], if the inverse image of every pre-open set in Y is regular open set in X.

2. P_β-open set

Definition 2.1 A pre-open subset A of a topological space (X,τ) is said to be P_β-open if for each $x \in A$, there exists β-closed set F such that $x \in F \subseteq A$. The family of P_β-open is denoted by $P_\beta O(X)$.

Remark 2.2 Every P_β-open set is pre-open.

But the converse is not true as showing in the next example.

Example 2.3 Let $X = \{a,b,c\}$ equipped with topology $\tau = \{\phi,\{b\},X\}$, then $PO(X) = \{\phi,\{b\},\{a,b\},\{b,c\},X\}$, $\beta C(X) = \{\phi,\{c\},\{a\},\{b,c\},X\}$, and $P_\beta O(X) = \{\phi,X\}$. Clearly $\{b\}$ is pre-open but it is not P_β-open set.

Proposition 2.4 Every $p\theta$-open is P_β-open set.

Proof. Let A be any $p\theta$-open subset of a topological space (X,τ). If A is empty set, then there is nothing to proof. If A is non empty set. Let $x \in A$, then there exists a pre-open U such that $x \in U \subseteq pcl(U) \subseteq A$. Since $pcl(U)$ is pre-closed, then it is β-closed, and A is pre-open , since $A = \bigcup_{x\in A} U_x$, U_x is a pre-open set for all x. Hence A is P_β-open set.

Corollary 2.5 Every θ-open is P_β-open set.

Proof. Follows from Proposition (2.4) and Proposition (1.8).

Proposition 2.6 every δ-open is P_β-open set.

Proof. Let A be any δ-open subset of a topological space (X, τ). If A is empty set, then there is nothing to proof. If not , let $x \in A$, then there exists an open set G such that $x \in G \subseteq int\ cl(G) \subseteq A$. Since $int\ cl(G)$ is regular open , then it is β-closed and since A is pre-open , then A is P_β-open set.

However open and P_β-open sets are independent as showing in the following examples:

Example 2.7 Let $X = \{a, b, c\}$ equipped with topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, then $\tau = PO(X)$and $P_\beta O(X) = \{\phi, X\}$. Clearly $\{a\}$ is open but it is not P_β-open set.

Example 2.8 Let $X = \{a, b, c, d\}$ equipped with topology $\tau = \{\phi, \{c\}, \{a, d\}, \{a, c, d\}, X\}$, then $P_\beta O(X) = \{\phi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. Hence $\{a, b, c\}$ is P_β-open but it is not open set.

Proposition 2.9 If a topological space (X, τ) is β-regular, then $\tau \subseteq P_\beta O(X)$.

Proof. Let W be non-empty open set, then $x \in W$. But (X, τ) is β-regular. by Theorem (1.19), there exists β-open set U such that $x \in U \subseteq \beta cl(U) \subseteq W$ and since W is an open set, then W is pre-open. Hence W is P_β-open set.

Proposition 2.10 If a topological space (X, τ) is pre-T_1 space, then $PO(X) = P_\beta O(X)$

Proof. Clearly every P_β-open set is pre-open. On the other hand, let A is pre-open. If A is empty set , then the proof is done, if A is non-empty set, then $x \in \{x\} \subseteq A$. By Proposition (1.16), $\{x\}$ is pre-closed, so $\{x\}$ is β-closed. Hence A is P_β-open set.

Proposition 2.11 In submaximal topological space (X, τ), every P_β-open set is open

Proof. Follows from Proposition(1.18).

Proposition 2.12 Let $\{A_\gamma : \gamma \in I\}$ be a collection of P_β-open sets in topological space(X, τ), then $\bigcup\{A_\gamma : \gamma \in I\}$ is also P_β-open set.

