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ISI(G) Topological Index of Sum Graphs For Some Finite Groups

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Abstract

A topological index can convert a great deal of knowledge about a mathematical structure into a concrete number. In theoretical mathematics, the topological indices are crucial. A topological index is a graph-based Recently, unified approaches have been adopted to study these topological indices in groups. In this work, we look at the generalized ISI index since we can obtain some other topological indices as exceptional cases of sum graphs of groups. These generalized indices reduce the work for calculating some topological indexes differently.

Keywords: Generalized ISI index, topological index, degree, cyclic graph, connected graph.

مؤشر ISI (G) الطوبولوجي لبيان الجمع لبعض الزمر المنتهية

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الخلاصة

يمكن للمؤشر الطوبولوجي التقاط الكثير من المعلومات حول بنية رياضية إلى رقم فعلي. المؤشرات الطوبولوجية لها دور أساسي في الرياضيات النظرية. الفهرس الطوبولوجي يعتمد على الرسم البياني

في الآونة الأخيرة ، تم اعتماد نهج موحدة لدراسة هذه الطوبولوجية مؤشرات في مجموعات. في هذا العمل ، ننظر إلى مؤشر ISI يمكننا الحصول على بعض المؤشرات الطوبولوجية الأخرى كحالات استثنائية لمجموع الرسوم البيانية للمجموعات ؛ هذه المؤشرات المعممة تقلل من العمل لحساب بعض الفهرس الطوبولوجي بشكل مختلف .

الكلمات المفتاحية: المعمم، الفهرس الطوبولوجي، الدرجة، البيان الدائري، البيان المتصل. ISIمؤشر.

Introduction

Let G be a graph, the set vertices V , and edges E of G denoted as $G(V(G),E(G))$. In this paper, we only consider simple finite graphs. The vertex $u \in V\ (G)$ that is degree represented by $deg(u)$(or $d(u)$) of G. The vertices x and y are called adjacent if there is an edge connecting them that is denoted by $x{\sim}y$ (or xy). An edge xy degree is defined as the number of edges connected to the edge, that is, the number of edges connected to x and y except for the edge xy. By $G \in\ G_+(k,\ell)$ we mean that G is a graph with k vertices and ℓ edges. The terminologies and notations used but not identified in this article can be found in [1]. Topological indices are the numerical values that are associated with a graph structure.

Many topological indices have been proposed and studied over the years based on distance, degree, and other graph parameters. Some of them can be found in [7, 11]. Historically, indices of Zagreb can be considered the first degree-based topological indices, which apply to many scientific, chemical, physical and economic, etc. [22-25] Various studies related to group or ring theory aspects of the Z modulo via graph theory, see that [3-11,21,26] and the references therein. In 2018, Alwardi A. et al. studied the entire Zagreb indices of graphs [2].

The sum graph of the group (Z_n) is a finite group of order n, where the vertices of the graph represent the elements of (Z_n) such that there is an edge between the two vertices a and b if and only if $\mathcal{O}(a)+\mathcal{O}(b)>\mathcal{O}(Z_n)$ denoted by $G_+(Z_n)$, where $\mathcal{O}(Z_n)$ is the order of Z_n and $\mathcal{O}(a),\mathcal{O}(b)$ be the orders of vertices a and b . The Topological indices have been studied in the last few years; they may be found in [12-16]. In 2013, the index of Zagreb was re-defined as first, second, and third indices of Zagreb by Ranjini et al. [3]. Further, it can be considered as a particular case of the generalized inverse sum index $ISI_{(\gamma,\mu)}(G)$ of a graph G proposed by Buragohain et al. in [4].

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In this paper, we computed topological indices, the generalized inverse sum index $ISI_{(\gamma,\mu)}(G)$ of $G_+ where\ G = Z_{p^n}, Z_{p^n q}, Z_{pq^m}\ and\ Z_{p^n q^m}$, where p and q are prime numbers and $p < q$.

Basic Concepts and Terminology

In what follows, we focus on a topological index that will be needed in the subsequent considerations.

The first general Zagreb index (or general Zeroth-order Randic index) Q_γ is defined as

$Q_\gamma(G) = \sum_{i=1}^{k} d_i^\gamma = \sum_{i\sim j}\left(d_i^{\gamma-1} + d_j^{\gamma-1}\right)$, Where k is the order of vertices and $\gamma \in \mathbb{R}$ [14].

The generalized Randic index (or Connectivity index) R_γ, is defined as

$R_\gamma = R_\gamma(G) = \sum_{i\sim j}\left(d_i d_j\right)^\gamma$, And $\gamma \in \mathbb{R}$ [14].

The general Sum-Connectivity index H_μ as $H_\mu = H_\mu(G) = \sum_{i\sim j}\left(d_i + d_j\right)^\mu$, And $\mu \in \mathbb{R}$ [15].

The generalized inverse sum index $ISI_{(\gamma,\mu)}(G)$ of a graph G proposed by Buragohain et al. in [4]. $ISI_{(\gamma,\mu)} = \sum_{i\sim j}\left(d_i d_j\right)^\gamma\left(d_i + d_j\right)^\mu$, and $\gamma, \mu \in \mathbb{R}$.

