Solve the Position to Time Equation for an Object Travelling on a Parabolic Orbit in Celestial Mechanics

Mohammed S. Rasheed

Applied Sciences Department

University of Technology

Email: msr197575@yahoo.com

Abstract

in this paper, the two body problem equation in parabolic orbit in celestial mechanics is solved using new iterative method with quadratic convergence. Initial solution is suggested depending on the time $(t-\tau)$, earth gravitational constant μ and the angular distance P to be $d_o = 6\,\mu\sqrt{\frac{\mu}{P^3}}\,(t-\tau)$, $M > 0$. The proposed methods considerably to be improvement of Newton's method with less iteration are needed to reach the solution of two body problem in parabolic orbit.

Keywords: Parabolic orbit, Barker's formula, Two body problem, Iterative methods, Order of convergence, Astrophysics.

Introduction

The determination of the position and velocity in two-body orbits leads to the solution of transcendental equation commonly referred to as "Kepler's equation" which relates the dependence of position in orbit with time. In classical analysis, the shape of theses two-body orbits is described through the use of conics and corresponding to each conic Kepler's equation has a different form. A useful quantity in classifying conics is a constant e called eccentricity [1-3]. In virtually every decade from 1650 to the present there have appeared papers devoted to Kepler's problem and its solution [4,5,6,7,8]. One of the usual ways to Kepler's equation is by the mean of iterative algorithms [9,10]. Several numerical methods have been suggested and analyzed under certain conditions. These numerical methods have been constructed using different technique such as Laguerre algorithm [11], Baoubaker Polynomials Expansion Scheme [12], and Richardson [13], others can be found in [14-18]

Properties of Parabolic Orbits

If an object attains escape velocity, but is not directed straight away from the planet, then it will follow a curved path. Although this path does not form a closed shape, it is still considered an orbit. Assuming that gravity is the only significant force in the system, this object's speed at any point in the orbit will be equal to the escape velocity at that point. The shape of the orbit will be a parabola whose focus is located at the centre of mass of the planet. The parabola can be shown to be the limiting form of both the ellipse and hyperbola as (e) tends to unity. Here Kepler's equation is [3]

$$2\sqrt{\frac{\mu}{p^3}}\, t = \left\{\tan\left(\frac{f}{2}\right) - \tan\left(\frac{f_0}{2}\right)\right\} + \frac{1}{3}\left\{tan^3\left(\frac{f}{2}\right) - tan^3\left(\frac{f_0}{2}\right)\right\} \quad (1)$$

where

μ =G.M. where G= universal gravitational constant and M=the solar mass of the two bodies,

p is the semi-latus rectum or parameters ,

f called the true anomaly, is the angle between the radius vector and the direction of pericenter or point of closest approach of the two bodies .

As e approaches unity from either the hyperbola or ellipse approaches infinity . Define the variable D such that

$$Y = \frac{r_0}{q}(e-1)D^2 \quad (2)$$

Then Y tends to 0 (if $e \to 1$). Hence from the following equation [1]

$$\sqrt{\frac{\mu}{r_o^3}}\, t = B\,(D + C\,D^2 + \tfrac{1}{3}\,Y\,D^3) \qquad (3)$$

It is easily verified that both B and C approach unity

we have

$$\sqrt{\frac{\mu}{r_o^3}}\, t = D + \frac{\delta_o}{2\sqrt{r_o}} D^2 + \frac{1}{6} D^3 \qquad (4)$$

using the fact that for parabolic motion

$$\sin f_o = \frac{\sqrt{p}\,\delta_o}{r_o}, \quad \cos f_o = \frac{p}{r_o} - 1 \qquad (5)$$

and that the root of Eq. (4) is

$$D = \frac{\sqrt{2}\,\sin(\frac{f-f_o}{2})}{\cos(\frac{f}{2})} \qquad (6)$$

Then, substitution of Eq. (5) and Eq. (6) does indeed lead to Eq. (1). As a case in point; at pericenter $f_o = 0$; hence

$$D = \sqrt{2}\tan(\tfrac{f}{2}), \quad \delta_o = 0, \quad \frac{p}{r_o} = 2, \qquad (7)$$

therefore Eq. (4) becomes

$$\sqrt{\frac{\mu}{r_o^3}}\,(t-\tau) = \sqrt{2}\,tan(\tfrac{f}{2}) + \frac{\sqrt{2}}{3} tan^3(\tfrac{f}{2}) \qquad (8)$$

or

$$2\sqrt{\frac{\mu}{r_o^3}}\,(t-\tau) = tan(\tfrac{f}{2}) + \frac{1}{3} tan^3(\tfrac{f}{2})$$

(9)

which is Barker's formula. Therefore; as Y approaches to 0 $(e \to 1)$, the hyperbolic and elliptic forms reduce to the parabolic form.

where τ is the time of perihelion passage.

Define $\bar{n}$ by the equation

$$\bar{n}^2\, p^3 = \mu \qquad (10)$$

Let $d = \tan\frac{f}{2}$ hence, eq. (4) may be written as

$$d + \frac{d^3}{3} = 2\bar{n}\,(t-\tau) \qquad (11)$$

Equations (9) and (11) are versions of Barker's equation, which has been extensively used in studies of the orbits of comets and is now used in Astrodynamics [1].