Proof. By Proposition (1.7), $\cup\{A_\gamma : \gamma \in I\}$ is pre-open set. Let $x \in \cup\{A_\gamma : \gamma \in I\}$, then there exists $\gamma_o \in I$ such that $x \in A_{\gamma_o}$ and since $A_{\gamma_o}$is P_β-open set, then there exists β-closed set F_{γ_o} such that $x \in F_{\gamma_o} \subseteq A_{\gamma_o} \subseteq \cup\{A_\gamma : \gamma \in I\}$. Hence $\cup\{A_\gamma : \gamma \in I\}$ is P_β-open set.

Proposition 2.13 Let A and B are subsets of topological space(X,τ). If $A \in P_\beta O(X)$ and $B \in \tau \cap \beta C\,(X)$, then $A \cap B \in P_\beta O(X)$.

Proof. By Proposition (1.10), $A \cap B \in PO(X)$. Let $x \in A \cap B$ and since A is P_β-open, then there exists β-closed F such that $x \in F \subseteq A$. Since the intersection of β-closed is also β-closed, then $F \cap B$ is β-closed set such that $x \in F \cap B \subseteq A \cap B$. Hence $A \cap B$ is P_β-open.

Proposition 2.14 Let $A \subseteq Y \subseteq X$ and (Y,τ_Y) be α-open subspace of topological space(X,τ). If $A \in P_\beta O(X,\tau)$, then $A \in P_\beta O(Y,\tau_Y)$.

Proof. Let A be P_β-open set in topological space(X,τ), then A is pre-open set and for each $x \in A$, there exists β-closed F such that $x \in F \subseteq A$. Since A is pre-open in topological space (X,τ) and $A \subseteq Y$,then by Proposition 1.11(1) A is pre-open set in Y. Since F is β-closed set in X, then $X \setminus F$ is β-open set in X and since Y is α-open in X, then by Proposition (1.12), $(X \setminus F) \cap Y = Y \setminus F$ is β-open in Y. Thus F is β-closed in Y. Hence A is P_β-open in subspace(Y,τ_Y).

Proposition 2.15 Let $A \subseteq Y \subseteq X$ and Y be α-open and β-closed subsets of topological space(X,τ). If $A \in P_\beta O(Y,\tau_Y)$, then $A \in P_\beta O(X,\tau)$.

Proof. If A isP_β-open set in subspace (Y,τ_Y), then A is pre-open set in Y and for each $x \in A$, there exists β-closed F in Y such that $x \in F \subseteq A$. Since A is pre-open in Y and Y is pre-open in X, then by Proposition 1.11(2) A is pre-open set in X. Since F is β-closed in Y and since Y be α-open and β-closed in topological space(X,τ), then by Proposition 1.14, F is β-closed in X. Hence A is P_β-open set in topological space (X,τ).

Definition 2.16 A subset A of topological space(X,τ) is called P_β-closed , if its complement of P_β-open set. The family of all P_β-closed subsets of topological space (X,τ) is denoted by $P_\beta C(X)$.

Proposition 2.17 A subset B of topological space(X,τ) is called P_β-closed if and only if B is pre-open and its intersection of P_β-open sets.

Proof. Straightforward.

Proposition 2.18 The intersection of any P_β-closed subsets of topological space(X,τ) is also P_β-closed.

Proof. Follows from the fact that the union of P_β-open sets is also P_β-open set.

The union of P_β-closed subsets of topological space(X,τ) need not be P_β-closed as showing in the following example.

Example 2.19 In Example 2.8, $P_\beta C(X) =$ $\{\phi,\{a\},\{b\},\{d\},\{a,b\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\},X\}$, then $\{a\},\{d\} \in P_\beta C(X,\tau)$. But $\{a\}\cup\{d\} = \{a,d\} \notin P_\beta C(X,\tau)$.

Proposition 2.20 If A is pre-regular subset of topological space (X,τ), then A is both P_β-open and P_β-closed set.

Proof. Straightforward.

Proposition 2.21 For any subset A of topological space (X,τ). if A is one of the following:

1) A is θ-closed.
2) A is $P\theta$-closed.
3) A is δ-closed.