1. The Sum graphs of groups $Z_{p^n}, Z_{p^n q}, Z_{pq^m}$ *and* $Z_{p^n q^m}$

The following section will investigate the concept of Z_n Group via graph theory by defining that by order laws. Moreover, we found the degree of the exceptional value of Z_n at $p^n, p^n q, pq^m, and\ p^n q^m$, where $p\ and\ q$ are the prime numbers to power the positive integers numbers n& m.

Definition 1.1 [19]: Let G be a finite cyclic group. The group sum graph, denoted by $G_+(V,E)$ of G_+ Is a graph with $V(G_+) = \bigcup_{x\in G}\langle x\rangle$ And two distinct vertices x and y are adjacent in G_+, denoted by $\langle x\rangle\sim\langle y\rangle$ if and only if $\mathcal{O}(x) + \mathcal{O}(y) > \mathcal{O}(G)$, where $\mathcal{O}(G)$ is the order of the group G. (i.e.) $V(G_+) = \bigcup_{x\in G}\langle x\rangle, E(G_+) = \{xy|\ \langle x\rangle\sim\langle y\rangle$ if and only if $\mathcal{O}(x) + \mathcal{O}(y) > \mathcal{O}(G)$, where $x, y \in G$ and $x \neq y\}$.

Remark 1.2 [19]:

If taking as definition 1.1, $\mathcal{O}(x) + \mathcal{O}(y) \leq \mathcal{O}(G)$, where G is a finite group of order n. We see that the graph is not connected because there exists at least one element a such that

$\mathcal{O}(a) = \mathcal{O}(G)$. Therefore, $\mathcal{O}(a) + \mathcal{O}(a_i) > \mathcal{O}(G), \forall\, a_i \in G, 1 \le i \le n$, (i.e.) a is isolated-vertex. Hence G is not connected.

Remark 1.3: Every finite cyclic group holds the group sum graph are connected and cyclic graphs.

Theorem 1.4 [19]: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then

$$\deg_{u \in V(Z_{p^n})}(u) = \begin{cases} p^{n-1}(p-1) & \text{if } \mathcal{O}(u) \neq p^n \\ p^n - 1 & \text{if } \mathcal{O}(u) = p^n \end{cases},$$

Where $p \ge 3$ is the prime number and $n \ge 2$ is a positive integer number

Theorem 1.5 [20]: If $G_+\left(V(Z_{p^n q}), E(Z_{p^n q})\right)$, then

$$\deg_{u \in V(Z_{p^n q})}(u) = \begin{cases} p^{n-1}(pq - (p+q) + 1 & \text{if } \mathcal{O}(u) \neq p^n q \\ p^n q - 1 & \text{if } \mathcal{O}(u) = p^n q \end{cases},$$

where, $2 < p < q, n \ge 1, n \in \mathbb{Z}^+$

Theorem 1.6 [27]: If $G_+\left(V(Z_{pq^m}), E(Z_{pq^m})\right)$, then

$$\deg_{u \in V(Z_{pq^m})}(u) = \begin{cases} q^{m-1}(pq - (p+q) + 1 & \text{if } \mathcal{O}(u) \neq pq^m \\ pq^m - 1 & \text{if } \mathcal{O}(u) = pq^m \end{cases},$$

where, $2 < p < q, m \ge 1, m \in \mathbb{Z}^+$.

Theorem 1.7 [21]: If $G_+\left(V(Z_{p^n q^m}), E(Z_{p^n q^m})\right)$, then

$$\deg_{u \in V(Z_{p^n q^m})}(u) = \begin{cases} p^{n-1} q^{m-1}(pq - (p+q) + 1 & \text{if } \mathcal{O}(u) \neq p^n q^m \\ p^n q^m - 1 & \text{if } \mathcal{O}(u) = p^n q^m \end{cases},$$

where, $2 < p < q, n, m \ge 1, n \& m \in \mathbb{Z}^+$.

2. $\boldsymbol{ISI_{(\gamma,\mu)}(G)}$ **Topological index of** G_+ *where* $G = Z_{p^n}, Z_{p^n q}, Z_{pq^m}$ *and* $Z_{p^n q^m}$

Notice in this section that we will compute the $\boldsymbol{ISI_{(\gamma,\mu)}(G)}$ Topological index with exceptional cases.

Theorem 2.1: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then $ISI_{(1,1)} =$

$$p^{n-1}(p-1)(p^n-1)\left[(p^n-1)^2\left(p^{n-1}(p-1)-1\right) + p^{n-1}\left(p^{n-1}(p-1)\right)\left(p^n-1 + p^{n-1}(p-1)\right)\right].$$

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}, \alpha = p^{n-1}(p-1), S \subseteq Z_{p^n}, a_i \in Z_{p^n}$, where $\mathcal{O}(a_i) = p^n, \forall\ 1 \le i \le \alpha$.