Two Step New Iterative Method

The objective of this section is based on suggesting a new iterative method for solving Eq. (11) as follows

Rewrite Eq. (11) in the following form

$$f(d) = d^3 + 3d - 6\bar{n}(t - \tau) \qquad (12)$$

Now we suggest the following algorithm for solving Eq. (12)

INPUT initial approximate solution $d_o = 6\,\mu\sqrt{\frac{\mu}{p^3}}\,(t - \tau),\quad M > 0$

tolerance ε , maximum number of iterations N.

OUTPUT approximate solution d_{n+1}.

Step 1: Set $n = 0$ and $i = 1$.

Step 2: While $i \leq N_0$ do steps 3-5.

Step 3: Calculate

$$y_n - d_n - \frac{2\,f(d_n)}{3\,f'(d_n)}$$

$$d_{n+1} = d_n - \frac{2\,f(d_n)}{f'(d_n) + f'(y_n)}, \qquad \text{for} \quad n = 0,1,2,\ldots$$

Step 4: If $|d_{n+1} \quad d_n| < \varepsilon;$ then OUTPUT (d_{n+1}) and stop.

Step 5: Set n=n+1; i=i+1 and go to Step 2.

Step 6: OUTPUT.

The convergence analysis of iterative technique given by the above algorithm will be discussed.

Expanding $f(d_n)$ and $f'(d_n)$ about α , to get

$$f(d_n) = f(\alpha) = +(d_n - \alpha)f'(\alpha) + \frac{(d_n - \alpha)^2}{2!}f^{(2)}(\alpha) + \frac{(d_n - \alpha)^3}{3!}f(\alpha) + \cdots$$

then

$$f(d_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots] \qquad (13)$$

$$f'(d_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots] \quad (14)$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k = 1,2,3, \ldots$ and $e_n = d_n - \alpha$

From Eqs. (13) and (14), we have

$$\frac{f(d_n)}{f'(d_n)} = [e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + \cdots] \quad (15)$$

and $$\frac{2\,f(d_n)}{3\,f'(d_n)} = \frac{2}{3}e_n - \frac{2}{3}c_2e_n^2 + \frac{4}{3}(c_2^2 - c_3)e_n^3 + \cdots] \quad (16)$$

using Eq. (15) and (16), yields

$$y_n = \alpha + \frac{1}{3}e_n + \frac{2}{3}c_2e_n^2 + \left(\frac{4}{3}c_3 - \frac{4}{3}c_2^2\right)e_n^3 + \cdots \quad (17)$$

By Taylor's series, we have

$$f(y_n) = f'(\alpha)[\frac{1}{3}e_n + \frac{7}{9}c_2e_n^2 + \left(\frac{37}{27}c_3 - \frac{8}{9}c_2^2\right)e_n^3 + \cdots \quad (18)$$

and

$$f'(y_n) = f'(\alpha)[(1 + \frac{2}{3}c_2e_n + \left(\frac{4}{3}c_2^2 + \frac{1}{3}c_3\right)e_n^2 + \left(4c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}.7c_4\right)e_n^3 + \cdots] \quad (19)$$

obtain $f'(d_n) + f'(y_n)$ using Eqs. (14) and (19)

Hence

$$[f'(d_n) + f'(y_n)] = f'(\alpha)[2 + \left(2c_2 + \frac{2}{3}c_2\right)e_n + \left(3c_3 + \frac{4}{3}c_2^2 + \frac{1}{3}c_3\right)e_n^2 + (4c_4 + 4c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}.7c_4)e_n^3 \quad (20)$$

$$[f'(d_n) + f'(y_n)] = f'(\alpha)[2 + \left(\frac{8}{3}c_2\right)e_n + \left(\frac{4}{3}c_2^2 + \frac{10}{3}c_3\right)e_n^2 + \left(4c_2c_3 - \frac{8}{3}c_2^3 + \frac{136}{27}c_4\right)e_n^3 + \cdots \quad (21)$$

$$\frac{2\,f(d_n)}{f'(d_n) + f'(y_n)} = e_n - \frac{1}{3}c_2\,e_n^2 + 0(e_n^3)$$

$$d_{n+1} = \alpha + \frac{1}{3}c_2\,e_n^2 + 0(e_n^3)$$

or

$$e_{n+1} - \frac{1}{3}c_2\,e_n^2 + 0(e_n^3) \quad (22)$$

Thus, we observe that the proposed algorithm has quadratic order convergence.