Then it isP_β-closed.

Proof. Obvious.

Proposition 2.22 Let A and B are any subsets of topological space(X,τ). If $A \in P_\beta C(X)$ and $B \in C(X)\cap P_\beta O(X)$, then $A\cap B \in P_\beta C(X)$.

Proof. Follows from Proposition (2.13).

Proposition 2.23 Let $B \subseteq Y \subseteq X$ and Y be α-closed subspace of topological space(X,τ). If $A \in P_\beta C(X,\tau)$, then $A \in P_\beta C(Y,\tau_Y)$.

Proof. Follows from Proposition (2.14).

Proposition 2.24 Let $A \subseteq Y \subseteq X$ and Y be α-closed and β-open subset of topological space(X,τ). If $A \in P_\beta C(Y,\tau_Y)$, then $A \in P_\beta C(X,\tau)$.

Proof. Follows from Proposition (2.15).

Definition 2.25 If A is any subset of topological space (X,τ), then P_β-interior of A is the largest P_β-open set contained in A. Which is denoted by $P_\beta int(A)$.

Definition 2.26 Let A is any subset of topological space (X,τ), a point p of X is said to be P_β-interior point of A, if there exists a P_β-open set U containing p such that $p \in U \subseteq A$.

Proposition 2.27 For any subset A of topological space(X,τ), $P_\beta int(A) \subseteq Pint(A) \subseteq \beta int(A)$.

Proof. Obvious.

Here some properties of P_β-interior operator.

Proposition 2.28 If A and B are any two subsets of topological space (X,τ), then

1) $P_\beta int(A) \subseteq A$.
2) A isP_β-open if and only if $A = P_\beta int(A)$.
3) If $A \subseteq B$, then $P_\beta int(A) \subseteq P_\beta int(B)$.
4) $P_\beta int(A) \cup P_\beta int(B) \subseteq P_\beta int(A \cup B)$.
5) $P_\beta int(A \cap B) \subseteq P_\beta int(A) \cap P_\beta int(B)$.

Proof. Obvious.

Definition 2.29 Let (X,τ) be a topological space and let $B \subseteq X$, then P_β-closure of B is the intersection of allP_β-closed sets which containing B or equivalently the smallest P_β-closed set containing B which is denoted by $P_\beta cl(B)$.

For the following proposition, routine proof so it is omitted.

Proposition 2.30 Let B be any subset of a topological space(X,τ). Then $x \in P_\beta cl(B)$ if and only if for each P_β-open set U containing x, $U \cap B \neq \phi$.

Proposition 2.31 For any subset B of topological space(X,τ), $pcl(B) \subseteq P_\beta cl(B)$.

Proof. Let $x \notin P_\beta cl(B)$, then by Proposition 2.30, there exists P_β-open set U containing x such that $U \cap B = \phi$. But U is P_β-open set, thus U is a pre-open containing x. This implies $x \notin pcl(B)$. Hence $pcl(B) \subseteq P_\beta cl(B)$.

Proposition 2.32 If A is a P_β-open and B is any subset of a topological space(X, τ). then $A \cap B = \phi$ if and only if $P_\beta int(A) \cap P_\beta cl(B) = \phi$.

Proof. Suppose that $A \cap B = \phi$, then $A \subseteq X \setminus B$. By Proposition 2.28 and lemma 2. $P_\beta int(A) \subseteq P_\beta int(X \setminus B) = X \setminus P_\beta cl(B)$. Therefore $P_\beta int(A) \cap P_\beta cl(B) = \phi$. Conversely, $P_\beta int(A) \cap P_\beta cl(B) = \phi$, this implies that $P_\beta int(A) \subseteq X \setminus P_\beta cl(B) = P_\beta int(X \setminus B)$. But A is a P_β-open, thus $A \subseteq P_\beta int(X \setminus B) \subseteq X \setminus B$. Hence $A \cap B = \phi$.