Since every a_i It is adjacent to all vertices belonging to $Z_{p^n}, \forall\ 1 \le i \le \alpha$, except itself.

$$ISI_{(1,1)} = \left[\sum_{k=1}^{\alpha} d(a_1)d(a_k)\big(d(a_1) + d(a_k)\big) - \big(d(a_1)d(a_1)\big(d(a_1) + d(a_1)\big)\right] +$$

$$\left[\underbrace{\begin{matrix} d(a_1)d(0)\big(d(a_1) + d(0)\big) + d(a_1)d(p)\big(d(a_1) + d(p)\big) + \\ \dots + d(a_1)d\left(p^{n-1}(p-1)\right)\left(d(a_1) + d\left(p^{n-1}(p-1)\right)\right) \end{matrix}}_{p^{n-1}-terms}\right]$$

$$+\left[\sum_{k=1}^{\alpha} d(a_2)d(a_k)\big(d(a_2) + d(a_k)\big) - \big(d(a_2)d(a_1)\big(d(a_2) + d(a_1)\big) + \big(d(a_2)d(a_2)\big(d(a_2) + d(a_2)\big)\right] +$$

$$\left[\underbrace{d(a_2)d(0)\big(d(a_2) + d(0)\big) + \dots + d(a_2)d\left(p^{n-1}(p-1)\right)\left(d(a_2) + d\left(p^{n-1}(p-1)\right)\right)}_{p^{n-1}-terms}\right] + \dots +$$

$$\left[\sum_{k=1}^{\alpha} d(a_k)d(a_k)\big(d(a_k) + d(a_k)\big) - \left(\sum_{i=1}^{\alpha} d(a_k)d(a_k)\left(d(a_k) + d(a_k)\right)\right)\right] +$$

$$\left[\underbrace{d(a_k)d(0)\big(d(a_k) + d(0)\big) + \dots + d(a_k)d\left(p^{n-1}(p-1)\right)\left(d(a_k) + d\left(p^{n-1}(p-1)\right)\right)}_{p^{n-1}-terms}\right]$$

Since $d(a_1) = d(a_2) = \dots = d(a_\alpha) = d(a) = p^n - 1$ and

$d(0) = d(p) = \dots = \left(p^{n-1}(p-1)\right) = p^{n-1}(p-1) = \alpha$

$$\Rightarrow ISI_{(1,1)} = \left[(\alpha-1)d^2(a)\big(2d(a)\big) + p^{n-1}d(a)\alpha\big(d(a) + \alpha\big)\right]$$
$$+\left[(\alpha-2)d^2(a)\big(2d(a)\big) + p^{n-1}d(a)\alpha\big(d(a) + \alpha\big)\right] +$$
$$\vdots$$
$$\left[(\alpha-\alpha)d^2(a)\big(2d(a)\big) + p^{n-1}d(a)\alpha\big(d(a) + \alpha\big)\right]$$

$= \left[d^2(a)2d(a)\sum_{i=1}^{\alpha}(\alpha-i) + \alpha\rho^{n-1}d(a)\alpha\big(d(a) + \alpha\big)\right]$, since $\sum_{i=1}^{\alpha}(\alpha - i) = \frac{\alpha(\alpha-1)}{2}$

$$\Longrightarrow ISI_{(1,1)} = \left[d^2(a)\big(2d(a)\big)\frac{\alpha(\alpha-1)}{2} + \alpha\rho^{n-1}d(a)\alpha\big(d(a) + \alpha\big)\right] \dots (1)$$

$$ISI_{(1,1)} = \alpha d(a)\left[d^2(a)(\alpha - 1) + p^{n-1}\alpha\big(d(a) + \alpha\big)\right]$$

$ISI_{(1,1)} = p^{n-1}(p-1)(p^n-1)\left[(p^n-1)^2 \cdot (p^{n-1}(p-1)-1) + p^{n-1}\left(p^{n-1}(p-1)(p^n-1+\right.\right.$
$p^{n-1}(p-1)]$

$ISI_{(1,1)} = p^{n-1}(p^n-1)(p-1)\left[(p^n-1)^2(p^{n-1}(p-1)-1) + p^{n-1}\left(p^{n-1}(p-1)(2p^n-p^{n-1}-1)\right)\right]$

Remark 2.2: The generalized inverse sum index $ISI_{(\gamma,\mu)}(G_+)$ of a graph G_+ it was proposed by Buragohain et al.

$ISI_{(\gamma,\mu)} = \sum_{i\sim j} (d_i d_j)^\gamma (d_i + d_j)^\mu$, and $\gamma, \mu \in \mathbb{R}$.

Theorem 2.3: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then

$ISI_{(\gamma,\mu)} = p^{n-1}(p-1)(p^n-1)^\gamma\left[2^{\mu-1}(p^n-1)^{\gamma+\mu}\left(p^{n-1}(p-1)-1\right) + p^{n-1}\left(p^{n-1}(p-1)\right)^\gamma\left(p^n-1+\right.\right.$
$\left.\left.p^{n-1}(p-1)\right)^\mu\right]$.

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}, \alpha = p^{n-1}(p-1), S \subseteq Z_{p^n}, a_i \in Z_{p^n}$, where $O(a_i) = p^n, \forall\ 1 \le i \le \alpha$.

Since every a_i it is adjacent to all vertices belonging to $Z_{p^n}, \forall\ 1 \le i \le \alpha$, except itself.