Application of the new method to solve parabolic orbit equation

Apply the suggested algorithm to solve the parabolic orbit equation Eq. (11) , $d + \frac{d^3}{3} = 2\bar{n}\,(t-\tau)$ with $(t-\tau) = 1.2025\ TU$ (time unit) , $p = 2\ AU$ (angular distance unit) and $\mu = 1$. Using Eq. (10) to obtain $\bar{n}$, ($\bar{n}^2\,p^3 = \mu$) that is $\bar{n} = 0.353553390593274$, therefore; $b = 6\,\bar{n}\,(t-\tau) = 2.550887713130470$. Take the suggested initial solution $d_o = 6\,\mu\sqrt{\frac{\mu}{p^3}}\,(t-\tau)$, $M > 0$, the numerical results for Eq. (11) for $M - 0.05 + i\ 0.05$; $i - 0$, 1, 2,…15 to get the solution of $d_n = 0.723865337018299$, are listed in the table (1)

We take $\varepsilon - 10^{-15}$ as tolerance. The following criteria is used for estimating the zero

$\sigma = |d_{n+1} - d_n| < \varepsilon$, $|f(d_n)| < \varepsilon$

For convergence criteria, it was required that σ the distance between two consecutive iterates was less than 10^{-15}, n represents the number of iterations and $f(d_n)$, the absolute value of the function. Also the computational order of convergence (ρ) can be approximated using the formula [10]

$$\rho = \frac{ln|(d_{n+1} - \alpha)/(d_n - \alpha)|}{ln|(d_n - \alpha)/(d_{n-1} - \alpha)|}$$

Table (1) shows the results using Newton and presented methods for different values of d_o

No. of cases	M	Initial guess d_o	NM	σ	ρ	Presented Method	σ	ρ
1	0.05	0.127544385656524	6	1.509e-010	1.76959	4	6.041e-011	1.58814
2	0.1	0.255088771313047	5	1.430e-011	1.88283	4	7.579e-011	1.70040
3	0.15	0.382633156969571	5	2.239e-013	1.95315	4	1.477e-009	1.81001
4	0.2	0.510177542626094	5	2.106e-008	1.98779	4	2.675e-012	1.90395
5	0.25	0.637721928282618	5	1.637e-011	1.99923	4	4.996e-015	1.97260
6	0.3	0.765266313939141	5	9.992e-016	1.86088	4	2.779e-013	2.04079
7	0.35	0.892810699595665	5	3.335e-009	1.99224	4	3.272e-012	2.04649
8	0.4	1.020355085252188	5	3.297e-014	1.97581	4	9.992e-016	2.08068
9	0.42	1.093237591341630	5	9.390e-013	1.96236	4	2.853e-009	2.09672
10	0.45	1.147899470908712	5	7.367e-012	1.95052	4	9.519e-009	2.10663
11	0.5	1.275443856565235	6	3.384e-010	1.91783	4	2.979e-013	4.85930
12	0.55	1.402988242221759	6	6.199e-009	1.87971	4	5.195e-014	2.12553
13	0.6	1.530532627878282	7	1.998e-015	1.83816	4	9.620e-013	2.11821
14	0.65	1.658077013534806	7	7.794e-014	1.79494	4	1.123e-011	2.10115
15	0.7	1.785621399191329	7	1.771e-012	1.75148	4	9.155e-011	2.07606
16	0.75	1.913165784847853	7	5.496e-014	1.97411	4	9.992e-016	2.04483

Conclusions

The solution of two body problem in a parabolic orbit, where discussed where the true anomaly f (as a function of the time) can be obtained by solving a cubic equation for $\tan(\frac{f}{2})$ named Barker's formula. We have suggested and analyzed two step iterative method which works well for Barker's formula with suitable suggested initial solution for the iterative. We proved that the convergence of the new method is quadratic and showed that the proposed method provided that only the first derivative of the function exist, and it is not required to compute second or higher derivatives of the function to carry out iterations. The results in table (1) demonstrated that the proposed two step method is better than Newton's method and we can see accuracy and efficiency of our two step method when compared with the Newton's method. Note that only four iterations are needed to reach the exact solution with small tolerance, while Newton's method requires five, six or seven iterations.

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Mohammed S. Rasheed

حل معادلة الموقع الى الزمن لأي جسم يتنقل على مدار القطع المكافيء في الميكانيك السماوي

محمد سهام رشيد
الجامعة التكنولوجية
قسم العلوم التطبيقية
الجامعة التكنولوجية- ص.ب: 35317 - بغداد- العراق.
Email: msr197575@yahoo.com

الخلاصة

في هذه البحث، تم حل مسألة جسمين متواجدين في مدار القطع المُكافيء في الميكانيك السماوي بأستخدام طريقة تكرارية جديدة ذات أقتراب تربيعي. تم أقتراح حل ابتدائي يعتمد على الزمن $(t-\tau)$ ، وثابت الجاذبية الارضية μ والمسافة الزاوية P ليكون $(t-\tau)\sqrt{\frac{\mu}{P^{3}}}\,\mu\, 6 = d_o\ 0 < M$, . الطريقة المُقترحة تُعتبر تحسين لطريقة نيوتن وتحتاج الى تكرارات أقل للوصول لحل مسألة الجسمين المتواجدين في مسار القطع المُكافيء.

الكلمات المفتاحية: المدار المكافىء, صيغة باركر, مشكلة جسمين أثنين, الطرق التكرارية, رتبة الاقتراب، الفيزياء الفلكية.