Definition 2.34 Let A be any subset of a topological space(X, τ). A point $x \in X$ is called P_β-limit point of A, if for each P_β-open set U containing x, $U \cap (A \setminus \{x\}) \neq \phi$. The set of all P_β-limit points of A is called P_β-derived set of A and denoted by $P_\beta D(A)$.

Proposition 2.35 Let A be any subset of a topological space(X, τ). If F is a β-closed set in topological space(X, τ) containing x such that $F \cap (A \setminus \{x\}) \neq \phi$, then $x \in P_\beta D(A)$.

Proof. Let U be an P_β-open set in topological space(X, τ) containing x, then U is pre-open and there exists β-closed set F such that $x \in F \subseteq U$, for each $x \in U$. By hypothesis $F \cap (A \setminus \{x\}) \neq \phi$, thus $U \cap (A \setminus \{x\}) \neq \phi$. Hence $x \in P_\beta D(A)$.

Proposition 2.36 If a subset B of a topological space(X, τ) is P_β-closed, then B contains all of its P_β-limit points.

Proof. Suppose that B is P_β-closed, then $X \setminus B$ is P_β-open set and since $(X \setminus B) \cap B = \phi$. Therefore $(X \setminus B) \cap B - \{x\} = \phi$ for each $x \in X \setminus B$. Thus no point of $X \setminus B$ is limit point of B. Hence B contains all of its P_β-limit points.

Proposition 2.37 Let A be any subset of a topological space(X, τ). If A is P_β-closed, then $P_\beta D(A) \subseteq A$.

Proof. Let A be P_β-closed subset of a topological space $(X, \tau)$and let $x \notin A$, then $X \setminus A$ is a P_β-open set and since$(X \setminus A) \cap A - \{x\} = \phi$. Thus $x \notin P_\beta D(A)$. Therefore $P_\beta D(A) \subseteq A$.

The following theorem state some basic properties of P_β-derived set

Theorem 2.38 Let A and B be any two subsets of topological space (X, τ). Then

1) $P_\beta D(\phi) = \phi$.
2) If $A \subseteq B$, then $P_\beta D(A) \subseteq P_\beta D(B)$.
3) $P_\beta D(A \cap B) \subseteq P_\beta D(A) \cap P_\beta D(B)$.
4) $P_\beta D(A) \cup P_\beta D(B) \subseteq P_\beta D(A \cup B)$.
5) $x \in P_\beta D(A)$ Implies that $x \in P_\beta D(A \setminus \{x\})$.

Proof. Obvious.

Theorem 2.39 Let A be any subset of a topological space(X, τ). Then

1) $P_\beta D\left(P_\beta D(A)\right) \setminus A \subseteq P_\beta D(A)$.
2) $P_\beta D\left(A \cup P_\beta D(A)\right) \subseteq A \cup P_\beta D(A)$.

Proof. 1) Let $x \in P_\beta D\left(P_\beta D(A)\right) \setminus A$, then for any P_β-open set U containing x, $U \cap \left(P_\beta D(A) \setminus \{x\}\right) \neq \phi$, let $y \in U \cap \left(P_\beta D(A) \setminus \{x\}\right)$. Since U is a P_β-open set containing y and since $y \in P_\beta D(A) \setminus \{x\}$, then $U \cap (A \setminus \{y\}) \neq \phi$. Let $p \in U \cap (A \setminus \{y\})$. But $p \in A$ and $x \notin A$, thus $p \neq x$. It follows that $p \in U \cap (A \setminus \{x\}) \neq \phi$. Hence $x \in P_\beta D(A)$.