$ISI_{(1,1)} = p^{n-1}(p^n-1)(p-1)\left[(p^n-1)^2\left(p^{n-1}(p-1)-1\right) + p^{n-1}\left(p^{n-1}(p-1)\left(2p^n-p^{n-1}-1\right)\right)\right]$

So, in general, for the formula $ISI_{(\gamma,\mu)} = \sum_{i\sim j} (d_i d_j)^\gamma (d_i + d_j)^\mu$

We can have $ISI_{(\gamma,\mu)}$ for $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, the proof follows immediately from eq.(1), we get

$$ISI_{(\gamma,\mu)} = \left[(d^2(a))^\gamma(2d(a))^\mu\left(\frac{\alpha(\alpha-1)}{2} + \alpha p^{n-1}(d(a)+\alpha)^\gamma(d(a)+\alpha)^\mu\right)\right]$$

$ISI_{(\gamma,\mu)} = \alpha(d(a))^\gamma[2^{\mu-1}(d(a))^{\gamma+\mu}(\alpha-1) + p^{n-1}(\alpha)^\gamma(d(a)+\alpha)^\mu]$

$ISI_{(\gamma,\mu)} = p^{n-1}(p-1)(p^n-1)^\gamma\left[2^{\mu-1}(p^n-1)^{\gamma+\mu}\left(p^{n-1}(p-1)-1\right) + p^{n-1}((p^{n-1}(p-\right.$
$\left.1))^\gamma(p^n-1+p^{n-1}(p-1))^\mu\right]$.

$ISI_{(\gamma,\mu)} = p^{n-1}(p-1)(p^n-1)^\gamma[2^{\mu-1}(p^n-1)^{\gamma+\mu}(p^{n-1}(p-1)-1) +$
$p^{n-1}((p^{n-1}(p-1))^\gamma(2p^n-p^{n-1}-1)^\mu]$

In particular, we can find exceptional cases:

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(1) $ISI_{(1,-1)}$ is an inverse sum index equal

$$p^{n-1}(p-1)(p^n-1)\left[\frac{p^{n-1}(p-1)-1}{4}+\frac{p^{n-1}\left(p^{n-1}(p-1)\right)}{(2p^n-p^{n-1}-1)}\right]$$

(2) $ISI_{(-1,1)}=\frac{p^{n-1}(p-1)}{(p^n-1)}\left[\left(p^{n-1}(p-1)-1\right)+\frac{p^{n-1}(2p^n-p^{n-1}-1)}{p^{n-1}(p-1)}\right]=p^n$.

(3) $2ISI_{(1/2,-1)}=\mathrm{GA}(Z_{p^n})$, where GA is a first geometric-arithmetic index

$$=2\left[p^{n-1}(p-1)\sqrt{(p^n-1)}\right]\left[\frac{p^{n-1}(p-1)-1}{4\sqrt{p^n-1}}+\frac{p^{n-1}\sqrt{p^{n-1}(p-1)}}{(2p^n-p^{n-1}-1)}\right]$$

(4) $ISI_{(1,1)}=\mathrm{ReZG}_3$, where $\mathrm{ReZG}_3(G)$ is a Redefined third Zagreb index

$$=p^{n-1}(p^n-1)(p-1)\left[(p^n-1)^2\left(p^{n-1}(p-1)-1\right)+p^{n-1}\left(p^{n-1}(p-1)\right)\left(2p^n-p^{n-1}-1\right)\right]$$

Example 2.4.

$Z_{3^3}=\{0,1,2,\dots,26\}, p=3, n=3, \mathcal{O}(Z_{3^3})=27.$

(1) $ISI_{(1,-1)}$ =3712.0909

(2) $ISI_{(-1,1)}=27=3^3$.

(3) $2ISI_{(1/2,-1)}=312.2998$.

(4) $ISI_{(1,1)}$= 8714160.

Theorem 2.5: If $G_+\left(V(Z_{p^nq}),E(Z_{p^nq})\right)$, then

$$ISI_{(1,1)}=\sum_{i\sim j}(d_id_j)(d_i+d_j)=$$

$$p^{n-1}(pq-(p+q)+1)(p^nq-1)\left[(p^nq-1)^2\left(p^{n-1}(pq-(p+q)+1)-1\right)\right.$$

$$\left.+p^{n-1}(p+q-1)p^{n-1}(pq-(p+q)+1)\left(p^{n-1}q(2p-1)-p^{n-1}(p-1)-1\right)\right].$$

Proof: Suppose that $S=\{a_1,a_2,\dots,a_\alpha\}, \alpha=p^{n-1}(pq-(p+q)+1), S\subseteq Z_{p^nq}, a_i\in Z_{p^nq}$, where $\mathcal{O}(a_i)=p^nq, \forall\ 1\le i\le\alpha$ Since, every a_i it is adjacent to all vertices belonging to $Z_{p^nq}, \forall\ 1\le i\le\alpha$, except itself.

$$ISI_{(1,1)}=\left[\sum_{k=1}^{\alpha}d(a_1)d(a_k)\left(d(a_1)+d(a_k)\right)-d(a_1)d(a_1)\left(d(a_1)+d(a_1)\right)\right]$$

$$+\left[\underbrace{d(a_1)d(0)\big(d(a_1)+d(0)\big)+d(a_1)d(p)\big(d(a_1)+d(p)\big)+\cdots}_{p^{n-1}(p+q-1)-terms}\right]$$

$$\left[\underbrace{\ldots+d(a_1)d\left(\alpha_n p^{n-1}q\right)\left(d(a_1)+d\left(\alpha_n p^{n-1}q\right)\right)}_{p^{n-1}(p+q-1)-terms}\right]$$