2) Let $x \in P_\beta D\left(A \cup P_\beta D(A)\right)$. If $x \in A$, then $P_\beta D\left(A \cup P_\beta D(A)\right) \subseteq A \subseteq A \cup P_\beta D(A)$. Let $x \in P_\beta D\left(A \cup P_\beta D(A)\right) \setminus A$, then for any P_β-open set U containing x, $U \cap \left(\left(A \cup P_\beta D(A)\right) \setminus \{x\}\right) \neq \phi$. It follows that either $U \cap (A \setminus \{x\}) \neq \phi$ or $U \cap \left(P_\beta D(A) \setminus \{x\}\right) \neq \phi$. Similarly for (1), we only have $U \cap (A \setminus \{x\}) \neq \phi$. Thus $x \in P_\beta D(A)$. In both cases we have $P_\beta D\left(A \cup P_\beta D(A)\right) \subseteq A \cup P_\beta D(A)$.

Proposition 2.40 Let A be any subset of a topological space(X, τ). Then $P_\beta int(A) = A \setminus P_\beta D(X \setminus A)$.

Proof. Let $x \in A \setminus P_\beta D(X \setminus A)$, then $x \notin P_\beta D(X \setminus A)$, so there exists a P_β-open set U containing x such that $U \cap (X \setminus A) = \phi$ and so $x \in U \subseteq A$. Hence $x \in P_\beta int(A)$. On the other hand, let $x \in P_\beta int(A)$ and since $P_\beta int(A)$ is a P_β-open set such that$P_\beta int(A) \cap (X \setminus A) = \phi$, then Let $x \in A \setminus P_\beta D(X \setminus A)$. Hence$P_\beta int(A) = A \setminus P_\beta D(X \setminus A)$.

3. P_β- Irresolute and P_β^*- Irresolute functions

Definition 3.1 a function $f:(X,\tau) \longrightarrow (Y,\xi)$ is said to be P_β- Irresolute, if the inverse image of every P_β- open set in Y is a P_β- open set in X.

Definition 3.2 A function $f:(X,\tau) \longrightarrow (Y,\xi)$ is said to be P_β^*- Irresolute, if the inverse image of every pre-open set in Y is a P_β- open set in X.

Remark 3.3 every P_β^*- Irresolute function is P_β- Irresolute.

In general the converse is not true as showing in the next example.

Example 3.4 Let $X = \{a,b,c\}$ equipped with topology $\tau = \{\phi, \{a\}, X\}$. Consider the identity map $f:(X,\tau) \longrightarrow (X,\tau)$, then f isP_β- Irresolute. But it is not P_β^*- Irresolute mapping, since $\{a\}$ is a pre-open and $f^{-1}(\{a\}) = \{a\}$ is not P_β- open set.

The next proposition determine the relation between pre-irresolute and P_β^*- Irresolute

Proposition 3.5 Let $f:(X,\tau) \longrightarrow (Y,\xi)$ be any function and let X be a pre-T_1 space then f is P_β^*- Irresolute if and only if it is pre-irresolute.

Proof. Clearly every P_β^*- Irresolute is pre-irresolute. Conversely, suppose f is pre-irresolute and let U be a pre-open set in Y, then $f^{-1}(U)$ is pre-open set in X. But X is pre-T_1 space, thus for each $x \in f^{-1}(U)$, $\{x\}$ is pre-closed and hence its β-closed. It follows that $f^{-1}(U)$ is P_β-open set in X. Hence f is P_β^*- Irresolute.

Proposition 3.6 for a function $f:(X,\tau) \longrightarrow (Y,\xi)$, the following are equivalent:

1) f is P_β^*- Irresolute
2) For each pre-open set V in Y containing $f(x)$, there exists a P_β- open set U containing x such that $f(U) \subseteq V$.
3) The inverse image of every pre-closed set in Y is P_β-closed set in X.

Proof. 1)⟹2) Suppose that f is P_β^*- Irresolute and let V be pre-open set in Y containing $f(x)$, then $f^{-1}(V) = U$ is a P_β-open set in X containing x such that $f(U) = f(f^{-1}(V)) \subseteq V$.