$+\left[\sum_{k=1}^{\alpha} d(a_2)d(a_k)\big(d(a_2)+d(a_k)\big)-d(a_2)d(a_1)\big(d(a_2)+d(a_1)\big)+\right.$
$\left. d(a_2)d(a_2)\big(d(a_2)+d(a_2)\big)\right]+$

$$[\underbrace{d(a_2)d(0)\big(d(a_2)+d(0)\big)+d(a_2)d(p)\big(d(a_2)+d(p)\big)+\cdots}_{p^{n-1}(p+q-1)-terms}$$

$$\underbrace{\ldots+d(a_2)d\left(\alpha_n p^{n-1}q\right)\left(d(a_2)+d\left(\alpha_n p^{n-1}q\right)\right)}_{p^{n-1}(p+q-1)-terms}]+\cdots+\left[\sum_{k=1}^{\alpha} d(a_\alpha)d(a_k)\big(d(a_\alpha)+\right.$$

$$\left.d(a_k)\big)-\left(\sum_{k=1}^{\alpha} d(a_\alpha)d(a_k)\big(d(a_\alpha)+d(a_k)\big)\right)\right]+$$

$$[\underbrace{d(a_\alpha)d(0)\big(d(a_\alpha)+d(0)\big)+d(a_\alpha)d(p)\big(d(a_\alpha)+d(p)\big)+\cdots}_{p^{n-1}(p+q-1)-terms}$$

$$\underbrace{\ldots+d(a_\alpha)d(\alpha_n p^{n-1}q)\big(d(a_\alpha)+d(\alpha_n p^{n-1}q)\big)}_{p^{n-1}(p+q-1)-terms}]$$

Since $d(a_1)=d(a_2)=\cdots=d(a_\alpha)=d(a)=p^nq-1$.

$d(0)=d(p)=\cdots=d\left(\alpha_n p^{n-1}q\right)=p^{n-1}\big(pq-(p+q)+1\big)=\alpha.$

$$\mathrm{ISI}_{(1,1)}=\Big[(\alpha-1)d^2(a)\big(2d(a)\big)+p^{n-1}(p+q-1)d(a)\alpha(d(a)+\alpha)\Big]+$$
$$\Big[(\alpha-2)d^2(a)\big(2d(a)\big)+p^{n-1}(p+q-1)d(a)\alpha(d(a)+\alpha)\Big]+\cdots+\Big[(\alpha-\alpha)d^2(a)\Big(2d(a)+$$
$$p^{n-1}(p+q-1)d(a)\alpha(d(a)+\alpha)\Big]$$

$$=\Big[d^2(a)\big(2d(a)\big)\sum_{i=1}^{\alpha}(\alpha-i)+\alpha p^{n-1}(p+q-1)d(a)\alpha(d(a)+\alpha)\Big]\ldots(2)$$

since $\sum_{i=1}^{\alpha}(\alpha-i)=\frac{\alpha(\alpha-1)}{2}$.

$$\Rightarrow ISI_{(1,1)}=\Big[d^2(a)\cdot\big(2d(a)\big)\left(\frac{\alpha(\alpha-1)}{2}\right)+\alpha p^{n-1}(p+q-1)d(a)\alpha(d(a)+\alpha)\Big]\ldots(3)$$

$$ISI_{(1,1)}=\alpha d(a)\Big[d^2(a)(\alpha-1)+p^{n-1}(p+q-1)\alpha(d(a)+\alpha)\Big]$$

$$ISI_{(1,1)} = p^{n-1}\big(pq-(p+q)+1\big)\big(p^nq-1\big)\Big[\big(p^nq-1\big)^2\Big(p^{n-1}\big(pq-(p+q)+1\big)-1\Big) +$$
$$p^{n-1}\big(p+q-1\big)p^{n-1}\big(pq-(p+q-1)+1\big)\Big(p^nq-1+p^{n-1}\big(pq-(p+q)+1\big)\Big]$$
$$ISI_{(1,1)} = p^{n-1}\big(pq-(p+q)+1\big)\big(p^nq-1\big)\Big[\big(p^nq-1\big)^2\big(p^{n-1}\big(pq-(p+q)+1\big)-1\big)$$
$$+p^{n-1}\big(p+q-1\big)p^{n-1}\big(pq-(p+q)+1\big)\Big(p^{n-1}q\big(2p-1\big)-p^{n-1}\big(p-1\big)-1\Big)\Big].$$

So, in general, for the formula $ISI(\gamma,\mu) = \sum_{i\sim j}\,(d_id_j)^{\gamma}(d_i+d_j)^{\mu}, \gamma,\mu \in \mathbb{R}$.

We can have $ISI_{(\gamma,\mu)}$ for $G_+\left(V\left(Z_{p^nq}\right), E\left(Z_{p^nq}\right)\right)$. The proof follows immediately from Eq.(3). We get

$$\mathrm{ISI}(\gamma,\mu) = \left[\left(d^2(a)\right)^{\gamma}(2d(a))\left(\frac{\alpha(\alpha-1)}{2} + \alpha p^{n-1}(p+q-1)(d(a)\alpha)\right)^{\gamma} - (d(a)+\alpha)^{\mu}\right] \ldots (4)$$

$$\mathrm{ISI}(\gamma,\mu) = \alpha(d(a))^{\gamma}\left[2^{\mu-1}\left(d(a)\right)^{\gamma+\mu}(\alpha-1)\,p^{n-1}(p+q-1)(\alpha)^{\gamma}(d(a)+\alpha)^{\mu}\right]$$

$$\mathrm{ISI}(\gamma,\mu) = \left(p^{n-1}\big(pq-(p+q)+1\big)\right)(p^nq-1)^{\gamma}\Big[2^{\mu-1}(p^nq-1)^{\gamma+\mu}\Big(p^{n-1}\big(pq-(p+q)+$$
$$1\big)-1+p^{n-1}(p+q-1)\left(p^{n-1}\big(pq-(p+q)+1\big)\right)^{\gamma}\left(p^nq-1+\left(p^{n-1}\big(pq-(p+q)+1\big)\right)\right)^{\mu}\Big].$$

In particular, we can find exceptional cases

(1) $ISI_{(1,-1)}$ = where $ISI_{(1,-1)}$ is an inverse sum index

$$ISI_{(1,-1)} = \left(p^{n-1}\big(pq-(p+q)+1\big)\right)(p^nq-1)\Bigg[\frac{1}{4}\left(p^{n-1}\big(pq-(p+q)+1\big)\right)-1+$$
$$\frac{p^{n-1}(p+q-1)\left(p^{n-1}\big(pq-(p+q)+1\big)\right)}{(p^nq-1)+\left(p^{n-1}\big(pq-(p+q)+1\right)}\Bigg].$$

(2) $$ISI_{(-1,1)} = \frac{\left(p^{n-1}\big(pq-(p+q)+1\big)\right)}{(p^nq-1)}\Bigg[\left(p^{n-1}\big(pq-(p+q)+1\big)-1\right)+$$
$$\frac{p^{n-1}(p+q-1)\left((p^nq-1)+\left(p^{n-1}\big(pq-(p+q)+1\big)\right)\right)}{p^{n-1}\big(pq-(p+q)+1\big)}\Bigg].$$

(3) $2ISI_{(1/2,-1)} = \mathrm{GA}\left(Z_{p^n}\right)$, where GA is a first geometric-arithmetic index

$$= 2\left[p^{n-1}(pq-(p+q)+1)\sqrt{p^nq-1}\left[\frac{p^{n-1}(pq-(p+q)+1)-1}{4\sqrt{p^nq-1}} + \frac{p^{n-1}(p+q-1)\sqrt{p^{n-1}(pq-(p+q)+1)}}{(p^nq-1)+\left(p^{n-1}(pq-(p+q)+1\right)}\right]\right].$$

(4) $ISI_{(1,1)}$ = ReZG3 , where ReZG3(G) is a Redefined third Zagreb index

$$=\left(p^{n-1}(pq-(p+q)+1)\right)(p^nq-1)\left[(p^nq-1)^2\left(p^{n-1}(pq-(p+q)+1)-1\right)+p^{n-1}(p+q-1)\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^nq-1+\left(p^{n-1}(pq-(p+q)+1)\right)\right)\right].$$

Example 2.6. $\mathbb{Z}_{3^35} = \{0,1,2,\dots,134\}, p = 3, q = 5, n = 3, \mathcal{O}(\mathbb{Z}_{3^35}) = 135.$

(1) $ISI_{(1,-1)} = 383695.32.$

(2) $ISI_{(-1,1)} = 135.$

(3)2 $ISI_{(1/2,-1)} = GA(Z_{p^nq}) = 4343.43.$

(4) $ISI_{(1,1)}$ = ReZG3= 12425215392.

Theorem 2.7: If $G_+\left(V(Z_{pq^m}),E(Z_{pq^m})\right)$, then $ISI_{(1,1)} = \sum_{i\sim j}(d_id_j)(d_i+d_j)$

$$= q^{m-1}(pq-(p+q)+1)(pq^m-1)\left[(pq^m-1)^2\left(q^{m-1}(pq-(p+q)+1)-1\right)+q^{m-1}(p+q-1)\cdot\left(q^{m-1}(pq-(p+q)+1)\left(pq^m-1+q^{m-1}(pq-(p+q)+1)\right]\right..$$

Proof: Suppose that $S = \{a_1,a_2,\dots,a_\alpha\}, \alpha = q^{m-1}(pq-(p+q)+1)$

$S \subseteq Z_{pq^m}, a_i \in Z_{pq^m}$, where $\mathcal{O}(a_i) = pq^m, \forall\, 1 \le i \le \alpha$

Since every a_i It is adjacent to all vertices belonging to $Z_{pq^m}, \forall 1 \le i \le \alpha$, except itself.