2) ⟹3) Let F be pre-closed in Y, then $Y \setminus F$ is pre-open set in Y. If $f^{-1}(Y \setminus F) = \phi$, then the proof is done. If $f^{-1}(Y \setminus F) \neq \phi$. Let $x \in f^{-1}(Y \setminus F)$ and so $f(x) \in Y \setminus F$. by 2) there

exists P_β-open set U containing x such that $f(U) \subseteq Y \setminus F$ this implies that $x \in U \subseteq f^{-1}(Y \setminus F)$. So $f^{-1}(Y \setminus F)$ is P_β- open. But $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$, thus $f^{-1}(F)$ is P_β-closed set in X.

3) ⟹1) Let U be a pre-open set in Y, then $Y \setminus U$ is a pre-closed in Y. By 3) $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is P_β-closed in X and so, $f^{-1}(U)$ is P_β-open set in X. Hence f is P_β^*- Irresolute.

Proposition 3.7 for a function $f:(X,\tau) \longrightarrow (Y,\xi)$, the following are equivalent:

1) f is P_β- Irresolute.
2) For each P_β-open set V in Y containing $f(x)$, there exists a P_β- open set U containing x such that $f(U) \subseteq V$.
3) The inverse image of every P_β-closed set in Y is P_β-closed set in X.

Proof. Similar to the proof of proposition (3.6).

Theorem 3.8 A function $f:(X,\tau) \longrightarrow (Y,\xi)$ is P_β- Irresolute if and only if $f\left(P_\beta cl(A)\right) \subseteq P_\beta cl(f(A))$.

Proof. Suppose that f is P_β- Irresolute. For any subset A of topological space (X,τ), $P_\beta cl(f(A))$ is P_β-closed set in topological space(Y,ξ). Since f is P_β- Irresolute, then by Proposition (3.7), $f^{-1}\left(P_\beta cl(f(A))\right)$ is P_β-closed set in X and since $f(A) \subseteq P_\beta cl(f(A))$, then $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(P_\beta cl(f(A))\right)$. Therefore, $P_\beta cl(A) \subseteq f^{-1}\left(P_\beta cl(f(A))\right)$. Hence $f\left(P_\beta cl(A)\right) \subseteq P_\beta cl(f(A))$.

Conversely, let F be a P_β-closed set in Y, then $f^{-1}(F)$ is any subset of X. By hypothesis, $f\left(P_\beta cl(f^{-1}(F))\right) \subseteq P_\beta cl\left(f(f^{-1}(F))\right) = P_\beta cl(F) = F$, So $f^{-1}\left(f\left(P_\beta cl(f^{-1}(F))\right)\right) \subseteq f^{-1}(F)$. Thus $P_\beta cl(f^{-1}(F)) \subseteq f^{-1}(F)$ this implies that $f^{-1}(F)$ is P_β-closed set in X. Hence f is P_β- Irresolute.

Proposition 3.9 Let $f:(X,\tau) \longrightarrow (Y,\xi)$ be an injective function. If f is pre-irresolute and β-irresolute, then f is P_β- Irresolute.

Proof. Straightforward.

Proposition 3.10 Every completely pre-irresolute is P_β^*- Irresolute.

Proof. Follow from the fact that every regular open set is P_β-open set.

Theorem 3.11 If a function $f:(X,\tau)\longrightarrow(Y,\xi)$ is P_β- Irresolute and $g:(Y,\xi)\longrightarrow(Z,\rho)$ is P_β^*-Irresolute, then $g\circ f:(X,\tau)\longrightarrow(Z,\rho)$ is P_β^*- Irresolute.

Proof. Let W be a pre-open set in Z. Since g is P_β^*- Irresolute, then $g^{-1}(W)$ is P_β- open set in Y. But f is P_β-irresolute, therefore $f^{-1}\left(g^{-1}(W)\right)=(g\circ f)^{-1}(W)$ is P_β- open set in X. Hence $g\circ f$ is P_β^*- Irresolute.