$$ISI_{(1,1)} = \left[\sum_{k=1}^{\alpha} d(a_1)d(a_k)\left(d(a_1)+d(a_k)\right)-\left(d(a_1)d(a_1)\right)\left(d(a_1)d(a_1)\right)\right]$$

$$+\left[\underbrace{d(a_1)d(0)(d(a_1)+d(0))+\dots+d(a_1)d\left(\alpha_{2m}pq^{m-1}\right)\left(d(a_1)+d\left(\alpha_{2m}pq^{m-1}\right)\right)}_{q^{m-1}(p+q-1)-terms}\right]$$

$$+\left[\sum_{k=1}^{\alpha} d(a_2)d(a_k)\left(d(a_2)+d(k)\right)-\left(d(a_2)d(a_1)\left(d(a_2)+d(a_1)+d(a_2)d(a_2)\left(d(a_2)+d(a_2)\right)\right]+\right.$$

$$\left[\underbrace{\begin{array}{c} d(a_2)d(0)\big(d(a_2)+d(0)\big)+d(a_2)d(p)\big(d(a_2)+d(p)\big) \\ +\cdots+d(a_2)d\Big(\alpha_{2m}pq^{m-1}\Big)\Big(d(a_2)+d\Big(\alpha_{2m}pq^{m-1}\Big)\Big) \end{array}}_{q^{m-1}(p+q-1)-terms}\right]$$

$$+\cdots+\left[\sum_{k=1}^{\alpha} d(a_k)d(a_k)\big(d(a_k)+d(a_k)\big)-\Big(\sum_{k=1}^{\alpha}\big(d(a_\alpha)d(a_k)\big)\big(d(a_\alpha)+d(a_k)\big)\Big)\right]$$

$$+\left[\underbrace{d(a_\alpha)d(0)(d(a_\alpha)+d(0))+\cdots+d(a_\alpha)d\Big(\alpha_{2m}pq^{m-1}\Big)\Big(d(a_\alpha)+d\Big(\alpha_{2m}pq^{m-1}\Big)\Big)}_{q^{m-1}(p+q-1)-terms}\right]$$

Since $d(a_1)=d(a_2)=\cdots=d(a_\alpha)=d(a)=pq^m-1$, and

$d(0)=d(p)=\cdots=d\Big(\alpha_{2m}pq^{m-1}\Big)=q^{m-1}(pq-(p+q)+1)=\alpha$

$$\begin{aligned} ISI_{(1,1)} = {} & \big[(\alpha-1)d^2(a)\big(2d(a)\big)+\beta d(a)\alpha(d(a)+\alpha)\big]+ \\ & \big[(\alpha-2)d^2(a)\big(2d(a)\big)+\beta d(a)\alpha(d(a)+\alpha)\big]+ \\ & \quad\vdots \\ & \big[(\alpha-\alpha)d^2(a)(2d(a))+\beta d(a)\alpha(d(a)+\alpha)\big] \end{aligned}$$

where $\beta=q^{m-1}(p+q-1)$

$=d^2(a)(2d(a))\sum_{i=1}^{\alpha}(\alpha-i)+\alpha\beta d(a)\alpha(d(a)+\alpha)$,Since $\sum_{i=1}^{\alpha}(\alpha-i)=\frac{\alpha(\alpha-1)}{2}$

$\Rightarrow ISI_{(1,1)}=\left[d^2(a)(2d(a))\left(\frac{\alpha(\alpha-1)}{2}\right)+\alpha\beta d(a)\alpha(d(a)+\alpha)\right]\cdots(5)$

$\Rightarrow ISI_{(1,1)}=\alpha d(a)\big[d^2(a)(\alpha-1)+\beta\alpha(d(a)+\alpha)\big]$

$$=q^{m-1}\big(pq-(p+q)+1\big)\big(pq^m-1\big)\Big[\big(pq^m-1\big)^2\Big(q^{m-1}\big(pq-(p+q)+1\big)-1\Big)+$$
$$q^{m-1}\big(p+q-1\big)\cdot\Big(q^{m-1}\big(pq-(p+q)+1\big)\Big(pq^m-1+q^{m-1}\big(pq-(p+q)+1\big)\Big]$$

So, in general, for the formula $ISI_{(\gamma,\mu)}=\sum_{i\sim j}\big(d_id_j\big)^{\gamma}\big(d_i+d_j\big)^{\mu}$

We can have $ISI_{(\gamma,\mu)}$ for $G_+\Big(V\big(Z_{pq^m}\big),E\big(Z_{pq^m}\big)\Big)$. The proof follows immediately from $Eq(5)$, we get

$$ISI_{(\gamma,\mu)}=\left[(d^2(a))^{\gamma}(2d(a))^{\mu}\left(\frac{\alpha(\alpha-1)}{2}+\alpha\beta(d(a)(\alpha))^{\gamma}(d(a)+\alpha)^{\mu}\right]\right.$$

$$ISI_{(\gamma,\mu)}=\alpha(d(a))^{\gamma}\Big[2^{\mu-1}(d(a))^{\gamma+\mu}\big(\alpha-1\big)+\beta(\alpha)^{\gamma}(d(a)+\alpha)^{\mu}\Big]$$

$$ISI_{(\gamma,\mu)} = q^{m-1}(pq-(p+q)+1)(pq^m-1)^{\gamma} \cdot \left[2^{\mu-1}(pq^m-1)^{\gamma+\mu}\left(q^{m-1}(pq-(p+q)+1)-1\right) + \beta\left(q^{m-1}(pq-(p+q)+1)\right)^{\gamma}\left(pq^m-1+q^{m-1}(pq-(p+q)+1)\right)^{\mu}\right]$$

where $\beta = q^{m-1}(p+q-1), \gamma \& \mu \in \mathbb{R}$.