Theorem 3.12 If a function $f:(X,\tau)\longrightarrow(Y,\xi)$ is P_β^*- Irresolute and $g:(Y,\xi)\longrightarrow(Z,\rho)$ is pre-Irresolute, then $g\circ f:(X,\tau)\longrightarrow(Z,\rho)$ is P_β^*- Irresolute.

Proof. Similar to proof of the Theorem (3.11).

Theorem 3.13 If a function $f:(X,\tau)\longrightarrow(Y,\xi)$ is P_β- Irresolute and $g:(Y,\xi)\longrightarrow(Z,\rho)$ is P_β-Irresolute , then $g\circ f:(X,\tau)\longrightarrow(Z,\rho)$ is P_β- Irresolute.

Proof. Similar to proof of the Theorem 3.11.

Definition 3.14 A function $f:(X,\tau)\longrightarrow(Y,\xi)$ is said to be P_β-open , if the image of every P_β-open set in X is P_β-open set in Y.

Proposition 3.15 If a function $f:(X,\tau)\longrightarrow(Y,\xi)$ is P_β-open and onto, $g:(Y,\xi)\longrightarrow(Z,\rho)$ be any function, and $g\circ f:(X,\tau)\longrightarrow(Z,\rho)$ is P_β- Irresolute, then $g:(Y,\xi)\longrightarrow(Z,\rho)$ is P_β-Irresolute.

Proof. Let W be P_β-open set in Z. Since $g\circ f$ is P_β- Irresolute, then $(g\circ f)^{-1}(W)=f^{-1}\left(g^{-1}(W)\right)$ is P_β-open set in X. But f is P_β-open and onto, thus $g^{-1}(W)$ is P_β-open set in Y. Hence g is P_β- Irresolute.

Proposition 3.16 If a function $f:(X,\tau)\longrightarrow(Y,\xi)$ is P_β-open and onto, $g:(Y,\xi)\longrightarrow(Z,\rho)$ be any function, and $g\circ f:(X,\tau)\longrightarrow(Z,\rho)$ is P_β^*- Irresolute, then $g:(Y,\xi)\longrightarrow(Z,\rho)$ is P_β^*-Irresolute.

Proof. Similar to proof of Proposition (3.15).

Theorem 3.17 If $X = T \cup S$, where T and S are α-open and β-closed. If $f:(X,\tau) \longrightarrow (Y,\xi)$ is any function such that $f\,|_T$ and $f\,|_S$ are P_β- Irresolute, then f is P_β- Irresolute.

Proof. Let W be a P_β-open set in Y. Since $f\,|_T$ and $f\,|_S$ are P_β- Irresolute, then $(f\,|_T)^{-1}(W)$ and $(f\,|_S)^{-1}(W)$ are P_β-open set in T and S respectively. And by Proposition 2.15, $(f\,|_T)^{-1}(W)$ and $(f\,|_S)^{-1}(W)$ are P_β-open set in X. Therefore $f^{-1}(W) = (f\,|_T)^{-1}(W) \cup (f\,|_S)^{-1}(W)$ is P_β-open set in X. Hence f is P_β- Irresolute.

Theorem 3.18 If a function $f:(X,\tau) \longrightarrow (Y,\xi)$ is P_β- Irresolute and A be any regular open subset of X, then $f\,|_A : (A,\tau_A) \longrightarrow (Y,\xi)$ is P_β- Irresolute.

Proof. Let W be a P_β-open set in Y. Since f is P_β- Irresolute, then $f^{-1}(W)$ is P_β-open set in X. But A is regular open set in X, thus by Proposition 2.13, $f^{-1}(W) \cap A = (f\,|_A)^{-1}(W)$ is P_β-open set in X and by Proposition 2.14, $(f\,|_A)^{-1}(W)$ is P_β-open set in A. Hence $f\,|_A$ is P_β-Irresolute.

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