In particular, we can find exceptional cases:

(1) $ISI_{(1,-1)}$ = where $ISI_{(1,-1)}$ is an inverse sum index

$$ISI_{(1,-1)} = \left(q^{m-1}(pq-(p+q)+1)\right)(pq^m-1) \cdot \left[\frac{\left(q^{m-1}(pq-p+q)+1\right)-1)}{4} + \frac{q^{m-1}(p+q-1)\left(q^{m-1}(pq-(p+q)+1)\right)}{\left(pq^m-1+q^{m-1}(pq-(p+q)+1\right)}\right]$$

$$(2) ISI_{(-1,1)} = \frac{q^{m-1}(pq-(p+q)+1)}{pq^m-1} \cdot \left[\left(q^{m-1}(pq-(p+q)+1)-1\right) + \frac{q^{m-1}(p+q-1)\left(pq^m-1+q^{m-1}(pq-(p+q)+1)\right)}{\left(q^{m-1}(pq-(p+q)+1)\right)}\right] = pq^m$$

(3) $2ISI_{(1/2,-1)} = \mathrm{GA}(Z_{p^n})$, where GA is a first geometric-arithmetic index

$$= 2\left[q^{m-1}(pq-(p+q)+1)\sqrt{pq^m-1} \cdot \left[\frac{\left(q^{m-1}(pq-(p+q)+1)\right)-1}{4\sqrt{pq^m-1}} + \frac{q^{m-1}(p+q-1)\sqrt{q^{m-1}(pq-(p+q)+1)}}{\left(pq^m-1+q^{m-1}(pq-(p+q)+1)\right)}\right]\right]$$

(4) $ISI_{(1,1)} = \mathrm{ReZG_3}$, where $\mathrm{ReZG_3(G)}$ is a Redefined third Zagreb index

$$= \left(q^{m-1}(pq-(p+q)+1)\right)(pq^m-1)\left[(pq^m-1)^2\left(q^{m-1}(pq-(p+q)+1)-1\right) + q^{m-1}(p+q-1)\left(q^{m-1}(pq-(p+q)+1)\right)\left(pq^m-1+\left(q^{m-1}(pq-(p+q)+1)\right)\right)\right].$$

Example 2.8.

$Z_{35^3} = \{0,1,2,\dots,374\}, p=3, q=5, m=3, \mathcal{O}(Z_{35^3}) = 375.$

(1) $ISI_{(1,-1)} = 82822748.$

(2) $ISI_{(-1,1)} = pq^m = 375.$

(3) $2ISI_{(1/2,-1)} = GA(Z_{pq^m}) = 53253.1569.$

(4) $ISI_{(1,1)} = \mathrm{ReZG_3} = 3584814235200.$

Theorem 2.9: If $G_+\left(V\left(Z_{p^nq^m}\right), E\left(Z_{p^nq^m}\right)\right)$, then

$ISI_{(1,1)} = \sum_{i\sim j}\left(d_id_j\right)\left(d_i + d_j\right) = p^{n-1}q^{m-1}(pq-(p+q)+1)\left(p^nq^m-1\right)\left[\left(p^nq^m-1\right)^2 \cdot \left(p^{n-1}q^{m-1}(pq-(p+q)+1)-1\right) + p^{n-1}q^{m-1}(p+q-1) \cdot p^{n-1}q^{m-1}(pq-(p+q)+1)\left(p^nq^m-1+p^{n-1}q^{m-1}(pq-(p+q)+1)\right)\right]$,

where $2 < p < q$, and n, m are positive integers numbers, $n, m \geq 1$.

Proof: The proof follows immediately from Theorem 4.5 and 4.7. We have the result.

Theorem 2.10: The general inverse sum index $ISI_{(1,1)}(G)$ of $G_+\left(V\left(Z_{p^nq^m}\right), E\left(Z_{p^nq^m}\right)\right)$, defined as $ISI_{(\gamma,\mu)} = \sum_{i\sim j}\left(d_id_j\right)^{\gamma}\left(d_i + d_j\right)^{\mu} =$

$ISI_{(\gamma,\mu)} = p^{n-1}q^{m-1}(pq-(p+q)+1)\left(p^nq^m-1\right)^{\gamma} \cdot \left[2^{\mu-1}\left(p^nq^m-1\right)^{\gamma+\mu}\left(p^{n-1}q^{m-1}(pq-(p+q)+1)-1\right) + \beta\left(p^{n-1}q^{m-1}(pq-(p+q)+1)\right)^{\gamma}\left(p^nq^m-1+p^{n-1}q^{m-1}(pq-(p+q)+1)\right)^{\mu}\right]$

where $\beta = p^{n-1}q^{m-1}(p+q-1)$, $2 < p < q$, and n, m are positive integers numbers, $n, m \geq 1, \gamma, \mu \in R$

Proof: The proof follows immediately from Theorem 4.5 and 4.7. we have the result.

Conclusions

In the study, we will compute the $ISI_{(\gamma,\mu)}(G)$ Topological index with exceptional cases of graphs of groups Z_{p^n}, Z_{p^nq}, Z_{pq^m} *and* $Z_{p^nq^m}$, and some particular exceptional cases indices of a group Z_{p^n}, Z_{p^nq}, Z_{pq^m} *and* $Z_{p^nq^m}$.

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